Aakash
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Time : 4 hrs. January 19, 2020

SOLUTIONS

for

35th Indian National Mathematical Olympiad-2020

Instructions :

1. Calculators (in any form) and protractors are not allowed.
2. Rulers and compasses are allowed.
3. All questions carry equal marks. Maximum marks : 102.
4. Answer all the questions.
5. Answer to each question should start on a new page. Clearly indicate the question number.

1. Let τ_1 and τ_2 be two circles of unequal radii, with centres O_1 and O_2 respectively, in the plane intersecting in two distinct points A and B . Assume that the centre of each of the circles τ_1 and τ_2 is outside the other. The tangent to τ_1 at B intersects τ_2 again in C, different from B, the tangent to τ_2 at B intersects τ_1 again in D, different from B. The bisectors of $\angle DAB$ and $\angle CAB$ meet τ_1 and τ_2 again in X and Y, respectively, different from A. Let P and Q be the circumcenter of triangles ACD and XAY respectively. Prove that PQ is the perpendicular bisector of the segment O_1O_2.

Sol. Let $\angle O_1BD = \theta$

Then $\angle O_2BC = (\angle O_1BO_2 - 90°) = (\angle O_1BD + 90°) - 90° = \theta$

Also ΔO_1BD and ΔO_2BC are isosceles, hence

$\angle BO_1D = \angle BO_2C = 180° - 2\theta$

SO $\angle DAB = \angle CAB = 90° + \theta$

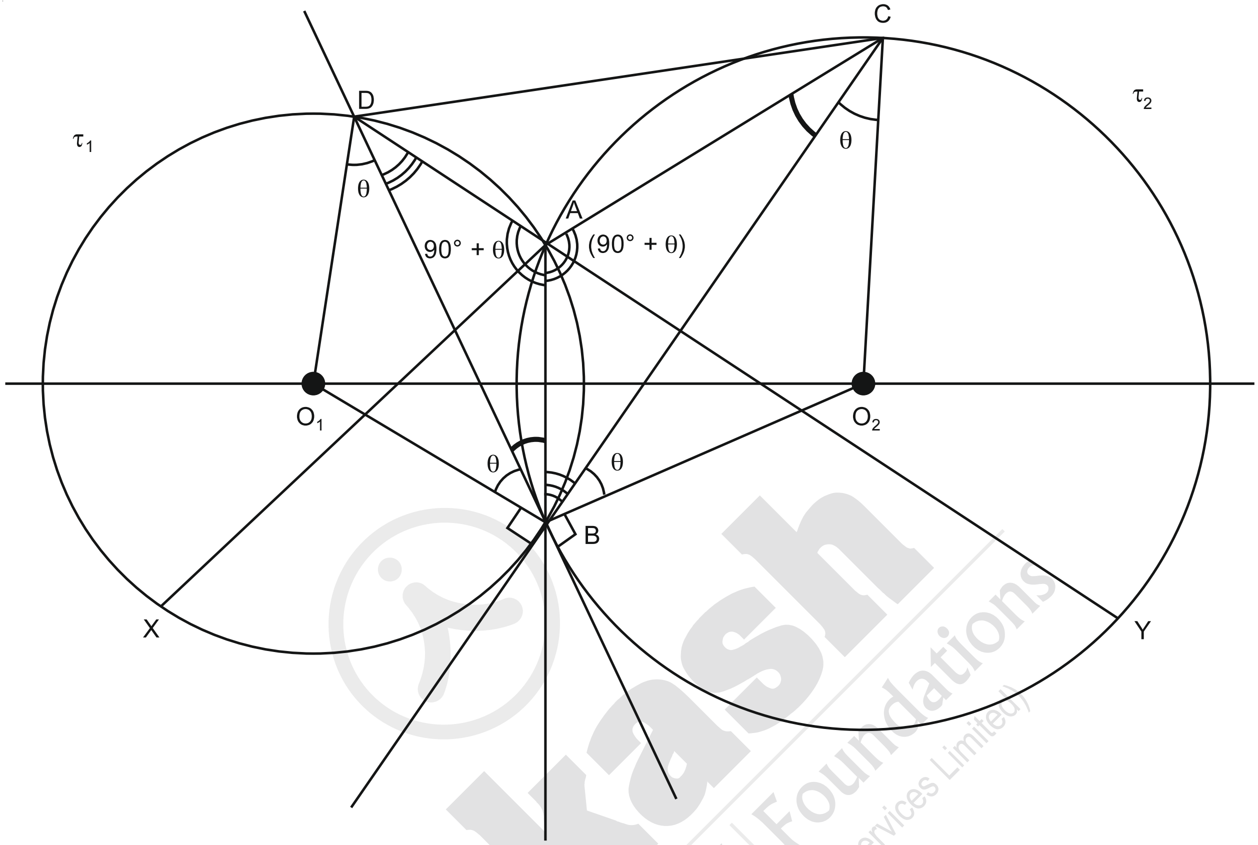
$\because$ P is the point of intersection of perpendicular bisectors of AD and AC, which means $PO_1 \perp AD$ and $PO_2 \perp AC$

$\because \angle PO_1O_2 = 90° - \theta = \angle PO_2O_1$

So ΔPO_1O_2 is isosceles

Similarly ΔQO_1O_2 is isosceles

Hence two isosceles triangles PO_1O_2 and QO_1O_2 are formed on same base O_1O_2, then PQ will be the perpendicular bisector of O_1O_2.



2. Suppose $P(x)$ is a polynomial with real coefficients satisfying the condition

$P(\cos\theta + \sin\theta) = P(\cos\theta - \sin\theta)$,

for every real θ. Prove that $P(x)$ can be expressed in the form

$P(x) = a_0 + a_1(1 - x^2)^2 + a_2(1 - x^2)^4 + \ldots + a_n(1 - x^2)^{2n}$.

for some real numbers $a_0, a_1, a_2, \ldots, a_n$ and nonnegative integer n.

Sol. $P(\cos\theta + \sin\theta) = P(\cos\theta - \sin\theta)\ \forall\theta\in R$

Put $\theta = \dfrac{\pi}{2}$

$\Rightarrow P(1) = P(-1) = C_0$

So, $x + 1 \mid P(x) - C_0$ and $x - 1 \mid P(x) - C_0$

$\Rightarrow (x^2 - 1) \mid P(x) - C_0$

Consider $P(x) - C_0 = (x^2 - 1)\,Q(x)$ ... (1)

Put $x = \cos\theta + \sin\theta$ and $x = \cos\theta - \sin\theta$

$\sin 2\theta . Q(\cos\theta + \sin\theta) = -\sin 2\theta . Q(\cos\theta - \sin\theta)\ \forall\theta\in R$

$\Rightarrow Q(\cos\theta + \sin\theta) = -Q(\cos\theta - \sin\theta)$

Put $\theta = 0 \Rightarrow Q(1) = 0 \Rightarrow x - 1 \mid Q(x)$

Put $\theta = \pi \Rightarrow Q(-1) = 0 \Rightarrow x + 1 \mid Q(x)$

$\Rightarrow (x^2 - 1) \mid Q(x)$

Consider $q(x) = (x^2 - 1)\,R(x)$ ... (2)

From (1) and (2),

$P(x) - C_0 = (x^2 - 1)^2.R(x)$ ... (3)

Again put $x = \cos\theta + \sin\theta$ and $x = \cos\theta - \sin\theta$

$R(\cos\theta + \sin\theta) = R(\cos\theta - \sin\theta)$

So $R(x)$ satisfies the same condition as $P(x)$ satisfies.

Hence $R(x) - d_0 = (x^2 - 1)^2. S(x)$ ... (4)

And so on.

From (3) and (4),

$P(x) - C_0 = (x^2 - 1)^2 (d_0 + (x^2 - 1)^2 .S(x))$

$\Rightarrow P(x) = C_0 + d_0(x^2 - 1)^2 + e_0(x^2 - 1)^4 + ...$

3. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$. Let $S \subseteq X$ be such that any positive integer n can be written as $p + q$ where the non-negative integers p, q have all their digits in S. Find the smallest possible number of elements in S.

Sol. First of all we should find a suitable set S for single digit integers n.

So $n = p + q$ where $n = 0,1, 2, 3, ..., 9$

For this 0 and 1 are compulsory elements of S, while 9 would be least important elements of S.

$0 = 0 + 0$

$1 = 1 + 0$

$2 = 1 + 1$

for $n = 3$ we need either 2 or 3 in S.

Case I: When 2 is present in S

$3 = 2 + 1$

$4 = 2 + 2$

for $n = 5, 6, 7, 8, 9$ we need any two as following (4, 5) or (4, 7).

Case II: When 3 is present in S

$3 = 3 + 0$

$4 = 3 + 1$

for $n = 5, 6, 7, 8, 9$ we need any two as following (4, 5) or (4, 6) or (4, 8) or (4, 9) and so on.

So in both cases we find $|S| \geq 5$.

If we assume $|S| = 4$

Let $x_1, x_2, x_3, x_4 \in S$

Then $\{x_1 + x_1, x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_2, x_2 + x_3, x_2 + x_4, x_3 + x_3, x_3 + x_4$ and $x_4 + x_4\}$ are the possible ten sums which must be a permutation of {0, 1, 2 ,3, 4, 5, 6, 7, 8, 9}.

Clearly $x_1 + x_1, x_2 + x_2, x_3 + x_3$ and $x_4 + x_4$ are even.

Some other should also be even.

WLOG let $x_1 + x_2$ is even, then remaining five sums must be odd, which is not possible

because if $x_1 \in$ odd $\Rightarrow x_2 \in$ odd

$\Rightarrow x_3 \in$ even (as $x_2 + x_3 \in$ odd)

and $x_4 \in$ even (as $x_2 + x_4 \in$ odd)

hence $x_3 + x_4$ will be even.

So, $|S| = 4$ is impossible.

Now if n has more than one digit. So we can split all digits of n into two parts using elements of S.

(For example if $S = \{0, 1, 2, 4, 5\}$ then $8 = 4 + 4$ and $6 = 2 + 4$ then $868 = 424 + 444$)

4. Let $n \geq 3$ be an integer and let $1 < a_1 \leq a_2 \leq a_3 \leq .... \leq a_n$ be n real numbers such that $a_1 + a_2 + a_3 + ...+ a_n = 2n$. Prove that

$$a_1a_2.......a_{n-1} + a_1a_2.....a_{n-2} + ...+a_1a_2 + a_1 + 2 \leq a_1a_1.....a_n$$

Sol. Let $P(n) \cdot a_1a_2...a_{n-1} + a_1a_2...a_{n-2} + ...+ a_1a_2 + a_1 + 2 \leq a_1 \cdot a_2 \cdot a_3 \,.\, ..a_n$ here $a_1 + a_2 + a_3 + ... + a_n = 2n$ and $1 \leq a_1 \leq a_2 \leq ... \leq a_n$

So clearly $a_n \geq 2$ and $a_1 \leq 2$

Now $P(2)$: $a_1 + 2 \leq a_1a_2$ where $a_1 + a_2 = 4$

$\Rightarrow a_1^2 - 3a_1 + 2 \leq 0$

$\Rightarrow a_1 \in [1, 2]$ which is true, so $P(2)$ is true

$P(3)$: from $P(2)$

$a_1 + 2 \leq a_1a_2 \Rightarrow a_1a_3 + 2a_3 \leq a_1a_2a_3$ (here $a_3 \geq 2$)

$\Rightarrow \quad a_1a_2 + a_3 + a_3 \leq a_1a_2a_3$

$\Rightarrow \quad a_1a_2 + a_1 + 2 \leq a_1a_2a_3$

So $P(3)$ is true

Let $P(k)$ is also true. So

$2 + a_1 + a_1a_2 + ... + a_1a_2a_3 \, ... \, a_{k-1} \leq a_1a_2a_3 \, ...a_k$

$P(k + 1)$: In $P(k)$ multiply a_{k+1} both sides.

$2a_{k+1} + a_1a_{k+1} + a_1a_2a_{k+1} + ... + a_1a_2a_3 \cdot a_{k-1}a_{k+1} \leq a_1a_2a_3...a_k \cdot a_{k+1}$

$\Rightarrow \quad a_{k+1} + a_{k+1} + a_1a_{k+1} + a_1a_2a_{k+1} + ...+a_1a_2...a_{k-1} \cdot a_{k+1} \leq a_1 \cdot a_2 \, ...a_{k+1}$

$\{\because \; a_{k+1} \geq 2$ and $a_1 \leq a_2 \leq a_3 \leq ... \leq a_{k+1}\}$

$\Rightarrow \quad 2 + a_1 + a_1a_2 + a_1a_2a_3 + ...+ a_1a_2 \, ...a_{k-1} \cdot a_k \leq a_1a_2 \, ... \, a_{k+1}$

So $P(k + 1)$ is also true

Hence $P(n)$ is true for all $n \in N$, $n \geq 3$

5. Infinitely many equidistant parallel lines are drawn in the plane. A positive integer $n \geq 3$ is called frameable if it is possible to draw a regular polygon with n sides all whose vertices lie on these lines and no line contains more than one vertex of the polygon.

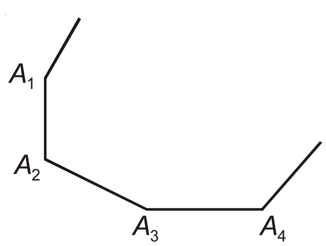
(a) Show that 3, 4, 6 are frameable

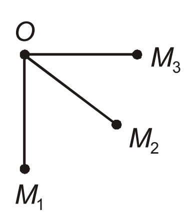
(b) Show that any integer $n \geq 7$ is not frameable

(c) Determine whether 5 is frameable

Sol. If a line segment has both ends on any two given lines, and another equal and parallel segment has one end on any given line, then its other vertex must lie on any one of the given line. (Statement : 1)

Let a n-sided regular polygon is frameable for $n \geq 7$.



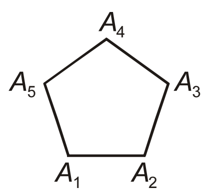


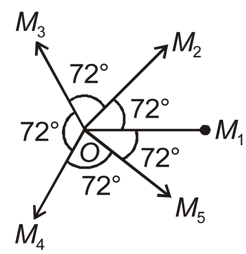
A_i's lie on the parallel lines. We draw line segments parallel and equal to sides of this polygon from point O (which lies on any one of these lines), then we will get line segments OM_1, OM_2, …, OM_n of equal lengths whose ends (i.e. M_i) lie on these parallel lines (from Statement : 1)

Now we got a regular polygon $M_1 M_2 M_3 \ldots M_n$. In this polygon angle subtended by each side on centre is less than 60° ($\because n \geq 7$) hence $M_i M_{i+1} < A_i A_{i+1}$. If we use infinite decent like this, we will get some regular polygon having side length less than the distance between parallel lines (which is impossible). Hence contradiction.

So the only possible value of n are 3, 4, 5 or 6.

For $n = 5$



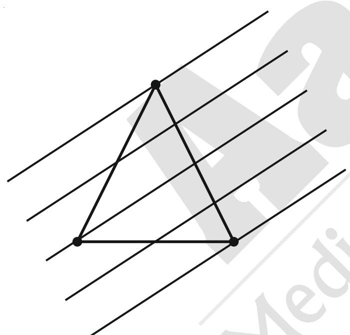


We formed a polygon $M_1M_2M_3M_4M_5$ whose all vertices are on given lines (as discussed earlier).

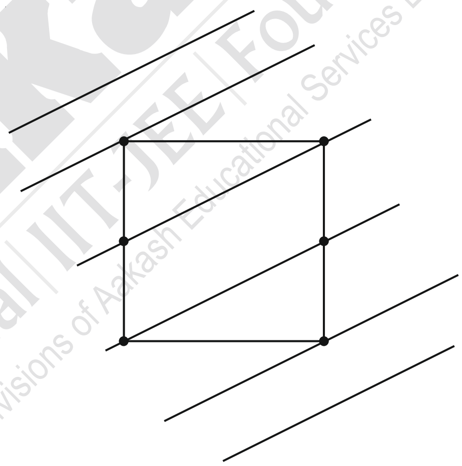
Let N_i be the reflection of M_i about O, hence $N_1N_2N_3N_4N_5$ is also such polygon, which implies $M_1N_1M_2N_2M_3N_3M_4N_4M_5N_5$ is a regular decagon whose vertices are on the given lines, which is impossible.

Hence 5 is not frameable.

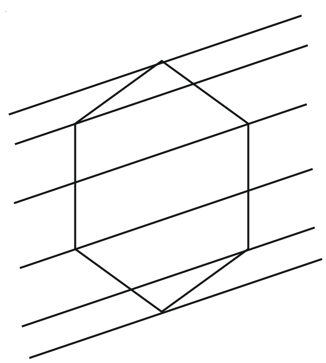
n = 3, 4 and 6 can easily be observed from the diagram.



$n = 3$



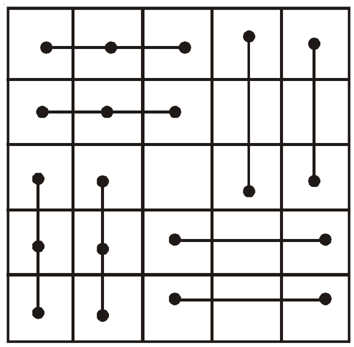
$n = 4$



$n = 6$

6. A stromino is a 3 × 1 rectangle. Show that a 5 × 5 board divided into twenty-five 1 × 1 squares cannot be covered by 16 strominos such that each stromino covers exactly three unit squares of the board and every unit square is covered by either one or two strominos. (A stromino can be placed either horizontally or vertically one the board)

Sol. First we try to fill the squares so that each square is overlapped by one stromino only. Hence we can use 8 strominoes and one square will be left vacant. One of the possible way to do this will be



For the second layer we first try to cover that square which was left vacant. In this way if we use seven strominoes, there will be no three vacant squares in a row, hence the last stromino will cover any other square which already has two layers.

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