Aakash
Medical | IIT-JEE | Foundations
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Time : 4 hrs. January 20, 2019

SOLUTIONS

for

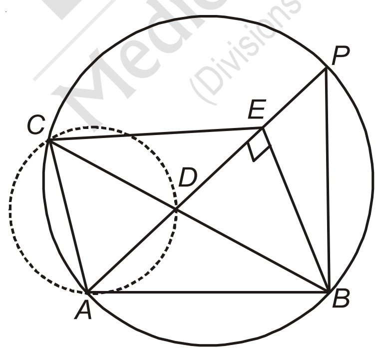
34th Indian National Mathematical Olympiad-2019

Instructions :

1. Calculators (in any form) and protractors are not allowed.
2. Rulers and compasses are allowed.
3. Answer all the questions.
4. All questions carry equal marks. Maximum marks : 102.
5. Answer to each question should start on a new page. Clearly indicate the question number.

1. Let ABC be a triangle with ∠BAC > 90°, let D be a point on the segment BC and E be a point on the line AD such that AB is tangent to the circumcircle of traingle ACD at A and BE is perpendicular to AD. Given that CA = CD and AE = CE, determine ∠BCA in degrees.

Sol.



We have, $\angle BAD = \angle ACD$ (Angles on alternate segments)

Angles on arc AB, $\angle APB = \angle ACB$

Hence, $\angle APB = \angle BAP$

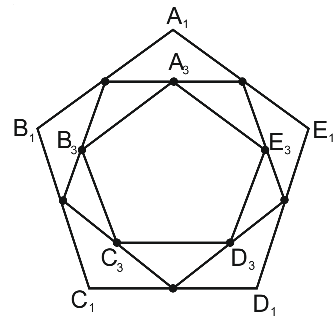
$\Rightarrow$ E is the midpoint of AP but $AE = CE$ is given $\Rightarrow AE = CE = EP$

$\Rightarrow$ ΔABP has its circumcentre E with AP diameter

$$\therefore \quad \angle BCA = \frac{180° - \angle ABP}{2} = 45°$$

2. Let $A_1B_1C_1D_1E_1$ be a regular pentagon. For $2 \le n \le 11$, let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoints of the sides of $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}$. All the 5 vertices of each of the 11 pentagons are arbitrarily coloured red or blue. Prove that four points among these 55 points have the same colour and form the vertices of a cyclic quadrilateral.

Sol.



Notice that pentagons $A_1B_1C_1D_1E_1$, $A_3B_3C_3D_3E_3$, $A_5B_5C_5D_5E_5$, ... , $A_{11}B_{11}C_{11}D_{11}E_{11}$ have same orientation. If any one of these 6 pentagons have at least 4 vertices are of same colour (say of blue) then we join these four vertices to get a cyclic quadrilateral with blue vertices.

If none among the 6 pentagons have at least 4 vertices of blue colour, then each of these 6 pentagons have at least 2 vertices of red colour. We join these two vertices to get 6 line segments having end points in red.

As any diagonal or side of a regular pentagon can have only 5 possible orientations, among the 6 line segments at least two are parallel (By pigeonhole principle). We join the end points of 2 parallel line segments to get a isosceles trapezium, (which is also cyclic) of vertices in red.

3. Let *m, n* be distint positive integers. Prove that

$\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) \le 2|m - n| + 1$

Further, determine when equality holds.

Sol. Without loss of generality, let us assume that m > n. Then we need to prove,

$\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) \le 2(m - n) + 1$

i.e $x + y + z \le 2k + 1$

where $x = \gcd(m, n) = \gcd(m - n, n)$

$y = \gcd(m + 1, n + 1) = \gcd(m - n, n + 1)$

$z = \gcd(m + 2, n + 2) = \gcd(m - n, n + 2)$

$K = m - n$

as n and $n + 1$ are consecutive integers.

So $\gcd(x, y) = 1$, $\gcd(y, z) = 1$ and $\gcd(x, z) = 2$ when x and z are even and $\gcd(x, z) = 1$ otherwise

<u>When gcd (x, z) = 1</u>

$x|(m - n) \Rightarrow x|k,\ y|k,\ z|k \Rightarrow xyz|k$

$\Rightarrow xyz \le K$

If $x = y = z = 1$ then $x + y + z = 3 = 2 + 1 = 2K + 1$ else $x + y + z < 2xyz + 1 \le 2K + 1$ as x, y, z are the integers not all equal to 1.

When gcd $(x, z) = 2$

$\Rightarrow$ gcd $(x', z') = 1$ where $2x' = x$ and $2z' = z$

$\Rightarrow x|k,\ y|k,\ z/k \Rightarrow 2x'yz'/k$

$\Rightarrow 2x'yz' \le k$

when $y > 1$ then $x + y + z = 2(x' + z') + y \le 4x'yz' + 1 = 2k + 1$

when $y = 1$ then at $x = 2 = z$, equality holds

$\therefore$ Equality holds when either m and n are consecutive integers or even positive integers

4. Let n and M be positive integers such that $M > n^{n-1}$. Prove that there are n distinct primes $p_1, p_2, p_3, \ldots p_n$ such that p_j divides M + j for $1 \le j \le n$.

Sol. Suppose there are only k(<n) distinct primes $p_1, p_2, \ldots p_k$ such that p_j divides M + j.

Let $M + j = p_1^{x_1} p_2^{x_2} \ldots\ldots p_r^{x_r} \ldots\ldots p_k^{x_k}$ .....(i)

Suppose $p_r^{x_r}$ is the maximum value out of all $p_i^{x_i}$ occuring in any M+ j , $1 \le j \le n$

$\therefore$ (M + t) – (M + s) would be divided by $p_r^{x_t}$ where $x_t \le x_r$

$p_r^{x_t} \mid (t - s) \le n - 1$ .....(ii)

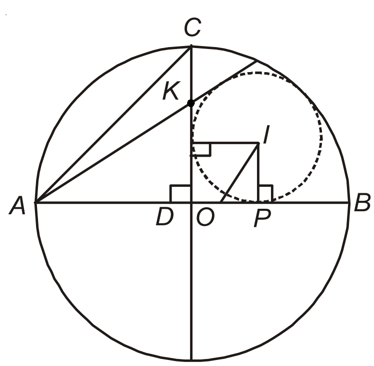
Also $p_r^{x_t} \ge (M + j)^{1/n-1}$ (as $p_r^{x_r}$ is the maximum term and total terms are $\le$ n – 1)

$\Rightarrow p_r^{x_t} > (n^{n-1} + j)^{1/n-1} \ge \left(n^{n-1}\right)^{\frac{1}{n-1}} = n$ .....(iii)

(ii) and (iii) leads to a contradiction. hence, there are n distinct primes such that p_j divides M + j for $1 \le j \le n$

5. Let AB be a diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB. Let K be a point of the segment CD such that AC is equal to the semiperimeter of the triangle ADK. Show that the excircle of triangle ADK opposite A is tangent to Γ.

Sol.



$$AC = \frac{AD + KD + AK}{2}$$

$$IP = \frac{KD + AK - AD}{2}$$

$$= \frac{AD + KD + AK}{2} - AK$$

$\Rightarrow$ $IP = AC - AK$

$OI^2 = OP^2 + PI^2$

$= (AC - AO)^2 + PI^2$

$= AO^2 + PI^2 + 2 \times AD \times AO - 2AC \times AO$

$\Rightarrow$ $OI^2 = (AO - IP)^2$

$\Rightarrow$ $OI = AO - IP$

$\Rightarrow$ Two circles touch internally.

6. Let f be a function defined from the set $\{(x, y) : x, y \text{ real}, xy \neq 0\}$ to the set of all positive real numbers such that

(i) $f(xy, z) = f(x, z)\, f(y, z)$, for all $x, y \neq 0$;

(ii) $f(x, 1 - x) = 1$, for all $x \neq 0,1$.

Prove that

(a) $f(x, x) = f(x, -x) = 1$, for all $x \neq 0$;

(b) $f(x,y)\, f(y, x) = 1$, for all x, y $\neq 0$.

Sol. We have, $f(1, x) = f(1, x) \,.\, f(1, x)$

$\Rightarrow f(1, x) = 1$

Also, $f(1, x) = f(-1, x).f(-1, x) = 1 \Rightarrow (-1, x) = 1$

from $f(1, x) = 1 \Rightarrow \lim_{n\to\infty} f\left(x^{\frac{1}{2^n}}, x\right) = 1$

$\Rightarrow f\left(x^{\frac{1}{2^{n-1}}}, x\right) = 1, \ldots.. \Rightarrow f(x^{1/2}, x) = 1 \Rightarrow f(x, x) = 1\ \forall x \in (0, \infty)$

$f(1, -x) = 1 \Rightarrow \lim_{n\to\infty}\left(x^{\frac{1}{2^n}}, x\right) = 1, \ldots.. \Rightarrow f(x, -x) = 1\ \ \forall x \in (0, \infty)$

for negative x, similar proof

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