Date: August 25, 2019 | Time: 10 AM to 1 PM

Number of Questions: 30 | Max Marks: 102

Aakash

Medical | IIT-JEE | Foundations

(Divisions of Aakash Educational Services Limited)

Regd. Office **:** Aakash Tower, 8, Pusa Road, New Delhi-110005
Ph.: 011-47623456

Answers & Solutions

for

MTA PRMO - 2019

INSTRUCTIONS

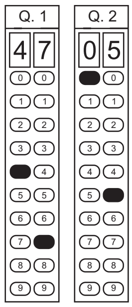
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a black pencil or a black or blue ball pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your PRMO score.
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS

1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.

WRONG METHODS	CORRECT METHOD
Ⓐ ● ⊙ Ⓓ	Ⓐ ● Ⓒ Ⓓ

4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 6 carry **2 marks** each; questions 7 to 21 carry **3 marks** each; questions 22 to 30 carry **5 marks** each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the Candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

1. Consider the sequence of numbers $\left[n+\sqrt{2n}+\frac{1}{2}\right]$ for $n \geq 1$, where [x] denotes the greatest integer not exceeding x. If the missing integers in the sequence are $n_1 < n_2 < n_3 < \ldots$. then find n_{12}.

Answer (91)

Sol. Let $x_n = n+\left[\sqrt{2n}+\frac{1}{2}\right]$ for $n \in N$

when $\sqrt{2n}+\frac{1}{2}$ will cross over some integer; an integer will be missed from the sequence $<x_n>$

Let $\left[\sqrt{2n}+\frac{1}{2}\right]=k$ when $k \in N$

$\Rightarrow \sqrt{2n}+\frac{1}{2} \in [k, k+1)$

$\Rightarrow n \in \left[\frac{k(k-1)}{2}+\frac{1}{8}, \frac{k(k+1)}{2}+\frac{1}{8}\right)$

Clearly $n = \frac{k(k+1)}{2}$

So, the integer $(x_n + 1)$ will not be available in the sequence $<x_n>$ for $n = \frac{k(k+1)}{2}$

Then $n_1 = x_1 + 1$, $n_2 = x_3 + 1$, $n_3 = x_6 + 1$, ...

$n_{12} = x_{78} + 1 = 78 + \left[\sqrt{156}+\frac{1}{2}\right]+1 = 91$

2. If $x = \sqrt{2}+\sqrt{3}+\sqrt{6}$ is a root of $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are integers, what is the value of $|a + b + c + d|$?

Answer (93)

Sol. $\left(z-(\sqrt{2}+\sqrt{3})\right)^2 = 6$

$z^2+\left(5+2\sqrt{6}\right)-2z\left(\sqrt{2}+\sqrt{3}\right)=6$

$z^2+5+2\sqrt{6}-2z\left(z-\sqrt{6}\right)=6$

$z^2-2z^2-1+2\sqrt{6}\,(z+1)=0$

$\Rightarrow 2\sqrt{6}\,(z+1)=z^2+1$

$\Rightarrow 24\,(z+1)^2 = z^4 + 2z^2 + 1$

$\Rightarrow 24\,(z^2 + 2z + 1) = z^4 + 2z^2 + 1$

$\Rightarrow z^4 - 22z^2 - 48z - 23 = 0$

$\therefore a = 0;\ b = -22;\ c = -48;\ d = -23$

$|a + b + c + d| = |-93| = 93$

3. Find the number of positive integers less than 101 that can not be written as the difference of two squares of integers.

Answer (25)

Sol. Let the two integers be x and y

$x^2 - y^2 = k \quad \Rightarrow \quad (x + y)(x - y) = k$

$\because$ $(x + y)$ and $(x - y)$ both are even or both are odd.

So, this equation will not have solution for x and y only if $k = 4\lambda + 2$ ($\lambda \in$ integer)

$\Rightarrow k = 2, 6, 10, \ldots, 98$

Number of such values of k = 25

4. Let $a_1 = 24$ and form the sequence a_n, $a \geq 2$ by $a_n = 100a_{n-1} + 134$. The first few terms are

24, 2534, 253534, 25353534,......

What is the least value of n for which a_n is divisible by 99?

Answer (88)

Sol. $\because$ $a_n = 11\lambda + (a_{n-1} + 2)$ and $a_n = 9\lambda + (a_{n-1} - 1)$

$\because a_1 \equiv 2 \pmod{11}$

$\Rightarrow a_2 \equiv 4 \pmod{11}$

$\Rightarrow a_3 \equiv 6 \pmod{11}$

and so on.

So $a_{11}, a_{22}, a_{33}, \ldots$ will be divisible by 11.

Again, $a_1 \equiv 6 \pmod 9$

$a_2 \equiv 5 \pmod 9$

$a_3 \equiv 4 \pmod 9$

and so on.

So, $a_7, a_{16}, a_{25}, a_{34}, \ldots$ will be divisible by 9.

The first term that is divisible by both 9 and 11 will be a_{88}.

5. Let N be the smallest positive integer such that N + 2N + 3N +....+ 9N is a number all whose digits are equal. What is the sum of the digits of N?

Answer (37)

Sol. Let x = N + 2N + 3N + ... + 9N

$\Rightarrow$ x = 45 N

Let all digits of x be a and x be a 'n' digit number.

$$\frac{a}{9}\left(10^n - 1\right) = 45\,N \quad \Rightarrow \quad 10^n - 1 = \frac{9\times 9\times 5}{a}N$$

$\because$ Last digit of $10^n - 1$ is 9 so a must be 5.

Now 9 × 9 × N = 999 9

$\Rightarrow$ 9 N = 1 1 1 1 1

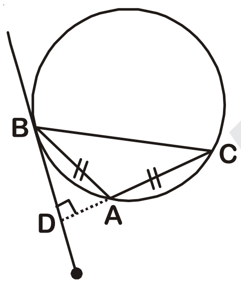
$\because$ 9N is divisible by 9 so least value of 9 N will be $9N = \underbrace{111111111}_{9\text{ times}}$

$\Rightarrow$ N = 12345679

6. Let ABC be a triangle such that AB = AC. Suppose the tangent to the circumcircle of ABC at B is perpendicular to AC. Find ∠ABC measured in degrees.

Answer (30)

Sol.



Let BD is tangent to the circle at B, and AC intersect it at D.

∠DBA = ∠ACB = ∠ABC = θ

then ∠BAD = 2θ (external angle of ΔABC)

In ⊥ ΔABD

∠ABD + ∠BAD = 90°

$\Rightarrow$ θ + 2θ = 90°

$\Rightarrow$ θ = 30° = ∠ABC

7. Let s(n) denote the sum of the digits of a positive integer n in base 10. If s(m) = 20 and s(33m) = 120, what is the value of s(3m) ?

Answer (60)

Sol. S(m) = 20 = S(10m)

$\because$ S(m + n) ≤ S(m) + S(n)

$\Rightarrow$ S(11m) ≤ S(10m) + S(m)

$\Rightarrow$ S(11m) ≤ 40

Similarly S(33m) ≤ S(11m) + S(11m) + S(11m)

$\Rightarrow$ S(33m) ≤ 120

Here equality holds, which is only possible when all digits of m are either 1 or 2

(with no two consecutive 2's)

S(3 m) = 3·S(m) = 3 × 20 = 60

8. Let $F_k(a, b) = (a + b)^k - a^k - b^k$ and let S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. For how many ordered pairs (a, b) with a, b∈S and a ≤ b is $\frac{F_5(a,b)}{F_3(a,b)}$ an integer?

Answer (22)

Sol. $$\frac{F_5(a,b)}{F_3(a,b)} = \frac{(a+b)^5 - a^5 - b^5}{(a+b)^3 - a^3 - b^3} = \frac{5}{3}.(a^2 + b^2 + ab)$$

$\because$ $a^2 + b^2 + ab$ is divisible by 3 then

a ≡ b (mod 3).

So (a, b) = (1, 4), (1, 7), (1, 10), (4, 7), (4, 10), (7, 10)
or
(2, 5), (2, 8), (5, 8)
or
(3, 6), (3, 9), (6, 9)

} when a < b

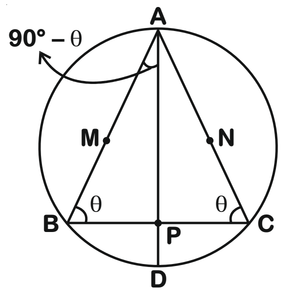
Also (1, 1), (2, 2), ... (10, 10) when a = b

Total number of ordered pairs = 22

9. The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is α° and the smaller one β° then what is the value of α – β?

Answer (90)

Sol.



Let AB = AC = ℓ

and M, N, P be the mid-points of sides as shown.

Circumcentre of ΔMNP lies on the perpendicular bisector of MN and also on the circumcircle of ΔABC; so the circumcentre of ΔMNP will be A or D.

But a circle with centre D and radius PD can't pass through M and N. Then A will be certainly the circumcentre.

AM = AP

$\Rightarrow \frac{\ell}{2} = \ell \sin\theta$

$\Rightarrow \theta = 30°$

So α = 120° and β = 30°

10. One day I went for a walk in the morning at x minutes past 5'O clock, where x is a two digit number. When I returned, it was y minutes past 6'O clock, and I noticed that (i) I walked exactly for x minutes and (ii) y was a 2 digit number obtained by reversing the digits of x. How many minutes did I walk?

Answer (42)

Sol. Let x = 10a + b then y = 10b + a

It is given that;

60 + y – x = x

$\Rightarrow$ 60 + 8b = 19a

$\Rightarrow$ 4(15 + 2b) = 19a

So, possible values of a are 4 or 8

When a = 4, b = 2

When a = 8, $b = \frac{13}{2}$ (Not possible)

So, x = 42

11. Find the largest value of a^b such that the positive integers a, b > 1 satisfy $a^b b^a + a^b + b^a = 5329$.

Answer (81)

Sol. $(a^b + 1)(b^a + 1) = 5330$

$\Rightarrow (a^b + 1)(b^a + 1) = 2 \times 5 \times 13 \times 41$

The possible unordered pairs of a^b and b^a are

(4, 1065), (12, 409), (40, 129), (9, 532), (25, 204) and (81, 64)

$\because$ a and b are positive integers, so only possibility is $a^b = 81$ and $b^a = 64 \Rightarrow a = 3, b = 4$

12. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set

$\{10a + b : 1 \le a \le 5, 1 \le b \le 5\}$

where a, b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?

Answer (47)

Sol.

11	12	13	14	15
21	22	23	24	25
31	32	33	34	35
41	42	43	44	45
51	52	53	54	55

From the set if 5 distinct numbers are chosen such that no two selected numbers has same unit digits and tens digits are

$5 \times 4 \times 3 \times 2 \times 1 = 120$

$\therefore$ Remainder = 47

13. Consider the sequence

1, 7, 8, 49, 50, 56, 57, 343, ...

which consists of sums of distinct powers of 7, that is, $7^0, 7^1, 7^0 + 7^1, 7^2, \ldots$, in increasing order. At what position will 16856 occur in this sequence?

Answer (36)

Sol. $16856 = 7^5 + 7^2$

Position of $7^0 = 1$

Position of $7^1 = 2$

Position of $7^2 = 4$

Position of $7^3 = 8$

Position of $7^4 = 16$

Position of $7^5 = 32$

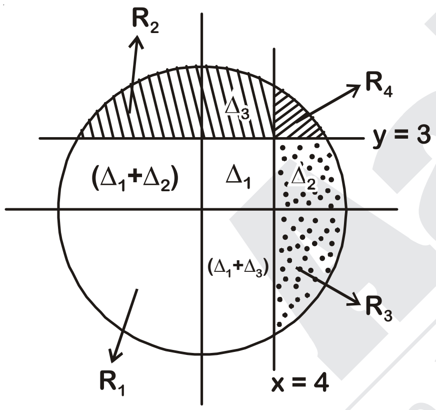
Next numbers will be $7^5 + 7^0$, $7^5 + 7^1$, $7^5 + 7^0 + 7^1$, $7^5 + 7^2$, ...

So, position of $7^5 + 7^2$ is 36^{th}

14. Let R denote the circular region in the xy-plane bounded by the circle $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ divide R into four regions R_i, i = 1, 2, 3, 4. If $[R_i]$ denotes the area of the region R_i and if $[R_1] > [R_2] > [R_3] > [R_4]$, determine $[R_1] - [R_2] - [R_3] + [R_4]$. [Here $[\Omega]$ denotes the area of the region Ω in the plane.]

Answer (48)

Sol. Let area of the quarter circle be Δ, and area of different regions are shown in the diagram.



$[R_1] = \Delta + 3\Delta_1 + \Delta_2 + \Delta_3$

$[R_2] = \Delta + \Delta_3 - \Delta_1 - \Delta_2$

$[R_3] = \Delta + \Delta_2 - \Delta_1 - \Delta_3$

$[R_4] = \Delta - \Delta_1 - \Delta_2 - \Delta_3$

So $[R_1] + [R_4] = 2\Delta + 2\Delta_1$ and

$[R_2] + [R_3] = 2\Delta - 2\Delta_1$

$\Rightarrow [R_1] - [R_2] - [R_3] + [R_4] = 4\Delta_1 = 4\ (3 \times 4) = 48$

15. In base –2 notation, digits are 0 and 1 only and the places go up in powers of –2. For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals number 7 in base 10. If the decimal number 2019 is expressed in base –2 how many non zero digits does it contain?

Answer (06)

Sol. Here we use, $2^n = (-2)^{n+1} + (-2)^n$ if n ε odd and $2(-2)^n = (-2)^{n+2} + (-2)^{n+1}$

Now $2019 = 2^0 + 2^1 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$

$\Rightarrow 2019 = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^5 + 2(-2)^6 + (-2)^7 + 2(-2)^8 + (-2)^9 + 2(-2)^{10}$

$\Rightarrow 2019 = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^5 + 2(-2)^7 + 3(-2)^8 + (-2)^9 + 2(-2)^{10}$

$\Rightarrow 2019 = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^5 + 4(-2)^8 + 2(-2)^9 + 2(-2)^{10}$

$\Rightarrow 2019 = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^5 + (-2)^{10} + (-2)^{10} + (-2)^{11} + 2(-2)^{10}$

$\Rightarrow 2019 = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^5 + (-2)^{11} + (-2)^{12}$

So $(2019)_{-2} = 1100000100111$

16. Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N?

Answer (69)

Sol. $\because \quad p > \sqrt{n} \Rightarrow p^2 > n$ and $p < 20$

$\Rightarrow$ p = {2, 3, 5, 7, 11, 13, 17, 19}

$\therefore$ N = 1 + 2 + 4 + 6 + 10 + 12 + 16 + 18

$\boxed{N = 69}$

17. Let a, b, c be distinct positive integers such that b + c – a, c + a – b and a + b – c are all perfect squares. What is the largest possible value of a + b + c smaller than 100?

Answer (91)

Sol. $b + c - a = m^2$

$c + a - b = n^2$

$a + b - c = p^2$

$\therefore \quad \boxed{a + b + c = m^2 + n^2 + p^2}$

$2c = m^2 + n^2$; $2a = n^2 + p^2$; $2b = m^2 + p^2$

$\therefore$ m, n, p are either all even or all odd

or $m = 2K_1$; $n = 2K_2$; $p = 2K_3$

$\Rightarrow c = 2(K_1^2 + K_2^2)$; $a = 2(K_2^2 + K_3^2)$

$b = 2(K_1^2 + K_3^2)$

Or let $m = 2K_1 + 1$; $n = 2K_2 + 1$; $n = 2K_3 + 1$

$\Rightarrow$ $c = 2K_1^2 + 2K_2^2 + (K_1 + K_2) + 1$

$\therefore$ It is clear that maximum value is possible when all m, n, p are odd number

$\because$ $a + b + c < 100$

Only one possibility left 9, 3, 1

$a + b + c = 81 + 9 + 1 = \boxed{91}$

18. What is the smallest prime number p such that $p^3 + 4p^2 + 4p$ has exactly 30 positive divisors?

Answer (43)

Sol. $p^3 + 4p^2 + 4p = p(p + 2)^2$

For exactly 30 factors

$(p + 2)^2$ must have 15 factors

$\therefore$ $p + 2$ must of type $p_1 \times p_2^2$ where p_1 and p_2 are different odd prime numbers

For least value of $p + 2$ let $p_2 = 3$ and $p_1 = 5$

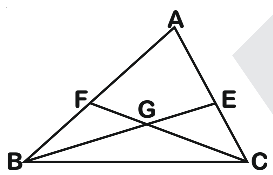
$\Rightarrow$ $p + 2 = 9 \times 5$

$\Rightarrow$ $p = 43$ is a prime number which is least.

19. If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer?

Answer (90)

Sol.



$\because$ BE = 15 and CF = 9

So, BG = 10 and CG = 6

Also, area (ΔABC)

$= 3\cdot$area (ΔBGC)

$\max[ABC] = 3\cdot\max[BGC]$

$$= 3\left(\frac{1}{2}\times 10\times 6\times \sin(\angle BGC)\right)$$

$= 90\cdot\sin(\angle BGC)$ {for maximum value let $\angle BGC = 90°$}

$= 90$

20. How many 4-digit numbers $\overline{abcd}$ are there such that $a < b < c < d$ and $b - a < c - b < d - c$?

Answer (07)

Sol. Let $b - a = x$

$c - b = y$

$d - c = z$

$x + y + z = d - a$ also $x < y < z$

The possible values of $x + y + z$ are 6, 7 and 8

Case I: If $x + y + z = 8$ that means $a = 1$ and $d = 9$

Also, $(x, y, z) = (1, 2, 5)$ or $(1, 3, 4)$

Possible four digit numbers are 1249 and 1259

Case II: If $x + y + z = 7$ then $(a, d) = (1, 8)$ or $(2, 9)$

Also $(x, y, z) = (1, 2, 4)$

Possible four digit numbers are 1248 and 2359

Case III: If $x + y + z = 6$ then $(a, d) = (1, 7)$ or $(2, 8)$ or $(3, 9)$

Also $(x, y, z) = (1, 2, 3)$

Possible four digit numbers are 1247; 2358 and 3469

Total 4 digit numbers possible = 7

21. Consider the set E of all positive integers n such that when divided by 9, 10, 11 respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E.

Answer (Bonus)

Sol. $(n_1, n_2, n_3) = (9, 10, 11)$

and remainders r_1, r_2, r_3 are in G.P.

So, $(r_1, r_2, r_3) = (8, 4, 2)$ or $(2, 4, 8)$

Now; $110x_1 \equiv 1 \pmod 9 \Rightarrow x_1 = 5$

$99x_2 \equiv 1 \pmod{10} \Rightarrow x_2 = -1$

and $90x_3 \equiv 1 \pmod{11} \Rightarrow x_3 = 6$

Using Chinese remainder theorem;

$x \equiv 550r_1 - 99r_2 + 540r_3 \pmod{990}$

For each triplet of (r_1, r_2, r_3)

$x \equiv 134 \pmod{990}$

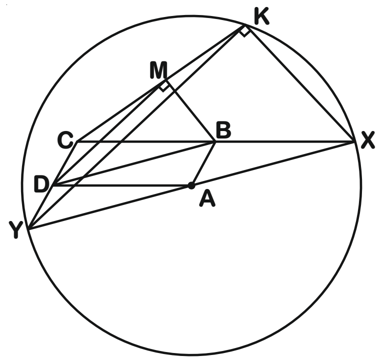
and $x \equiv 74 \pmod{990}$

The largest number x can't be found, however the smallest possible value of x is 74.

22. In parallelogram ABCD, AC = 10 and BD = 28. The points K and L in the plane of ABCD move in such a way that AK = BD and BL = AC. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$?

Answer (02)

Sol. First we are going to find $\angle BMD$.



We draw a circle having centre A and radius 28. K will be any point on the circle.

Extend CB to intersect the circle (at X) as shown.

$\because$ AX = BD = 28 and BX || AD

hence ADBX is a parallelogram.

Similarly extend CD to intersect the circle (at Y) as shown.

$\because$ AY = BD = 28 and DY || AB

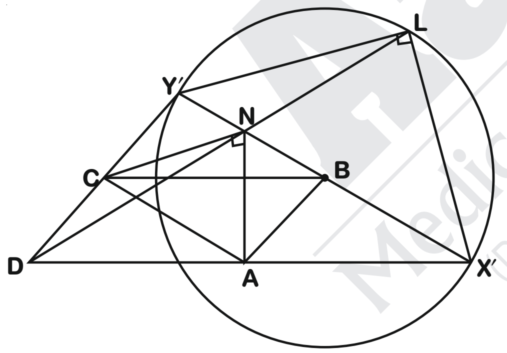
hence ABDY is also a parallelogram.

Now let a point K on the circle. M is mid point of CK.

In ΔYCK ; DM || YK and in ΔXCK ; BM || KX

So $\angle BMD = \angle XKY = 90°$

Now we are going to find $\angle ANC$.



We draw a circle having centre B and radius 10. L will be any point on the circle.

Extend DA to intersect the circle (at X′) as shown.

AX′ || BC and AC = BX′ = 10

hence ACBX′ is a parallelogram.

Similarly extend DC to intersect the circle (at Y′) as shown.

CY′ || AB and AC = BY′

hence CABY′ is also a parallelogram.

Now let a point L on the circle. N be the mid-point of DL.

In DLY′ ; CN || Y′L and in $\Delta DLX'$; AN || LX′

So $\angle ANC = \angle X'LY' = 90$

Finally $\tan^2\left(\frac{\angle ANC}{2}\right) + \cot^2\left(\frac{\angle BMD}{2}\right) = 2$

23. Let t be the area of a regular pentagon with each side equal to 1. Let P(x) = 0 be the polynomial equation with least degree, having integer coefficients, satisfied by x = t and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of P(x). What is the integer closest to $\sqrt{M}$? $\left(\sin 18° = \left(\sqrt{5}-1\right)/2.\right)$

Answer (Bonus)

Sol. Give a = 1, n = 5

Area of pentagon = $\left(\frac{na^2 \times \cot\left(\frac{\pi}{n}\right)}{4}\right)$

$\therefore$ Area = $\frac{1\times 5\times \cot\left(\frac{\pi}{5}\right)}{4} = \frac{5}{4}\cot 36° = t$

$\Rightarrow$ t = $\frac{5}{4}\times 1.376381$

$\Rightarrow$ $t \approx 1.7125$

Let, t = 1.7125

$\Rightarrow$ $100t \approx 171$

$\because$ t is root.

$\Rightarrow$ 100x – 171 = 0

will be a polynomial

Now, $M \approx 100 + |-171| \approx 271$

$\Rightarrow$ 16 < M < 17 ($\because$ $\sqrt{M} \approx 16.47$)

$\therefore$ Nearest integer can be 16

Also since value of $\sin 18° = \frac{\sqrt{5}-1}{2}$ is wrong given in question paper

$\Rightarrow$ This question should be Bonus.

But Answer **16** is calculated according to $\sin 18° = \frac{\sqrt{5}-1}{4}$ actual value

24. For $n \geq 1$, let a_n be the number beginning with n 9's followed by 744; e.g., a_4 = 9999744. Define $f(n) = \max\{m \in \mathbb{N} \mid 2^m \text{ divides } a_n\}$, for $n \geq 1$. Find f(1) + f(2) + f(3) + ... + f(10).

Answer (75)

Sol.

$a_1 = 9744 = 10^4 - 256$

$a_2 = 99744 = 10^5 - 256$

$a_5 = 10^8 - 256 = (10^4)^2 - (2^4)^2 = (10^4 + 2^4)(10^4 - 2^4)$

$= 2^4 \times (626)(10^2 - 2^2)(10^2 + 2^2)$

$= 2^{13} \times (313 \times 13 \times 3)$

$a_6 = 10^9 - 256$

$\vdots$

$a_{10} = 10^{13} - 256$

Now. $F(1) = 4$

$F(2) = 5$

$F(3) = 6$

$F(4) = 7$

$F(5) = 13$

and $F(6) = ... = F(10) = 8$

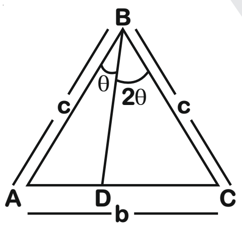
$\therefore$ $F(1) + F(2) + ... + F(10) = 40 + 35$

$= 75$

25. Let ABC be an isosceles triangle with AB = BC. A trisector of $\angle B$ meets AC at D. If AB, AC and BD are integers and AB – BD = 3, find AC.

Answer (26)

Sol.



$\because$ $b = 2c\sin\left(\frac{3\theta}{2}\right)$ (By projection formula)

Now $\text{area} = \frac{1}{2}c \times c\sin 3\theta = \frac{1}{2}c(c-3)$

$[\sin\theta + \sin 2\theta]$

$\Rightarrow$ $\boxed{c = \frac{3}{4}\text{cosec}^2\frac{\theta}{2}}$ Now $\boxed{\text{cosec}\frac{\theta}{2} = 2m}$

$\boxed{c = 3m^2}$

$\because$ c is integer, $m \in$ Integer

$\therefore$ $b = 2 \times 3m^2$ $\left[3\sin\frac{\theta}{2} - 4\sin^3\frac{\theta}{2}\right]$

$\boxed{b = 9m - \frac{3}{m}}$

$\therefore$ $\boxed{m = 3}$ or $\boxed{m = 1}$

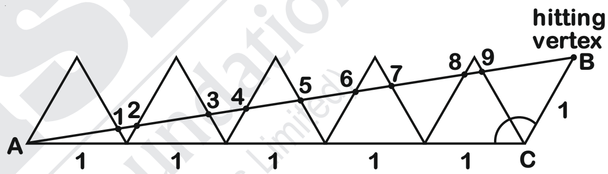
But at m = 1, c = 3 (which is not possible)

$\Rightarrow$ $b = 27 - 1 = \boxed{26}$

26. A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equals the angle of reflection. The distance traveled by the ball in meters is of the form $\sqrt{N}$, where N is an integer. What is the value of N?

Answer (31)

Sol.



Let AB = x

$\therefore$ In Δ ACB

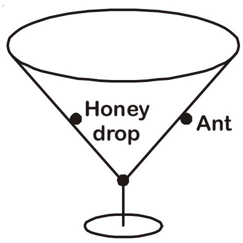
$\text{Cos } 120° = \frac{25 + 1^2 - x^2}{2 \times 5 \times 1}$

$-5 = 26 - x^2$

$\Rightarrow$ $\therefore x = \sqrt{31}$

$\Rightarrow$ $\boxed{N = 31}$

27. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the mid point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass. (See the figure). If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d? (Ignore the thickness of the glass).



Answer (36)

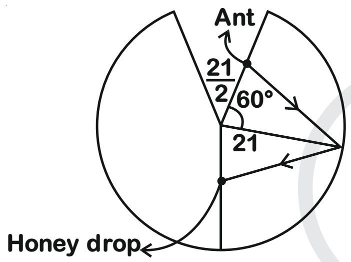
Sol. If we cut the cone along the slant height. It can be opened as a sector of circle. The radius of this arc will be slant height of the cone and perimeter of base circle of cone is now the length of this arc.

$2\pi \times 14 = 21 \times \theta$

$\Rightarrow \quad \theta = \frac{4\pi}{3}$

The ant has to cover the angle $\frac{2\pi}{3}$ and cross over the circular edge of the arc.

The shortest path is shown in the diagram.



The shortest path length

$$= 2\left[\sqrt{\left(\frac{21}{2}\right)^2 + (21)^2 - 2\cdot\left(\frac{21}{2}\right)(21)\cdot\cos 60^\circ}\right]$$

$\Rightarrow \quad d = 2 \times 21\sqrt{\frac{1}{4} + 1 - \frac{1}{2}}$

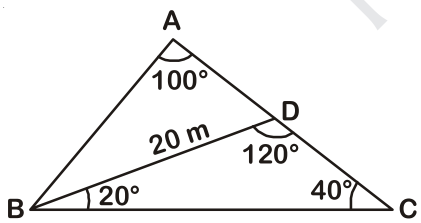
$\Rightarrow \quad d = 21\sqrt{3}$

$\Rightarrow \quad [d] = 36$

28. In a triangle ABC, it is known that $\angle A = 100°$ and AB = AC. The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that sin 10° ≈ 0.174.

Answer (27)

Sol.



In ΔBDC;

$$\frac{BC}{\sin 120°} = \frac{20}{\sin 40°}$$

$$\Rightarrow \quad BC = 20\left[\frac{\sqrt{3}}{\cos 10° + \sqrt{3}\sin 10°}\right]$$

$\because \quad \sin 10° \approx \frac{\sqrt{3}}{10}$

$$\Rightarrow \quad BC \approx \frac{20\sqrt{3}}{\frac{\sqrt{97}}{10} + \frac{3}{10}} \approx \frac{200\sqrt{3}}{3 + \sqrt{97}}$$

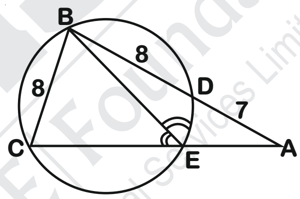
$$\Rightarrow \quad BC \approx \frac{347}{13} \approx 27$$

29. Let ABC be an acute angled triangle with AB = 15 and BC = 8. Let D be a point on AB such that BD = BC. Consider points E on AC such that $\angle DEB = \angle BEC$. If α denotes the product of all possible values of AE, find $[\alpha]$ the integer part of α.

Answer (68)

Sol. There are only two possible locations of point E.

Case I :

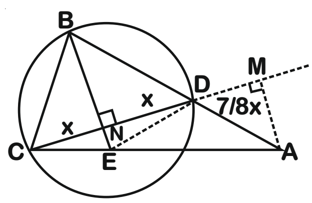


Point E lies on the circumcircle of ΔBCD because chords of equal length subtend equal angles on circumference.

So AE × AC = 7 × 15 = 105

$\Rightarrow \quad \ell_1 \times AC = 105 \qquad$ ...(i)

Case II :



E lies on the angle bisector of $\angle$B, because ΔBCE and ΔBDE will be congruent.

M is the foot of perpendicular from A to CD extended.

$\because \quad$ CN = DN = x

and $\Delta BND \sim \Delta AMD$

$\Rightarrow \quad DM = \frac{7}{8}x$

Now $\frac{AE}{AC} = \frac{MN}{MC} \qquad (\because \Delta CNE \sim \Delta CMA)$

$$\Rightarrow \frac{\ell_2}{AC} = \frac{x+\frac{7}{8}x}{2x+\frac{7}{8}x} = \frac{15}{23} \quad \text{...(ii)}$$

(i) × (ii)

$$\ell_1 \cdot \ell_2 = \frac{105 \times 15}{23} = \alpha$$

$\Rightarrow$ [α] = 68

30. For any real number x, let [x] denote the integer part of x; {x} be the fractional part of x ({x} = x – [x]). Let A denote the set of all real numbers x satisfying $\{x\} = \frac{x+[x]+\left[x+\left(\frac{1}{2}\right)\right]}{20}$.

If S is the sum of all numbers in A, find [S].

Answer (21)

Sol.

$$\{x\} = \frac{x+[x]+\left[x+\frac{1}{2}\right]}{20}$$

$$\Rightarrow \{x\} = \frac{2[x]+\left[x+\frac{1}{2}\right]}{19}$$

If $\{x\} \in \left[0, \frac{1}{2}\right)$

then $\{x\} = \frac{3}{19}[x]$

Clearly $\frac{3}{19}[x] \in \left[0, \frac{1}{2}\right)$

$$\Rightarrow [x] \in \left[0, \frac{19}{6}\right)$$

[x] = 0, 1, 2, 3 and corresponding value of

$$\{x\} = 0, \frac{3}{19}, \frac{6}{19}, \frac{9}{19}$$

If $\{x\} \in \left[\frac{1}{2}, 1\right)$

then $\{x\} = \frac{3[x]+1}{19}$

Clearly $\frac{3[x]+1}{19} \in \left[\frac{1}{2}, 1\right)$

$$\Rightarrow [x] \in \left[\frac{17}{6}, 6\right)$$

[x] = 3, 4, 5

and corresponding value of

$$\{x\} = \frac{10}{19}, \frac{13}{19}, \frac{16}{19}$$

S = 21

❑ ❑ ❑