Date: August 11, 2019
Time: 10 AM to 1 PM
Number of Questions: 30
Max Marks: 102

Aakash
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Answers & Solutions
for
PRMO - 2019

INSTRUCTIONS

1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is STRICTLY PROHIBITED. Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a black pencil or a black or blue ball pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your PRMO score.

4. Incompletely, Incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions 1 to 6 carry 2 marks each; questions 7 to 21 carry 3 marks each; questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate’s copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.
1. From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area removed to the nearest integer?

Answer (04)

Sol.

\[ 5 - 2a = a \sqrt{2} \]
\[ a = \frac{5}{2 + \sqrt{2}} = 5 \left(1 - \frac{1}{\sqrt{2}}\right) \]

Area removed = \[ 4 \times \frac{1}{2} a^2 \]
\[ = 2 \times 25 \left(\frac{\sqrt{2} - 1}{2}\right)^2 \]
\[ = 4.3 \]

Area to nearest integer is 4

2. Let \( f(x) = x^2 + ax + b \). If for all nonzero real \( x \)
\[ f \left( x + \frac{1}{x} \right) = f(x) + f \left( \frac{1}{x} \right) \]
and the roots of \( f(x) = 0 \) are integers, what is the value of \( a^2 + b^2 \)?

Answer (13)

Sol. \( f(x) = x^2 + ax + 6 \)
\[ f \left( x + \frac{1}{x} \right) = f(x) + f \left( \frac{1}{x} \right) \]
\[ \left( x + \frac{1}{x} \right)^2 + a \left( x + \frac{1}{x} \right) + b = x^2 + \frac{1}{x^2} + a \left( x + \frac{1}{x} \right) + 2b \]
\[ \Rightarrow b = 2 \]

Since \( f(x) = x^2 + ax + 2 \) and roots are integers
\[ \Rightarrow a \text{ must also be an integer} \]

For integral roots discriminant must be perfect square
\[ a^2 - 8 = k^2 \Rightarrow (a - k)(a + k) = 1.8 \]

3. Let \( x_n \) be a positive real number and for every integer \( n \geq 1 \) let \( x_{n+1} = x_n^2 - x_n \). If \( x_5 = 43 \), what is the sum of digits of the largest prime factor of \( x_6 \)?

Answer (13)

Sol. \( x_6 = 1 + x_1 x_2 x_3 x_4 x_5 \)
\[ x_5 = 1 + x_1 x_2 x_3 x_4 = 43 \]
\[ x_1 x_2 x_3 x_4 = 43 - 1 = 42 \]
\[ \therefore x_6 = 1 + 43 \times 42 = 1807 = 139.13 \]

Greatest prime factor = 139

Sum of its digits = 1 + 3 + 9 = 13

4. An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, what is the distance in feet it would have walked?

Answer (36)

Sol. Let \( \theta \) be the angle by which path of ant deviates from its previous path at each step \((\theta = 160°)\)

Net displacement of ant horizontally after \( n \) steps
\[ = 4 \left[ 1 + \cos \theta + \cos 2\theta + \ldots + \cos (n-1)\theta \right] \]
\[ = \frac{4 \cos \left( \frac{n-1}{2} \theta \right) \sin \left( \frac{n\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} = 0 \ldots (i) \]

Net displacement of ant vertically after \( n \) steps
\[ = 4 \left[ 0 + \sin \theta + \sin 2\theta + \ldots + \sin (n-1)\theta \right] \]
\[ = \frac{4 \sin \left( \frac{n-1}{2} \theta \right) \sin \left( \frac{n\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} = 0 \ldots (ii) \]

Hence, \( \sin \left( \frac{n\theta}{2} \right) = 0 \),

So, \( 80n \) must be multiple of 180.
\[ \therefore n = 9 \]

Distance covered = \( 4 \times 9 = 36 \).
5. Five persons wearing badges with numbers 1, 2, 3, 4, 5 are seated on 5 chairs around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other? (Two arrangements obtained by rotation around the table are considered different.)

Answer (10)

Sol.

Let 2 be fixed (we can also fix 3 or 4)
Now, we can put only 4 or 5 at place of A and B

Hence, total number of arrangement as per the question = 2 \times 5 = 10.

6. Let abc be a three digit number with nonzero digits such that \(a^2 + b^2 = c^2\). What is the largest possible prime factor of abc?

Answer (29)

Sol. Let abc be 3 digit number with non-zero digits satisfying \(a^2 + b^2 = c^2\) as \(a, b, c \in \{1, 2, 3, ..., 9\}\)
So, possible cases are
\[
a = 3, b = 4, c = 5
\]

OR
\[
a = 4, b = 3, c = 5
\]

\text{i.e., possible numbers are}
\[
345 = 3 \times 5 \times 23 \quad \text{...(i)}
\]
\[
435 = 3 \times 5 \times 29 \quad \text{...(ii)}
\]
\text{i.e., largest prime factor is 29.}

7. On a clock, there are two instants between 12 noon and 1 PM, when the hour hand and the minute hand are at right angles. The difference in minutes between these two instants is written as \(\frac{b}{c}\), where \(a, b, c\) are positive integers, with \(b < c\) and \(\frac{b}{c}\) in the reduced form. What is the value of \(a + b + c\)?

Answer (51)

Sol. Difference between angle made by minute and hour hand in one minute.
\[
\left(6 - \frac{1}{2}\right)^\circ
\]
For \(x\) minute, \(6x - \frac{x}{2} = \frac{11x}{2} = 90^\circ\) or \(270^\circ\)
\[
\Rightarrow x = \frac{90 \times 2}{11}, \frac{270 \times 2}{11}
\]
\[
\text{Difference} = \frac{360}{11}
\]
\[
= 32 + \frac{8}{11}
\]
\(a = 32\)
\(b = 8\)
\(c = 11\)
\(\text{i.e.} \quad \frac{a+b+c}{51}
\)

8. How many positive integers \(n\) are there such that \(3 \leq n \leq 100\) and \(x^{2n} + x + 1\) is divisible by \(x^2 + x + 1\)?

Answer (49)

Sol. If \(x^2 + x + 1 = 0\), then \(x = \omega, \omega^2\) where \(1 + \omega + \omega^2 = 0\) and \(\omega^3 = 1\)
Now, \(x^{2n} + x + 1 = (\omega^{2n} + \omega + 1)\) becomes zero
Iff \(2^n\) is divided by 3, gives remainder 2.
so, \(n = 3, 5, 7, ..., 99\)
The number of integers = 49

9. Let the rational number \(\frac{p}{q}\) be closest to but not equal to \(\frac{22}{7}\) among all rational numbers with denominator < 100. What is the value of \(p - 3q\)?

Answer (14)

Sol. \(\because \quad \frac{22}{7} = 3 + \frac{1}{7}\)

We need to find closest rational number to \(\frac{1}{7}\).
\[
\frac{1}{7} = \frac{14}{98}
\]
So closest fraction to \(\frac{1}{7}\) will be \(\frac{14}{99}\)
Then \(\frac{p}{q} = 3 - \frac{14}{99} = \frac{311}{99}\)
\(p - 3q = 14\)
10. Let \( \triangle ABC \) be a triangle and let \( \Omega \) be its circumcircle. The internal bisectors of angles \( A, B \) and \( C \) intersect \( \Omega \) at \( A_1, B_1 \) and \( C_1 \) respectively and the internal bisectors of angles \( A_1, B_1 \) and \( C_1 \) of the triangle \( A_1B_1C_1 \) intersect \( \Omega \) at \( A_2, B_2 \) and \( C_2 \), respectively. If the smallest angle of triangle \( ABC \) is 40°, what is the magnitude of the smallest angle of triangle \( A_2B_2C_2 \) in degrees?

Answer (55)

Sol.

\[
\begin{align*}
2x + 2y + 2z &= 180° \\
\Rightarrow x + y + z &= 90° \\
\angle AA_1C_1 &= z \\
\angle AA_1B_1 &= x \\
\therefore \angle A &= x + z \\
\text{Similarly,} \\
\angle B &= z + y \quad \text{and} \\
\angle C &= x + y
\end{align*}
\]

11. How many distinct triangles \( \triangle ABC \) are there, up to similarity, such that the magnitudes of angles \( A, B \) and \( C \) in degrees are positive integers and satisfy

\[
\cos A \cos B + \sin A \sin B \sin kC = 1
\]

for some positive integer \( k \), where \( kC \) does not exceed 360°?

Answer (06)

\[
12. A \text{ natural number } k > 1 \text{ is called good if there exist natural numbers } a_1 < a_2 < \ldots < a_k \\
\text{such that } \\
\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \ldots + \frac{1}{\sqrt{a_k}} = 1
\]

Let \( f(n) \) be the sum of the first \( n \) good numbers, \( n \geq 1 \). Find the sum of all values of \( n \) for which \( f(n + 5)/f(n) \) is an integer.

Answer (18)
The first good number is 3 because
\[
\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{36}} = 1
\]
Now we can split \(\frac{1}{\sqrt{9}}\) as \(\frac{1}{\sqrt{16}} + \frac{1}{\sqrt{144}}\)
So next good number will be 4.
Similarly we can split \(\frac{1}{\sqrt{144}}\) as \(\frac{1}{\sqrt{256}} + \frac{1}{\sqrt{48^2}}\),
then next good number will be 5 and so on
\[
f(n) = 3 + 4 + 5 + ... n \text{ terms } = \frac{n(n+5)}{2}
\]
\[
\Rightarrow \frac{f(n+5)}{f(n)} = \frac{n+10}{n} = 1 + \frac{10}{n}
\]
\[
\Rightarrow \text{ Possible values of n are } 1, 2, 5, 10
\]
Therefore, their sum = 18

Each of the numbers \(x_1, x_2, \ldots, x_{101}\) is \(\pm 1\). What is the smallest positive value of \(\sum_{1 \leq i < j \leq 101} x_i x_j\)?
Answer (10)

Sol. Using \((x_1 + x_2 + \ldots + x_{17})^2 = \sum x_i^2 + 2 \sum x_i \cdot x_j\)
\[
\Rightarrow (\sum x_i)^2 = 101 + 2 \sum x_i \cdot x_j
\]
\[1 \leq i < j \leq 101\]

For minimum possible positive value of \(\sum x_i \cdot x_j\),
We have
\[
\sum x_i = 11, \text{ for 56 positive and 45 negative value of } x_i
\]
We have, \(\sum x_i \cdot x_j = 10\)
\[1 \leq i < j \leq 101\]

Find the smallest positive integer \(n \geq 10\) such that \(n + 6\) is a prime and \(9n + 7\) is a perfect square.
Answer (53)

Sol. Since \(n + 6 = p\) is prime and
\[9n + 7 = k^2 \text{ is perfect square}\]
\[
\Rightarrow 9(n + 6) - 54 + 7 = k^2
\]
\[
\Rightarrow 9p = k^2 + 47
\]
Since LHS is odd \(\Rightarrow k^2\) and hence \(k\) must be even
more over \(k^2 = 9n + 7 > 97\)

Hence possible values of \(k^2 = 10^2, 12^2, 14^2, 16^2, 18^2, 20^2, 22^2\), for which \(k^2 + 47\) must be divisible by 9 and \(\frac{k^2 + 47}{9}\) must be prime. This gives \(k = 22\)
\[
\Rightarrow p = n + 6 = 59
\]
\[
\Rightarrow n = 53
\]

In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected?
Answer (45)

Number of parallel diagonals is 4 and 3 when it is symmetrical about line AB.

Number of symmetry in decagon is 5
\[
\Rightarrow \text{ Total number of pair of parallel diagonals } = 5 \times (\binom{4}{2} + \binom{3}{2})
\]
\[
= 45
\]

A pen costs ₹13 and a note book costs ₹17. A school spends exactly ₹10000 in the year 2017-18 to buy \(x\) pens and \(y\) note books such that \(x\) and \(y\) are as close as possible (i.e., \(|x - y|\) is minimum). Next year, in 2018-19 the school spends a little more than ₹10000 and buys \(y\) pens and \(x\) note books. How much more did the school pay?
Answer (40)

Sol. \(13x + 17y = 10000 \ldots (i)\)
\[
17x + 13y = 10000 + a \ldots (ii)
\]
Add (i) and (ii) and subtract (i) from (ii), then
\[
30(x + y) = 20000 + a \ldots (iii)
\]
\[
4(x - y) = a
\]
\[
\Rightarrow x - y = \frac{a}{4}
\]
Let \(a = 4k\)
But \(x + y = \frac{20000 + 4k}{30} = 666 + \frac{10 + 2k}{15}\)
\[
\Rightarrow k = 10 \quad \text{[} \because |x - y| \text{ is minimum}]\]
\[
\Rightarrow a = 40
\]
\[
\Rightarrow \text{ School had to pay ₹}40 \text{ extra in 2018-19.}
\]
17. Find the number of ordered triples \((a, b, c)\) of positive integers such that \(30a + 50b + 70c \leq 343.\)

Answer (30)

Sol. Let \(30a + 50b + 70c = 10n\)

Here \(a, b, c, n \in \mathbb{N}\) and \(n \leq 34\)

\[3a + 5b + 7c = n\]

\[3x + 5y + 7z = m\]

where \(m = n - 15\)

So \(m \in [0, 19]\)

Number of solutions = Sum of coeff. of all powers of \(x\) less than 20 in the expansion of

\[(1 + x^3 + x^6 + \ldots + x^{18}) (1 + x^5 + x^{10} + x^{15}) \]

\[(1 + x^7 + x^{14})\]

\[= (1 + x^3 + x^6 + x^9 + x^{12} + x^{15} + x^{18})\]

\[\vdots\]

\[= 9 + 7 + 5 + 4 + 3 + 1 + 1\]

\[= 30\]

18. How many ordered pairs \((a, b)\) of positive integers with \(a < b\) and \(100 \leq a, b \leq 1000\) satisfy \(\gcd(a, b) : \text{lcm}(a, b) = 1 : 495?\)

Answer (20)

Sol. Let the \(\gcd\) be \(x\), so \(\text{lcm}\) will be \(495x\)

and \(ab = 495x^2\)

\[\vdash 495 \cdot x^2 \leq 10^6 \]

\[\Rightarrow x \leq 44\]

Also, \(495 = 5 \times 11 \times 3^2\)

So one out of \(a\) and \(b\) is divisible by 5 and other by 11.

Let \(a = \lambda_1 x\) and \(b = \lambda_2 x\) (where \(x\) is \(\gcd\) of \(a\) and \(b\))

\[\Rightarrow \lambda_1 \cdot \lambda_2 = 495 = 5 \times 11 \times 3^2\]

(also \(\lambda_1 < \lambda_2\))

and \(\lambda_1\) and \(\lambda_2\) are co-prime.

\[
\begin{array}{|c|c|}
\hline
\lambda_1 & \lambda_2 & x \\
\hline
9 & 55 & 12, 13, \ldots, 18 (7 \text{ values}) \\
11 & 45 & 10, 11, 12, \ldots, 22 (13 \text{ values}) \\
\hline
\end{array}
\]

Total number of ordered pairs of \((a, b)\) = 20

19. Let \(AB\) be a diameter of a circle and let \(C\) be a point on the segment \(AB\) such that \(AC : CB = 6 : 7.\) Let \(D\) be a point on the circle such that \(DC\) is perpendicular to \(AB.\) Let \(DE\) be the diameter through \(D.\) If \([XYZ]\) denotes the area of the triangle \(XYZ,\) find \([ABD]/[CDE]\) to the nearest integer.

Answer (13)

Sol. Let the diameter of circle be \(26\ell\)

\[AC = 12\ell, \quad BC = 14\ell\]

\[CD = 12\ell\tan\theta = 14\ell\cot\theta \quad \Rightarrow \quad \tan^2\theta = \frac{7}{6}\]

\[\cos\theta = -\frac{1}{13}\]

Now, \(\frac{[ABD]}{[CDE]} = \frac{1}{2} \times 26\ell \times 12\ell \tan\theta \times \sin(2\theta - 90^\circ)\)

\[\Rightarrow \frac{[ABD]}{[CDE]} = \frac{-1}{\cos2\theta} = 13\]

20. Consider the set \(E\) of all natural numbers \(n\) such that when divided by 11, 12, 13 respectively, the remainders, in that order, are distinct prime numbers in an arithmetic progression. If \(N\) is the largest number in \(E,\) find the sum of digits of \(N.\)

Answer (*)

Sol. Case : I \(N \equiv 3 \pmod{11}\)

\(N \equiv 5 \pmod{12}\)

\(N \equiv 7 \pmod{13}\)

Using Chinese remainder theorem

\[156 x_1 \equiv 1 \pmod{11} \quad \Rightarrow \quad x_1 = 6\]

\[143 x_2 \equiv 1 \pmod{12} \quad \Rightarrow \quad x_2 = -1\]

\[132 x_3 \equiv 1 \pmod{13} \quad \Rightarrow \quad x_3 = 7\]

\(N \equiv 8561 \pmod{1716} \equiv 1697 \pmod{1716}\)

Case : II \(N \equiv 7 \pmod{11}\)

\(N \equiv 5 \pmod{12}\)

\(N \equiv 3 \pmod{13}\)

Using Chinese remainder theorem

\(N \equiv 8609 \pmod{1716} \equiv 29 \pmod{1716}\)
Case: III  \[ N \equiv 3 \pmod{11} \]
\[ N \equiv 7 \pmod{12} \]
\[ N \equiv 11 \pmod{13} \]

Using Chinese remainder theorem;
\[ N \equiv 11971 \pmod{1716} \]
\[ N \equiv 1675 \pmod{1716} \]

*The minimum possible value of \( N \) is 29 but maximum possible value can not be calculated.

21. Consider the set \( E = \{5, 6, 7, 8, 9\} \). For any partition \( \{A, B\} \) of \( E \), with both \( A \) and \( B \) non-empty, consider the number obtained by adding the product of elements of \( A \) to the product of elements of \( B \). Let \( N \) be the largest prime number among these numbers. Find the sum of the digits of \( N \).

**Answer (17)**

**Sol.** As \( A, B \) are partitions of \( E \), so \( A \cap B = \emptyset \)

Let product of elements of \( A \) be \( p_1 \) and that of \( B \) be \( p_2 \).

\[ p_1 \cdot p_2 = 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \]

\( \therefore p_1 + p_2 \) is prime number \( N \).

so if \( 6 \in A \) then \( 8, 9 \in A \)

So possible combinations of \( p_1 \) and \( p_2 \) are

\( (6, 8, 9), (5, 7) \) or \( (6, 8, 9, 5), (7) \) or \( (6, 8, 9, 7), (5) \)

The value of \( N \) may be 467 or 2167 or 3029 but 2167 and 3029 are not prime. So \( N = 467 \).

22. What is the greatest integer not exceeding the sum \( \sum_{n=1}^{1599} \frac{1}{\sqrt{n}} \)?

**Answer (78)**

**Sol.**

\[ \frac{\sqrt{n} + \sqrt{n-1}}{2} < \sqrt{n} < \sqrt{n+1} + \sqrt{n} \]

\[ \Rightarrow 2 \left( \sqrt{n+1} - \sqrt{n} \right) < \frac{1}{\sqrt{n}} < 2 \left( \sqrt{n} - \sqrt{n-1} \right) \]

\[ \Rightarrow 2 \left( \sqrt{1600} - 1 \right) < \sum_{n=1}^{1599} \frac{1}{\sqrt{n}} < 2 \left( \sqrt{1599} - 0 \right) \]

\[ \Rightarrow S \in (78, 80) \]

Again, \[ \sum_{n=2}^{1600} \frac{1}{\sqrt{n}} < 2 \left( \sqrt{1600} - 1 \right) = 78 \]

So, \( S = 78 + 1 - \frac{1}{\sqrt{1600}} = 79 - \frac{1}{40} \)

Clearly \( S \in (78, 79) \)

\[ \Rightarrow [S] = 78. \]

23. Let \( ABCD \) be a convex cyclic quadrilateral. Suppose \( P \) is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of \( ABCD \) is the least. If

\[ \{PA, PB, PC, PD\} = \{3, 4, 6, 8\}, \]

what is the maximum possible area of \( ABCD \)?

**Answer (55)**

**Sol.** Consider 5 points \( A, B, C, D \) and \( P \) in a plane.

Let distances of \( A, B, C \) and \( D \) from \( P \) be \( a, b, c, d \) respectively.

Area of quadrilateral \( ABCD \)

\[ \Delta = [APB] + [BPC] + [CPD] + [DPA] \]

\[ \Delta = \frac{1}{2} ab \sin \alpha + \frac{1}{2} bc \sin \beta + \frac{1}{2} cd \sin \gamma + \frac{1}{2} da \sin \delta \]

\( \Delta \) will be maximum when \( \alpha = \beta = \gamma = \delta = 90^\circ \)

\[ \Delta_{\text{max}} = \frac{1}{2} (a + c)(b + d) \]

Also \( ac = bd \) (cyclic quadrilateral)

As \( \{a, b, c, d\} = \{3, 4, 6, 8\} \)

\( \Rightarrow \{(a, c), (b, d)\} = \{(3, 8), (4, 6)\} \)

So, \( \Delta_{\text{max}} = \frac{1}{2} \times 11 \times 10 = 55 \)

24. A \( 1 \times n \) rectangle \( (n \geq 1) \) is divided into \( n \) unit \((1 \times 1)\) squares. Each square of this rectangle is coloured red, blue or green. Let \( f(n) \) be the number of colourings of the rectangle in which there are an even number of red squares. What is the largest prime factor of \( f(9)/f(3) \)? (The number of red squares can be zero.)

**Answer (37)**

**Sol.**

\[ f(n) = \binom{n}{0} \cdot 2^n + \binom{n}{2} \cdot 2^{n-2} + \binom{n}{4} \cdot 2^{n-4} + \ldots \]

\[ f(n) = \frac{(2+1)^n + (2-1)^n}{2} = 3^n + 1 \]

\[ f(9) = \frac{(3^9 + 1)/2}{2^{27 + 1}} + \frac{27^3 + 1}{2^{27^2 + 1}} - 27 = 703 - 19 \times 37 \]

\[ f(3) = \frac{(3^3 + 1)/2}{2^{27 + 1}} \]
25. A village has a circular wall around it, and the wall has four gates pointing north, south, east and west. A tree stands outside the village, 16 m north of the north gate, and it can be just seen appearing on the horizon from a point 48 m east of the south gate. What is the diameter, in meters, of the wall that surrounds the village?

Answer (48)

Sol. 

Equation of line AT is $y - 16 - r = \frac{r - 16 - r}{48} (x - 0)$

Perpendicular distance from $O(0, 0)$ to line AT is $r$

$\Rightarrow \frac{16 + r}{\sqrt{(16 + 2r)^2 + 1}} = r$

$\Rightarrow r = 24$

$\Rightarrow d = 48$

26. Positive integers $x$, $y$, $z$ satisfy $xy + z = 160$. Compute the smallest possible value of $x + yz$.

Answer (50)

Sol. $xy + z = 160$

Let $x + yz = \lambda$

$\lambda$ is minimum when $x = 26$, $y = 6$, $z = 4$

$\lambda_{\text{minimum}} = 26 + (6 \times 4) = 50.$

27. We will say that a rearrangement of the letters of a word has no fixed letters if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, H B R A T A is a rearrangement with no fixed letters of B H A R A T. How many distinguishable rearrangements with no fixed letters does B H A R A T have? (The two As are considered identical.)

Answer (84)

Sol. B H A T R A

Number of ways of derangement of 6 letters

$= \binom{6}{1} \left[ \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} \right]$

$= 265$

Number of derangements when $A_1$ takes place of $A_2$ (or $A_2$ takes place of $A_1$) $= \frac{265}{5} = 53$

Number of derangements when $A_1$ and $A_2$ exchange their places $= 4 \left[ \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \right] = 9$

Using principle of inclusion-exclusion,

Number of derangements $= 265 - 53 - 53 + 9 = 168$

But in each derangement $A_1$ and $A_2$ are identical, so actual number of ways of derangements $= \frac{168}{2} = 84$

28. Let $ABC$ be a triangle with sides 51, 52, 53. Let $\Omega$ denote the incircle of $\triangle ABC$. Draw tangents to $\Omega$ which are parallel to the sides of $ABC$. Let $r_1$, $r_2$, $r_3$ be the inradii of the three corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$.

Answer (15)

Sol. 

All corner triangles are similar with $\triangle ABC$

$r_1 = r_1 \left( \frac{a}{51} \right), r_2 = r_2 \left( \frac{b}{52} \right)$ and $r_3 = \frac{c}{53}\left( \frac{c}{53} \right)$

$r_1 + r_2 + r_3 = \frac{a}{51} + \frac{b}{52} + \frac{c}{53} + \frac{c}{53}$

$\therefore \triangle AP_1P_2 \sim \triangle ABC$

$\frac{a}{51} = \frac{h_1 - 2r}{h_1} = 1 - \frac{2r}{h_1}$ (where $h_1$ is altitude form A)
29. In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If AD = 7 and BE = 9, find the integer nearest to the area of triangle ABC.

Answer (47)

Sol. Given AD = 7
    BE = 9

In \( \triangle ABD \)
    BM is angle bisector so

\[
\frac{AB}{AM} = \frac{BD}{MD} \quad \therefore \quad AM = MD = \frac{7}{2}
\]

Construct DN parallel to BE
Now \( \triangle AME \) and \( \triangle ADN \) are similar,
So ND = 18 – 2x
Now \( \triangle BEC \) and \( \triangle DNC \) are also similar
So
\[
\frac{2a}{9} = x = \frac{27}{4}
\]

Now from triangle ABM, by Pythagoras theorem

\[
a = \frac{\sqrt{925}}{4}
\]

So, \( \tan \alpha = \frac{7/2}{27/4} = \frac{14}{27} \)

Now area of triangle ABC = \( \frac{1}{2} \times AB \times BC \times \sin 2\alpha \)

\[
= \frac{1}{2} \times a \times 2a \times 7/2 \times x \times 15 = 7 \times 27/4 = 47.25
\]

So, nearest area = 47

30. Let E denote the set of all natural numbers n such that 3 < n < 100 and the set \( \{1, 2, 3, \ldots, n\} \) can be partitioned into 3 subsets with equal sums. Find the number of elements of E.

Answer (64)

Sol. Sum of all digits in a set \( \{1, 2, 3, \ldots, n\} \) = \( \frac{n(n+1)}{2} \)

So for dividing any set into 3 subsets of equal sum,
\( n(n+1) \) should be multiple of 3
So \( n \not\equiv 3l + 1 \)
\Rightarrow n \not= 4, 7, 10, \ldots, 97

Total number of possible values of n
\[= 96 - 32 = 64\]