Date: August 11, 2019 | Time: 10 AM to 1 PM

Number of Questions: 30 | Max Marks: 102

Aakash

Medical | IIT-JEE | Foundations

(Divisions of Aakash Educational Services Limited)

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Answers & Solutions

for

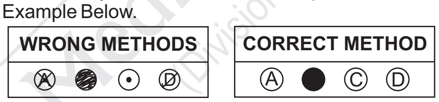
PRMO - 2019

INSTRUCTIONS

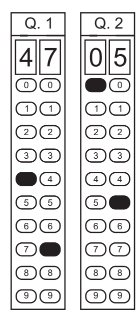
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a black pencil or a black or blue ball pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your PRMO score.
4. Incompletely, Incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS

1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.



4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.

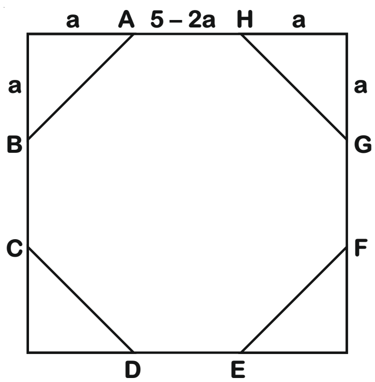


6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 6 carry 2 marks each; questions 7 to 21 carry 3 marks each; questions 22 to 30 carry 5 marks each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the Candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

1. From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area removed to the nearest integer?

Answer (04)

Sol.



$5 - 2a = a\sqrt{2}$

$a = \dfrac{5}{2+\sqrt{2}} = 5\left(1 - \dfrac{1}{\sqrt{2}}\right)$

Area removed $= 4 \times \dfrac{1}{2} a^2$

$= 2 \times 25 \dfrac{(\sqrt{2}-1)^2}{2}$

$\simeq 4.3$

Area to nearest integer is 4

2. Let $f(x) = x^2 + ax + b$. If for all nonzero real x

$f\left(x + \dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$

and the roots of $f(x) = 0$ are integers, what is the value of $a^2 + b^2$?

Answer (13)

Sol. $f(x) = x^2 + ax + 6$

$f\left(x + \dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$

$\left(x + \dfrac{1}{x}\right)^2 + a\left(x + \dfrac{1}{x}\right) + b = x^2 + \dfrac{1}{x^2} + a\left(x + \dfrac{1}{x}\right) + 2b$

$\Rightarrow$ b = 2

Since $f(x) = x^2 + ax + 2$ and roots are integers

$\Rightarrow$ a must also be an integer

For integral roots discriminant must be perfect square

$a^2 - 8 = k^2 \Rightarrow (a - k)(a + k) = 1.8$

Either

$$\Rightarrow \begin{array}{r} a - k = 1 \\ a + k = 8 \\ \hline 2a = 9 \end{array} \qquad \begin{array}{r} a - k = 2 \\ a + k = 4 \\ \hline 2a = 6 \end{array}$$

$a = \dfrac{9}{2}$ (rejected) $\qquad$ a = 3

$\therefore \; a^2 + b^2 = 3^2 + 2^2 = 13$

3. Let x_1 be a positive real number and for every integer $n \geq 1$ let $x_{n+1} = 1 + x_1x_2 \ldots x_{n-1}x_n$. If $x_5 = 43$, what is the sum of digits of the largest prime factor of x_6?

Answer (13)

Sol. $x_6 = 1 + x_1x_2x_3x_4x_5$

$x_5 = 1 + x_1x_2x_3x_4 = 43$

$x_1x_2x_3x_4 = 43 - 1 = 42$

$\therefore \; x_6 = 1 + 43 \times 42 = 1807 = 139.13$

Greatest prime factor = 139

Sum of its digits = 1 + 3 + 9 = 13

4. An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, what is the distance in feet it would have walked?

Answer (36)

Sol. Let θ be the angle by which path of ant deviates from its previous path at each step ($\theta = 160°$)

Net displacement of ant horizontally after n steps

$= 4[1 + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta]$

$= \dfrac{4\cos\dfrac{(n-1)}{2}\theta \sin\left(\dfrac{n\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)} = 0 \qquad \ldots(i)$

Net displacement of ant vertically after n steps

$= 4[0 + \sin\theta + \sin 2\theta + \ldots + \sin(n-1)\theta]$

$= \dfrac{4\sin\left(\dfrac{(n-1)\theta}{2}\right)\sin\left(\dfrac{n\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)} = 0 \qquad \ldots(ii)$

Hence, $\sin\left(\dfrac{n\theta}{2}\right) = 0$,

So, 80 n must be multiple of 180.

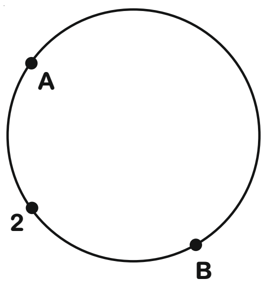
$\therefore$ n = 9

Distance covered = 4 × 9 = 36.

5. Five persons wearing badges with numbers 1, 2, 3, 4, 5 are seated on 5 chairs around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other? (Two arrangements obtained by rotation around the table are considered different.)

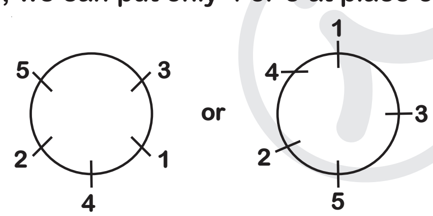
Answer (10)

Sol.



Let 2 is fixed (we can also fix 3 or 4)

Now, we can put only 4 or 5 at place of A and B



Hence, total number of arrangement as per the question = 2 × 5 = 10.

6. Let $\overline{abc}$ be a three digit number with nonzero digits such that $a^2 + b^2 = c^2$. What is the largest possible prime factor of $\overline{abc}$?

Answer (29)

Sol. Let $\overline{abc}$ be 3 digit number with non-zero digits satisfying $a^2 + b^2 = c^2$

as a, b, c $\in$ {1, 2, 3, ... 9}

So, possible cases are

a = 3, b = 4, c = 5

OR

a = 4, b = 3, c = 5

i.e., possible numbers are

345 = 3 × 5 × 23 ...(i)

435 = 3 × 5 × 29 ...(ii)

i.e., largest prime factor is 29.

7. On a clock, there are two instants between 12 noon and 1 PM, when the hour hand and the minute hand are at right angles. The difference in minutes between these two instants is written as $a + \frac{b}{c}$, where a, b, c are positive integers, with b < c and $\frac{b}{c}$ in the reduced form. What is the value of a + b + c?

Answer (51)

Sol. Difference between angle made by minute and hour hand in one minute.

$$= \left(6 - \frac{1}{2}\right)^{o}$$

For x minute, $6x - \frac{x}{2} = \frac{11x}{2} = 90° \text{ or } 270°$

$$\Rightarrow \quad x = \frac{90 \times 2°}{11}, \frac{270 \times 2°}{11}$$

Difference = $\frac{360°}{11}$

$$= 32 + \frac{8}{11}$$

a = 32

b = 8

c = 11

i.e. $\boxed{a + b + c = 51}$

8. How many positive integers n are there such that $3 \le n \le 100$ and $x^{2^n} + x + 1$ is divisible by $x^2 + x + 1$?

Answer (49)

Sol. If $x^2 + x + 1 = 0$,

then $x = \omega, \omega^2$

where $1 + \omega + \omega^2 = 0$

and $\omega^3 = 1$

Now, $x^{2^n} + x + 1 = \omega^{2^n} + \omega + 1$ becomes zero

Iff 2^n is divided by 3, gives remainder 2.

so, n = 3, 5, 7, ..., 99

The number of integers = 49

9. Let the rational number $\frac{p}{q}$ be closest to but not equal to $\frac{22}{7}$ among all rational numbers with denominator < 100. What is the value of p – 3q?

Answer (14)

Sol. $\because \quad \frac{22}{7} = 3 + \frac{1}{7}$

We need to find closest rational number to $\frac{1}{7}$.

$$\because \quad \frac{1}{7} = \frac{14}{98}$$

So closest fraction to $\frac{1}{7}$ will be $\frac{14}{99}$

Then $\frac{p}{q} = 3 + \frac{14}{99} = \frac{311}{99}$

p – 3q = 14

10. Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1 and C_1 respectively and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2, respectively. If the smallest angle of triangle ABC is 40°, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees?

Answer (55)

Sol.

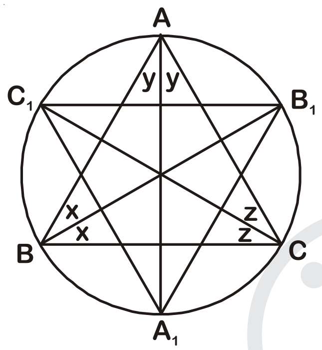


Fig-I

$2x + 2y + 2z = 180°$

$\Rightarrow\ x + y + z = 90°$

$\angle AA_1C_1 = z$

$\angle AA_1B_1 = x$

$\therefore\ \angle A_1 = x + z$

Similarly,

$\angle B_1 = z + y$ and

$\angle C_1 = x + y$

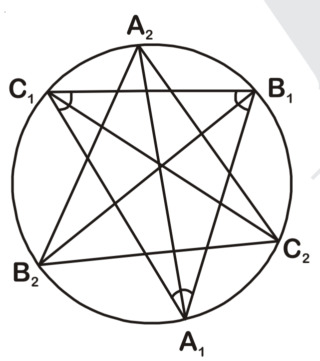


Fig-II

Again, in fig-II,

$\angle A_2 = \angle B_2A_2A_1 + \angle A_1A_2C_2$

$= \angle B_2B_1A_1 + \angle A_1C_1C_2$

$= \dfrac{z+x+y+y}{2} = \dfrac{x+y+z}{2} + \dfrac{y}{2}$

$= 45° + \dfrac{y}{2}$

Similarly, $\angle B_2 = 45° + \dfrac{x}{2}$

and $\angle C_2 = 45° + \dfrac{z}{2}$

let $\angle A$ be least

$\therefore\ y = 20°$

$\therefore\ \angle A_2$ is least

$\therefore\ \angle A_2 = 45° + 10° = 55°$

11. How many distinct triangles ABC are there, up to similarity, such that the magnitudes of angles A, B and C in degrees are positive integers and satisfy

cos A cos B + sin A sin B sin kC = 1

for some positive integer k, where kC does not exceed 360°?

Answer (06)

$\Rightarrow$ cosA.cosB + sinA.sinB.sinkC = 1

If kCE (180°, 360°)

then cosA.cosB = 1 – sinA.sinB.sinkC

$\Rightarrow$ cosA.cosB $\geq$ 1 + sinA.sinB (Impossible case)

If kCE (0°, 180°)

then cosA.cosB + sinA.sinB $\geq$ 1

$\Rightarrow$ cos(A – B) $\geq$ 1

$\Rightarrow$ A = B and kC = 90°

here C must be an even divisor of $90 = 2 \times 3^2 \times 5$

So, number of possible values of C = 2 x 3 = 6

Total number of possible triangles = 06

12. A natural number k > 1 is called good if there exist natural numbers

$$a_1 < a_2 < \ldots < a_k$$

such that

$$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \ldots + \frac{1}{\sqrt{a_k}} = 1$$

Let f(n) be the sum of the first n good numbers, $n \geq 1$. Find the sum of all values of n for which f(n + 5)/f(n) is an integer.

Answer (18)

Sol. The first good number is 3 because

$$\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{9}}+\frac{1}{\sqrt{36}}=1$$

Now we can split $\frac{1}{\sqrt{9}}$ as $\frac{1}{\sqrt{16}}+\frac{1}{\sqrt{144}}$

So next good number will be 4.

Similarly we can split $\frac{1}{\sqrt{144}}$ as $\frac{1}{\sqrt{256}}+\frac{1}{\sqrt{48^2}}$,

then next good number will be 5 and so on

$f(n) = 3 + 4 + 5 + \ldots n \text{ terms} = \frac{n(n+5)}{2}$

So; $\frac{f(n+5)}{f(n)}=\frac{n+10}{n}=1+\frac{10}{n}$

$\Rightarrow$ Possible values of n are 1, 2, 5, 10

Therefore, their sum = 18

13. Each of the numbers $x_1, x_2, \ldots, x_{101}$ is ± 1. What is the smallest positive value of $\sum_{1\le i<j\le 101} x_i x_j$?

Answer (10)

Sol. Using $(x_1+x_2+\ldots+x_{17})^2=\sum x_i^2+2\sum x_i \cdot x_j$

$$\Rightarrow \left(\sum x_i\right)^2=101+2\sum x_i x_j$$

$1\le i<j\le 101$

For minimum possible positive value of $\sum x_i \cdot x_j$,

We have

$\sum x_i = 11$, for 56 positive and 45 negative value of x_i

We have, $\sum x_i \cdot x_j = 10$

$1\le i<j\le 101$

14. Find the smallest positive integer $n \ge 10$ such that $n + 6$ is a prime and $9n + 7$ is a perfect square.

Answer (53)

Sol. Since $n + 6 = p$ is prime and

$9n + 7 = k^2$ is perfect square

$\Rightarrow$ $9(n + 6) - 54 + 7 = k^2$

$\Rightarrow$ $9p = k^2 + 47$

Since LHS is odd $\Rightarrow$ k^2 and hence k must be even

more over $k^2 = 9n + 7 > 97$

Hence possible values of $k^2 = 10^2, 12^2, 14^2, 16^2, 18^2, 20^2, 22^2, \ldots$ for which $k^2 + 47$ must be divisible by 9 and $\frac{k^2+47}{9}$ must be prime. This gives k = 22

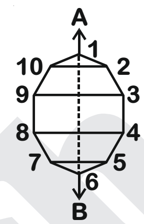
$\Rightarrow$ $p = n + 6 = 59$

$\Rightarrow$ $n = 53$

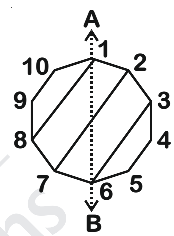
15. In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected?

Answer (45)

Sol.



or



Number of parallel diagonals is 4 and 3 when it is symmetrical about line AB.

Number of symmetry in decagon is 5

$\therefore$ Total number of pair of parallel diagonals

$= 5 \times ({}^4C_2 + {}^3C_2)$

$= 45$

16. A pen costs ₹13 and a note book costs ₹17. A school spends exactly ₹10000 in the year 2017-18 to buy x pens and y note books such that x and y are as close as possible (i.e., |x – y| is minimum). Next year, in 2018-19 the school spends a little more than ₹10000 and buys y pens and x note books. How much more did the school pay?

Answer (40)

Sol. $13x + 17y = 10000$ ...(i)

$17x + 13y = 10000 + a$ ...(ii)

Add (i) and (ii) and subtract (i) from (ii), then

$30(x + y) = 20000 + a$ ...(iii)

$4(x - y) = a$

$\Rightarrow$ $x - y = \frac{a}{4}$

Let a = 4k

But $x+y=\frac{20000+4k}{30}$

$=666+\frac{10+2k}{15}$

$\therefore$ k = 10 [$\because$ |x – y| is minimum]

$\Rightarrow$ a = 40

$\therefore$ School had to pay ₹40 extra in 2018-19.

17. Find the number of ordered triples (a, b, c) of positive integers such that 30a + 50b + 70c $\le$ 343.

Answer (30)

Sol. Let 30a + 50b + 70c = 10n

Here a, b, c, n $\in$ N and n $\le$ 34

$\Rightarrow$ 3a + 5b + 7c = n

$\Rightarrow$ 3x + 5y + 7z = m

where m = n – 15

So m$\in$ [0, 19]

Number of solutions = Sum of coeff. of all powers of x less than 20 in the expansion of

$(1 + x^3 + x^6 + ... + x^{18})(1 + x^5 + x^{10} + x^{15})$

$(1 + x^7 + x^{14})$

$= (1 + x^3 + x^6 + x^9 + x^{12} + x^{15} + x^{18})$

$[1 + x^5 + x^7 + x^{10} + x^{12} + x^{14} + x^{15} + x^{17} + x^{19}]$

= 9 + 7 + 5 + 4 + 3 + 1 + 1

= 30

18. How many ordered pairs (a, b) of positive integers with a < b and 100 $\le$ a, b $\le$ 1000 satisfy gcd(a, b) : lcm(a, b) = 1 : 495?

Answer (20)

Sol. Let the gcd be x, so lcm will be 495x

and ab = 495 x^2

$\because$ 495 $x_2 \le 10^6$

$\Rightarrow$ $x \le 44$

Also, 495 = 5 × 11 × 3^2

So one out of a and b is divisible by 5 and other by 11.

Let a = λ_1x and b = λ_2x (where x is gcd of a and b)

$\Rightarrow$ $\lambda_1 \cdot \lambda_2$ = 495 = 5 × 11 × 3^2 (also $\lambda_1 < \lambda_2$)

and λ_1 and λ_2 are co-prime.

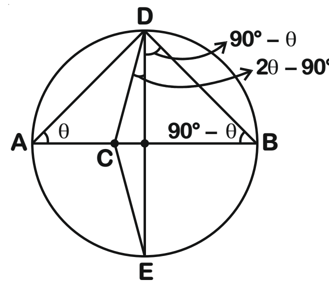
λ_1	λ_2	x
9	55	12, 13, ..., 18 (7 values)
11	45	10, 11, 12, ..., 22 (13 values)

Total number of ordered pairs of (a, b) = 20

19. Let AB be a diameter of a circle and let C be a point on the segment AB such that AC : CB = 6 : 7. Let D be a point on the circle such that DC is perpendicular to AB. Let DE be the diameter through D. If [XYZ] denotes the area of the triangle XYZ, find [ABD]/[CDE] to the nearest integer.

Answer (13)

Sol.



Let the diameter of circle be 26ℓ

AC = 12ℓ, BC = 14ℓ

CD = $12\ell\tan\theta = 14\ell\cot\theta \Rightarrow \tan^2\theta = \frac{7}{6}$

So, $\cos 2\theta = -\frac{1}{13}$

Now, $[ABD] = \frac{1}{2} \times 26\ell \times 12\ell \tan\theta$

and $[CDE] = \frac{1}{2} \times 26\ell \times 12\ell \tan\theta \times \sin(2\theta - 90°)$

$\Rightarrow \frac{[ABD]}{[CDE]} = \frac{-1}{\cos 2\theta} = 13$

20. Consider the set E of all natural numbers n such that when divided by 11, 12, 13 respectively, the remainders, in that order, are distinct prime numbers in an arithmetic progression. If N is the largest number in E, find the sum of digits of N.

Answer (*)

Sol. Case : I N $\equiv$ 3 (mod 11)

N $\equiv$ 5 (mod 12)

N $\equiv$ 7 (mod 13)

Using Chinese remainder theorem

156 $x_1 \equiv$ 1 (mod 11) $\Rightarrow$ x_1 = 6

143 $x_2 \equiv$ 1 (mod 12) $\Rightarrow$ x_2 = –1

132 $x_3 \equiv$ 1 (mod 13) $\Rightarrow$ x_3 = 7

N $\equiv$ 8561 (mod 1716) $\equiv$ 1697 (mod 1716)

Case : II N $\equiv$ 7 (mod 11)

N $\equiv$ 5 (mod 12)

N $\equiv$ 3 (mod 13)

Using Chinese remainder theorem

N $\equiv$ 8609 (mod 1716) $\equiv$ 29 (mod 1716)

Case : III $N \equiv 3 \pmod{11}$

$N \equiv 7 \pmod{12}$

$N \equiv 11 \pmod{13}$

Using Chinese remainder theorem;

$N \equiv 11971 \pmod{1716}$

$N \equiv 1675 \pmod{1716}$

*The minimum possible value of N is 29 but maximum possible value can not be calculated.

21. Consider the set E = {5, 6, 7, 8, 9}. For any partition {A, B} of E, with both A and B non-empty, consider the number obtained by adding the product of elements of A to the product of elements of B. Let N be the largest prime number among these numbers. Find the sum of the digits of N.

Answer (17)

Sol. As A, B are partitions of E, so $A \cap B = \phi$

Let product of elements of A be p_1 and that of B be p_2.

where $p_1.p_2 = 5\cdot6\cdot7\cdot8\cdot9$

$\because$ $p_1 + p_2$ is prime number N.

so if $6 \in A$ then $8, 9 \in A$

So possible combinations of p_1 and p_2 are

(6, 8, 9), (5, 7) or (6, 8, 9, 5), (7) or (6, 8, 9, 7), (5)

The value of N may be 467 or 2167 or 3029 but 2167 and 3029 are not prime. So N = 467.

22. What is the greatest integer not exceeding the sum $\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}$?

Answer (78)

Sol. $\frac{\sqrt{n}+\sqrt{n-1}}{2} < \sqrt{n} < \frac{\sqrt{n+1}+\sqrt{n}}{2}$

$\Rightarrow \quad 2\left(\sqrt{n+1}-\sqrt{n}\right) < \frac{1}{\sqrt{n}} < 2\left(\sqrt{n}-\sqrt{n-1}\right)$

$\Rightarrow \quad 2\left(\sqrt{1600}-1\right) < \sum_{n=1}^{1599} \frac{1}{\sqrt{n}} < 2\left(\sqrt{1599}-0\right)$

$\Rightarrow \quad S \in (78, 80)$

Again; $\sum_{n=2}^{1600} \frac{1}{\sqrt{n}} < 2\left(\sqrt{1600}-1\right) = 78$

So, $S = 78 + 1 - \frac{1}{\sqrt{1600}} = 79 - \frac{1}{40}$

Clearly $S \in (78, 79)$

$\Rightarrow \quad [S] = 78.$

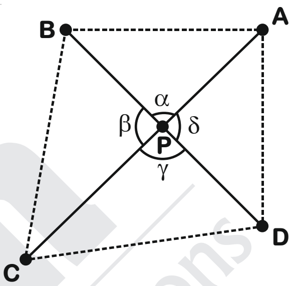
23. Let ABCD be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of ABCD is the least. If

{PA, PB, PC, PD} = {3, 4, 6, 8},

what is the maximum possible area of ABCD?

Answer (55)

Sol. Consider 5 points A, B, C, D and P in a plane. Let distances of A, B, C and D from P be a, b, c, d respectively.



Area of quadrilateral ABCD

$\Delta = [APB] + [BPC] + [CPD] + [DPA]$

$\Delta = \frac{1}{2}ab\sin\alpha + \frac{1}{2}bc\sin\beta + \frac{1}{2}cd\sin\gamma + \frac{1}{2}da\sin\delta$

Δ will be maximum when $\alpha = \beta = \gamma = \delta = 90°$

$\Delta_{max} = \frac{1}{2}(a+c)(b+d)$

Also ac = bd (cyclic quadrilateral)

As {a, b, c, d} = {3, 4, 6, 8}

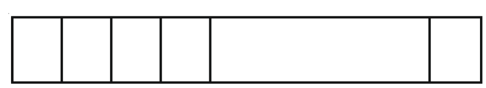
$\Rightarrow$ {(a, c), (b, d)} = {(3, 8), (4, 6)}

So; $\Delta_{max} = \frac{1}{2} \times 11 \times 10 = 55$

24. A 1 × n rectangle ($n \geq 1$) is divided into n unit (1 × 1) squares. Each square of this rectangle is coloured red, blue or green. Let f(n) be the number of colourings of the rectangle in which there are an even number of red squares. What is the largest prime factor of f(9)/f(3)? (The number of red squares can be zero.)

Answer (37)

Sol.



1 × n rectangle

$f(n) = {}^nC_0 \cdot 2^n + {}^nC_2 \cdot 2^{n-2} + {}^nC_4 \cdot 2^{n-4} + ...$

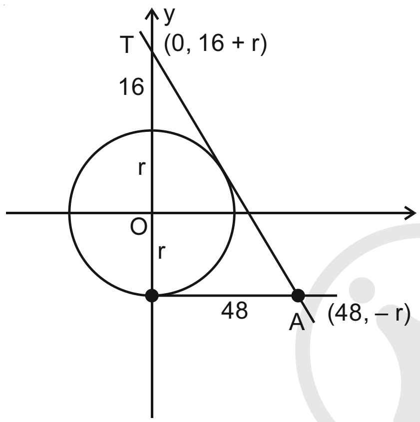
$f(n) = \frac{(2+1)^n + (2-1)^n}{2} = \frac{3^n+1}{2}$

$\frac{f(9)}{f(3)} = \frac{(3^9+1)/2}{(3^3+1)/2} = \frac{27^3+1}{27+1} = 27^2+1-27 = 703 = 19 \times 37$

25. A village has a circular wall around it, and the wall has four gates pointing north, south, east and west. A tree stands outside the village, 16 m north of the north gate, and it can be just seen appearing on the horizon from a point 48 m east of the south gate. What is the diameter, in meters, of the wall that surrounds the village?

Answer (48)

Sol.



Equation of line AT is $y - 16 - r = \frac{-r - 16 - r}{48}(x - 0)$

Perpendicular distance from O(0, 0) to line AT is r

$\Rightarrow \frac{16 + r}{\sqrt{\left(\frac{16 + 2r}{48}\right)^2 + 1}} = r$

$\Rightarrow$ r = 24

$\Rightarrow$ d = 48

26. Positive integers x, y, z satisfy xy + z = 160. Compute the smallest possible value of x + yz.

Answer (50)

Sol. xy + z = 160

Let x + yz = λ

λ is minimum when x = 26, y = 6, z = 4

$\lambda_{minimum}$ = 26 + (6 × 4) = 50.

27. We will say that a rearrangement of the letters of a word has no fixed letters if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, H B R A T A is a rearrangement with no fixed letters of B H A R A T. How many distinguishable rearrangements with no fixed letters does B H A R A T have? (The two As are considered identical.)

Answer (84)

Sol. B H A_1 R A_2 T

Number of ways of derangement of 6 letters

$= \lfloor 6 \left[\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \frac{1}{\lfloor 6}\right]$

= 265

Number of derangements when A_1 takes place of A_2 (or A_2 takes place of A_1) $= \frac{265}{5} = 53$

Number of derangements when A_1 and A_2 exchange their places $= \lfloor 4 \left[\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4}\right] = 9$

Using principle of inclussion – exclussion,

Number of derangements

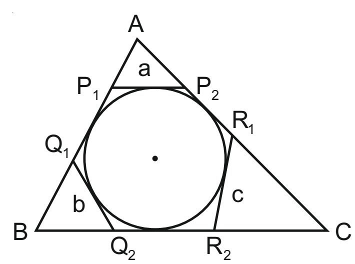
= 265 – 53 – 53 + 9 = 168

But in each derangement A_1 and A_2 are identical, so actual number of ways of derangements $= \frac{168}{2} = 84$

28. Let ABC be a triangle with sides 51, 52, 53. Let Ω denote the incircle of ΔABC. Draw tangents to Ω which are parallel to the sides of ABC. Let r_1, r_2, r_3 be the inradii of the three corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$.

Answer (15)

Sol.



$\because$ All corner triangles are similar with ΔABC

$r_1 = r \cdot \left(\frac{a}{51}\right), r_2 = r \cdot \left(\frac{b}{52}\right)$ and $r_3 = r \cdot \left(\frac{c}{53}\right)$

$r_1 + r_2 + r_3 = r\left[\frac{a}{51} + \frac{b}{52} + \frac{c}{53}\right]$

$\because \Delta AP_1P_2 \sim \Delta ABC$

$\frac{a}{51} = \frac{h_1 - 2r}{h_1} = 1 - \frac{2r}{h_1}$ (where h_1 is altitude form A)

So $r_1 + r_2 + r_3 = r\left[3 - 2r\left(\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3}\right)\right] = r$

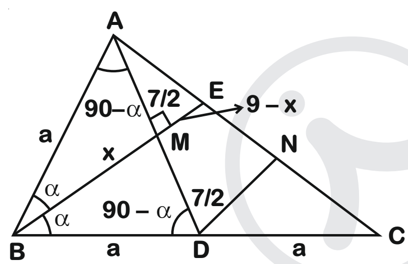
$= \frac{\Delta}{s} = \frac{1170}{78} = 15$

29. In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If AD = 7 and BE = 9, find the integer nearest to the area of triangle ABC.

Answer (47)

Sol. Given AD = 7

BE = 9



In ΔABD

BM is angle bisector so

$\frac{AB}{BD} = \frac{AM}{MD}$ $\left(\because \; AM = MD = \frac{7}{2}\right)$

AB = BD = a (Let)

Construct DN parallel to BE

Now ΔAME and ΔADN are similar,

So ND = 18 – 2x

Now ΔBEC and ΔDNC are also similar

So $\frac{2a}{9} = \frac{a}{18 - 2x} \Rightarrow x = \frac{27}{4}$

Now from triangle ABM, by Pythagoras theorem

$a = \frac{\sqrt{925}}{4}$

So, $\tan\alpha = \frac{7/2}{27/4} = \frac{14}{27}$

Now area of triangle ABC $= \frac{1}{2} \times AB \times BC \times \sin 2\alpha$

$= \frac{1}{2} \times a \times 2a \times 2 \times \frac{7/2}{a} \times \frac{x}{a}$

$= 7 \times \frac{27}{4} = 47.25$

So, nearest area = 47

30. Let E denote the set of all natural numbers n such that 3 < n < 100 and the set (1, 2, 3,..., n) can be partitioned into 3 subsets with equal sums. Find the number of elements of E.

Answer (64)

Sol. Sum of all digits in a set {1, 2, 3,....n) $= \frac{n(n+1)}{2}$

So for dividing any set into 3 subsets of equal sum,

n(n + 1) should be multiple of 3

So $n \notin 3\lambda + 1$

$\Rightarrow$ $n \neq 4, 7, 10, ..., 97$

Total number of possible values of n

= 96 – 32 = 64

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