Aakash
Medical | IIT-JEE | Foundations
(Divisions of Aakash Educational Services Limited)
Regd. Office : **Aakash Tower, 8, Pusa Road, New Delhi-110005. Ph.: 011-47623456**

Time : 3 Hrs.

Regional Mathematical Olympiad (RMO)-2019

Oct. 20, 2019

Instruction:
1. Calculators (in any form) and protractors are not allowed.
2. Rulers and compasses are allowed.
3. Answer all the questions.
4. All questions carry equal marks. Maximum marks: 102.
5. Answer to each question should start on a new page. Clearly indicate the question number.

1. Suppose x is a nonzero real number such that both x^5 and $20x + \frac{19}{x}$ are rational numbers. Prove that x is a rational number.

2. Let ABC be a triangle with circumcircle Ω and let G be the centroid of triangle ABC. Extend AG, BG and CG to meet the circle Ω again in A_1, B_1 and C_1, respectively. Suppose $\angle BAC = \angle A_1B_1C_1$, $\angle ABC = \angle A_1C_1B_1$ and $\angle ACB = \angle B_1A_1C_1$. Prove that ABC and $A_1B_1C_1$ are equilateral triangles.

3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+c^3+a^3}+\frac{c}{c^2+a^3+b^3} \le \frac{1}{5abc}.$$

4. Consider the following 3 × 2 array formed by using the numbers 1, 2, 3, 4, 5, 6:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}$$

Observe that all row sums are equal, but the sum of the squares is not the same for each row. Extend the above array to a 2 × k array $(a_{ij})_{3\times k}$ for a suitable k, adding more columns, using the numbers 7, 8, 9, …, $3k$ such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \text{ and } \sum_{j=1}^{k} \left(a_{1j}\right)^2 = \sum_{j=1}^{k} \left(a_{2j}\right)^2 = \sum_{j=1}^{k} \left(a_{3j}\right)^2.$$

5. In an acute angled triangle *ABC*, let *H* be the orthocenter, and let *D*, *E*, *F* be the feet of altitudes from *A*, *B*, *C* to the opposite sides, respectively. Let *L*, *M*, *N* be midpoints of segments *AH*, *EF*, *BC*, respectively. Let *X*, *Y* be feet of altitudes from *L*, *N* on to the line *DF*. Prove that *XM* is perpendicular to *MY*.

6. Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers *a*, *b*, *c*, *d* among them such that gcd(*a*, *b*) = gcd(*b*, *c*) = gcd(*c*, *d*) = gcd(*d*, *a*) = 1.

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SOLUTIONS

for

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1. Let $20x + \frac{19}{x} = a$ where $a \in Q$

$\Rightarrow$ $20x^2 = ax - 19$

Now, $x^5 \in Q$

$\Rightarrow$ $x^2 . x^2 . x \in Q$

$\Rightarrow$ $\left(\frac{ax - 19}{20}\right)^2 .x \in Q$

$\Rightarrow$ $\frac{1}{400}\left[a^2x^2 - 38ax + 361\right] x \in Q$

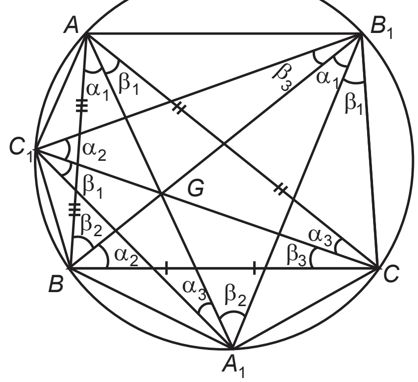
$\Rightarrow$ $a^2x^2 - 38ax + 361 \in Q$

$\Rightarrow$ $x\left[\frac{a^3}{20} - 38a\right] - \underbrace{\frac{19}{20}a^2}_{\text{rational number}} \in Q$

$\Rightarrow$ $x\left[\underbrace{\frac{a^3}{20} - 38a}_{\text{rational number}}\right] \in Q$

$\Rightarrow$ $x \in Q$

2. Refer to diagram,



$\Delta ABC \cong \Delta A_1B_1C_1$

$\angle A = \angle B_1$, $\angle B = \angle C_1$ and $\angle C = \angle A_1$

$\Rightarrow \quad \beta_1 = \beta_2 = \beta_3$

So, $\angle A_1B_1C = \beta_1 = \beta_2 = \Delta AA_1B_1$

Hence $AA_1 \parallel B_1C$

In ΔBB_1C;

$A_1G \parallel B_1C$ and A_1G bisects BC, So, AG also bisects BB_1. Similarly, G be the mid-point of AA_1.

$\therefore$ Centre is the only point inside a circle that bisects more than one chord.

So, G be the centre of circle Ω, hence ΔABC will be equilateral, so as $\Delta A_1B_1C_1$.

3. $\because \quad a + b + c = 1$

$\Rightarrow \quad a^2(a + b + c) = a^2$

$\Rightarrow \quad a^2 + b^3 + c^3 = a^3 + b^3 + c^3 + a^2b + a^2c$

Now, $\dfrac{a^3 + b^3 + c^3 + a^2b + a^2c}{5} \geq \left(a^7 . b^4 . c^4\right)^{\frac{1}{5}}$ (A.M. – G.M. Inequality)

$$\Rightarrow \quad \frac{a}{a^2 + b^3 + c^3} \leq \frac{1}{5\left(a^2b^4c^4\right)^{\frac{1}{5}}}$$

So, $\dfrac{a}{a^2 + b^3 + c^2} + \dfrac{b}{b^2 + c^3 + a^3} + \dfrac{c}{c^2 + a^3 + b^3} \leq \dfrac{1}{5}\left[\dfrac{1}{\left(a^2b^4c^4\right)^{\frac{1}{5}}} + \dfrac{1}{\left(b^2c^4a^4\right)^{\frac{1}{5}}} + \dfrac{1}{\left(c^2a^4b^4\right)^{\frac{1}{5}}}\right]$

$$\Rightarrow \quad \frac{a}{a^2 + b^3 + c^3} + \frac{b}{b^2 + c^3 + a^3} + \frac{c}{c^2 + a^3 + b^3} \leq \frac{1}{5abc}\left[a^{3/5}b^{1/5}c^{1/5} + b^{3/5}c^{1/5}a^{1/5} + c^{3/5}a^{1/5}b^{1/5}\right]$$

…(i)

WLOG, let $a \geq b \geq c$, so, $a^{1/5} \geq b^{1/5} \geq c^{1/5}$ and $a^{3/5} \geq b^{3/5} \geq c^{3/5}$ $\left(\text{as } a, b, c \in (0, 1)\right)$

Using rearrangement inequality, the sorted product is greatest among all permutations of product.

So, $a^{\frac{3}{5}}\cdot b^{\frac{1}{5}}\cdot c^{\frac{1}{5}}+b^{\frac{3}{5}}\cdot c^{\frac{1}{5}}\cdot a^{\frac{1}{5}}+c^{\frac{3}{5}}\cdot a^{\frac{3}{5}}\cdot b^{\frac{1}{5}}\le a^{\frac{3}{5}}\cdot a^{\frac{1}{5}}\cdot a^{\frac{1}{5}}+b^{\frac{3}{5}}\cdot b^{\frac{1}{5}}\cdot b^{\frac{1}{5}}+c^{\frac{3}{5}}\cdot c^{\frac{1}{5}}.\,c^{\frac{1}{5}}$

$$\Rightarrow \qquad a^{\frac{3}{5}}\cdot b^{\frac{1}{5}}\cdot c^{\frac{1}{5}}+b^{\frac{3}{5}}\cdot c^{\frac{1}{5}}\cdot a^{\frac{1}{5}}+c^{\frac{3}{5}}\cdot a^{\frac{1}{5}}\cdot b^{\frac{1}{5}}\le a+b+c=1 \qquad \ldots\text{(i)}$$

From (i) and (ii)

$$\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+c^3+a^3}+\frac{c}{c^2+a^3+b^3}\le\frac{1}{5abc}$$

4. Consider the 3 × k array as

$$\begin{pmatrix}1 & 6 & \_\,\_\,\_\\ 2 & 5 & \_\,\_\,\_\\ 3 & 4 & \_\,\_\,\_\end{pmatrix}_{3\times k}$$

Sum of squares of each row = $\frac{1}{3}\left[1^2+2^2+3^2+\ldots+(3k)^2\right]$

$= \dfrac{k(3k+1)(6k+1)}{2\times 3}$

As, $3\nmid 3k+1$ and $3\nmid 6k+1$

So, k should be divisible by 3, but $k\neq 3$ because first two columns are already decided here, so if we put 7, 8, 9 in third column in any order, the sum of each row will be different. Therefore the least possible value of k will be 6 and after that 9, 12, 15, … are the possible values of k.

We are going to form the required array when k = 6. So, integers, 1, 2, 3, …., 18 will be available in the array. We will denote the sum of elements of i^{th} row as R_i and sum of squares of elements of i^{th} row as S_i. As give 3 × 2 array suggests that

$S_1 = a^2 + (a + 5)^2 = a^2 + 10a + 25 = \lambda_a + 25$

$S_2 = (a + 1)^2 + (a + 4)^2 = a^2 + 10a + 17 = \lambda_a + 17$

And $S_3 = (a + 2)^2 + (a + 3)^2 = a^2 + 10a + 13 = \lambda_a + 13$

Here, $a = 1$

So, we can write the expression of similar kind for b

$b^2 + (b + 5)^2 = \lambda_b + 25$; $(b + 1)^2 + (b + 4)^2 = \lambda_b + 17$ and $(b + 2)^2 + (b + 3)^2 = \lambda_b + 13$ for $b = 7$

Till now integers from 7 to 12 are covered.

Again, $c^2 + (c + 5)^2 = \lambda_c + 25$

$(c + 1)^2 + (c + 4)^2 = \lambda_c + 17$

And $(c + 2)^2 + (c + 3)^2 = \lambda_c + 13$ for $c = 13$

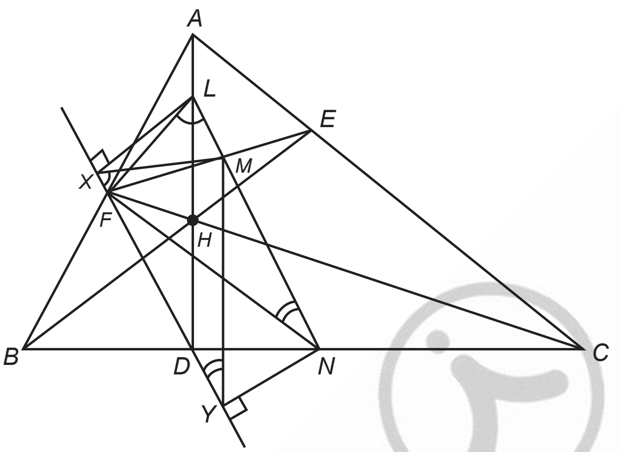
Here in each set of three relations sum of elements is fixed as $a + 5$, $b + 5$ and $c + 5$ respectively.

Now, we will add these relations in such a way that sum of squares of each row will contain $\lambda_a + \lambda_b + \lambda_c + (25 + 17 + 13)$.

So, the possible array will be

$$\begin{pmatrix} 1 & 6 & 8 & 11 & 15 & 16 \\ 2 & 5 & 9 & 10 & 13 & 18 \\ 3 & 4 & 7 & 12 & 14 & 17 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 6 & 9 & 10 & 14 & 17 \\ 2 & 5 & 7 & 12 & 15 & 16 \\ 3 & 4 & 8 & 11 & 13 & 18 \end{pmatrix}$$

5.



A circle having diameter BC passes through E and F. Also $AEHF$ is a cycle quadrilateral having AH as diameter. So, EF is the common chord of two above mentioned circles which will be perpndicularily bisected by line joining their centres (which is LN). So L, M and N are collinear. Now, we claim that ΔLFN and ΔXMY are similar, because $\angle MXY = \angle FLN$ ($\because XLMF$ is cyclic quadrilateral) and $\angle XYM = \angle LNF$ ($\because MFNY$ is cyclic quadrilateral) hence third angle of the two triangles will also be the same $\angle XMY = \angle LFN = 90°$ ($\because LN$ is the diameter of nine point circle passing through F)

$\Rightarrow \quad XM \perp MY$

6. Let $x_1, x_2, x_3 \ldots, x_{91}$ are 91 integers greater than one. Assume Ax_i be a set of integers which are coprime with x_i out of given 91 integers.

$Ax_i = \{\alpha, \beta, \gamma, ...\}$ where $\gcd(x_i, \alpha) = \gcd(x_i, \beta) = \gcd(x_i, \gamma) = ... = 1$

let $n(Ax_i) = y_i$

So, $\dfrac{y_1 + y_2 + y_3 + ... + y_{91}}{2} \geq 456$

$\Rightarrow \sum y_i \geq 912$ …(i)

Now we consider their exist no four integers a, b, c, d such that

$\gcd(a_1 b) = \gcd(b, c) = \gcd(c, d) = \gcd(a, d) = 1$

This is only possible if no two sets out of Ax_i's have atleast two elements in common. Because if we assume that sets Ax_1 and Ax_2 have two elements α and β in common, that means $\gcd(x_1,\alpha)=\gcd(\alpha,x_2)=\gcd(x_2,\beta)=\gcd(\beta,x_1)=1$ which is against our assumption.

So, if we calculate the number of ordered pairs, that can be formed from each of the sets Ax_i, no pair will be repeated and total number of such ordered pairs will be less that or equals to ${}^{91}C_2$

$${}^{y_1}C_2+{}^{y_2}C_2+\ldots+{}^{y_{91}}C_2\le{}^{91}C_2$$

$$\Rightarrow \sum_{i=1}^{91} y_i(y_i-1)\le 90\times 91$$

$$\Rightarrow \sum_{i=1}^{91} y_i^2-\sum_{i=1}^{91} y_i\le 90\times 91$$

$$\Rightarrow \frac{\left(\sum_{i=1}^{91} y_i\right)^2}{91}-\left(\sum_{i=1}^{91} y_i\right)\le 90\times 91$$

$$\Rightarrow \left(\sum_{i=1}^{91}(y_i)\right)^2-91\sum_{i=1}^{91} y_i\le 90\times 91^2=745290 \qquad \ldots\text{(ii)}$$

From (1) $\because \sum_{i=1}^{91} y_i\le 912$

So, $\left(\sum_{i=1}^{91} y_i\right)^2-91\left(\sum_{i=1}^{91} y_i\right)\ge 912(912-91)$

$$\Rightarrow \left(\sum_{i=1}^{91} y_i\right)^2-91\left(\sum_{i=1}^{91} y_i\right)\ge 748752 \qquad \ldots\text{(iii)}$$

As (ii) and (iii) are opposing each other, so our assumption is wrong.

Hence, there exist atleast four integers a, b, c, d such that

$\gcd(a,b)=\gcd(b,c)=\gcd(c,d)=\gcd(d,a)=1$

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