Date: 08/09/2024

Number of Questions: 30

Time: 3 Hours

Max Marks: 100

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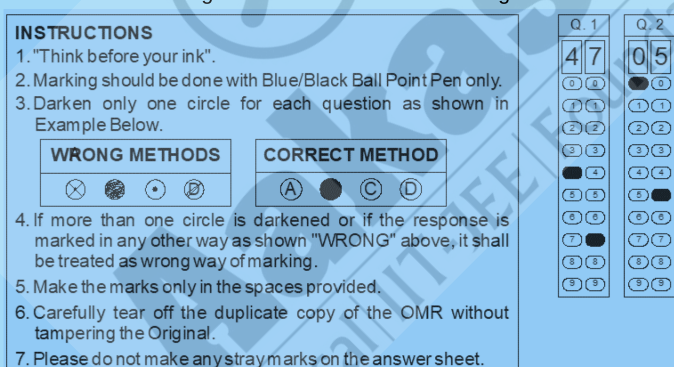
Answers & Solutions

for

IOQM – 2024-25

INSTRUCTIONS TO CANDIDATES

1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen.** Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The registration number and date of birth will be your login credentials for accessing your score.
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two-digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 & 30 carry 5 marks each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the Candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

Note:

1. gcd (a, b) denotes the greatest common divisor of integers a and b.
2. $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.
3. For a positive real number $m, \sqrt{m}$ denotes the positive square root of m. For example, $\sqrt{4} = +2$.
4. Unless otherwise stated all numbers are written in decimal notation.

1. The smallest positive integer that does not divide $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$ is:

Answer (11)

Sol. Let n be a smallest integer that does not divide $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$

Prime factorisation = $2^7 \times 3^4 \times 5 \times 7$

Clearly we can see that 2 and 7 factor of 2 will divide 9!

Also 3 and 4 factors of 3 will divide 9!

So number i.e. 2, 3, 2^2, 5. 2.3, 7, 2^3, 3^3, 5.2, will divide 9!

Therefore, least positive integer that does not divide 9! is 11

2. The number of four-digit odd numbers having digits 1, 2, 3, 4, each occurring exactly once, is:

Answer (12)

Sol. *abcd*

$\therefore$ *abcd* is odd number therefore d can take values as 1 or 3 $\Rightarrow$ 2 posibility and

a can take 3 value

b can take 2 value

c can take 1 value

Total values = $2 \times 3 \times 2 \times 1$

$\Rightarrow$ 12 numbers

3. The number obtained by taking the last two digits of 5^{2024} in the same order is:

Answer (25)

Sol. Last two digits of 5^{2024}

$\Rightarrow$ $5^{2024} \equiv a \pmod{100}$

$5^4 \equiv 25 \pmod{100}$

$5^{2024} \equiv 25 \pmod{100}$

$\Rightarrow$ last 2 digits in same order is 25

4. Let $ABCD$ be a quadrilateral with $\angle ADC = 70°, \angle ACD = 70°, \angle ACB = 10°$ and $\angle BAD = 110°$. The measure of $\angle CAB$ (in degrees) is:

Answer (70)

Sol. In ΔADC

$$70^\circ + 70^\circ + \angle DAC = 180^\circ$$

$\Rightarrow \quad \angle DAC = 40^\circ$

Now $\angle DAC + \angle CAB = 110^\circ$

$\Rightarrow \quad \angle CAB = 110^\circ - 40^\circ$

$= 70^\circ$

5. Let $a = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, let $b = \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$ and let $c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$. The value of $|ab - c|$ is:

Answer (1)

Sol. $a = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ ...(i)

and $b = \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$ ...(ii)

and $c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$

$= \left(a - \frac{z}{x}\right)\left(a - \frac{x}{y}\right)\left(a - \frac{y}{z}\right)$

$= a^3 - \left(\frac{z}{x} + \frac{x}{y} + \frac{y}{z}\right)a^2 + \left(\frac{z}{x} + \frac{x}{z} + \frac{y}{x}\right)a - 1$

$= a^3 - a^3 + ab - 1$

$\therefore \quad 1 = ab - c$

6. Find the number of triples of real numbers (a, b, c) such that $a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$.

Answer (6)

Sol. $a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$

If both equation have to satisfy

$\Rightarrow$ Either a or b or $c = \pm 1$

And other have to be 0.

So, if $a = 1$

$\Rightarrow \quad b = c = 0$

If $a = -1$

$\Rightarrow \quad b = c = 0$

Similarly, for b and c if $b = \pm 1$

$\Rightarrow \quad a = c = 0$

And if $c = \pm 1$

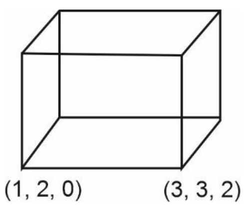
$\Rightarrow \quad a = b = 0$

Total triples = 6.

7. Determine the sum of all possible surface areas of a cube two of whose vertices are (1, 2, 0) and (3, 3, 2).

Answer (99)

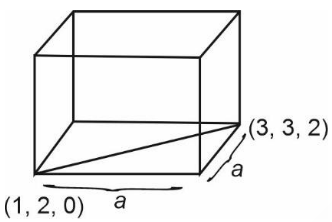
Sol. Case-I



$SA = 6 \times \left[\sqrt{4+1+4}\right]^2$

$= 54$ sq.unit^2

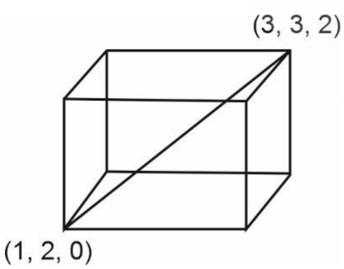
Case-II



$2a^2 + 9$

$a = \frac{3}{\sqrt{2}} \qquad SA = 6\left(\frac{9}{2}\right) = 27$ sq unit

Case-III



$\sqrt{3}a = 3$

$\Rightarrow a = \sqrt{3}$

$SA = 6a^2 = 6 \times 3 = 18$ sq.unit^2

$\therefore$ Possible $SA = 18 + 27 + 54 = 99$

8. Let *n*, be the smallest integer such that the sum of digits of *n* is divisible by 5 as sell as the sum of digits of (*0* + 1) is divisible by 5. What are the first two digits of *n* in the same order?

Answer (49)

Sol. Let sum of digits of *n*-digits number = 5λ

Let sum of digits of $n + 1$ number = 5k

$\underbrace{1, 2}_{1}\underbrace{, 3}_{1}, \ldots, \underbrace{9, 10}_{8}\underbrace{, 11}_{1}, \ldots, \underbrace{19, 20}_{8}, \ldots, \underbrace{99, 100}_{17}$

$\ldots, \underbrace{999, 1000}_{26}, \ldots, \underbrace{9999, 10{,}000}_{35}$

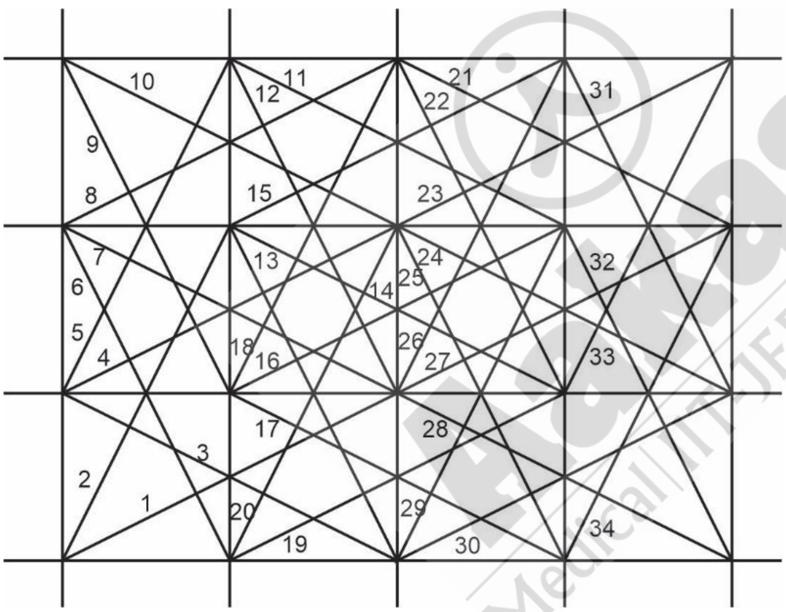
$\ldots, \underbrace{49999, 50000}_{\text{diff } 35}$

$\therefore$ Smallest number = 49,999

Firsts two digits = 49.

9. Consider the grid of points $X = \{(m, n) | 0 \le m, n \le 4\}$. We say a pair of points {(*a*, *b*), (*c*, *d*)} in *X* is a knight-move pair if $(c = a \pm 2 \text{ and } d = b \pm 1)$ or $(c = a \pm 1 \text{ and } d = b \pm 2)$. The number of knight-move pairs in *X* is:

Answer (48)



Sol.

Consider the grid as shown.

Each line represents the allowed move from one and point of line to other end point of line we can see the movement allowed in 3 × 2, 2 × 2 rectangle, vertical or horizontal. In each rectangle there are 4 moves allowed and total number of rectangle are 12.

$\therefore$ Total allowed moved = 12 × 4 = 48.

10. Determine the number of positive integral value of p for which there exists a triangle with side a, b and c which satisfy $a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = 0$

Answer (5)

Sol. $a^2 + 9b^2 - 6ab + p^2b^2 - 6pbc + 9c^2 = 0$

$(a - 3b)^2 + (pb - 3c)^2 = 0$

$a = 3b \quad pb = 3c$

$\Rightarrow \quad c = \dfrac{pb}{3}$

If a is the largest side

$b + \dfrac{pb}{3} > 3b$

$3 + p > 9$

$p > 6$ …(i)

If c is largest side

$3b + b > \dfrac{pb}{3}$

$p < 12$ …(ii)

so, $p = \{7, 8, 9, 10, 11\}$

$\Rightarrow$ Number of positive integral value of p is 5

11. The positive real numbers a, b, c satisfy:

$$\frac{a}{2b+1} + \frac{2b}{3c+1} + \frac{3c}{a+1} = 1$$

$$\frac{1}{a+1} + \frac{1}{2b+1} + \frac{1}{3c+1} = 2$$

What is the value of $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$?

Answer (12)

Sol. $\dfrac{3c}{a+1} + \dfrac{a}{2b+1} + \dfrac{2b}{3c+1} = 1$ …(i)

$\dfrac{1}{a+1} + \dfrac{1}{2b+1} + \dfrac{1}{3c+1} = 2$ …(ii)

Add (i) and (ii)

$\Rightarrow \quad \dfrac{3c+1}{a+1} + \dfrac{a+1}{2b+1} + \dfrac{2b+1}{3c+1} = 3$

$\Rightarrow \quad A.M. \geq G.M.$

$$\Rightarrow \quad \left(\frac{\frac{3c+1}{a+1}+\frac{a+1}{2b+1}+\frac{2b+1}{3c+1}}{3}\right) \geq (1)^{\frac{1}{3}}$$

$$\Rightarrow \quad \frac{3c+1}{a+1}=1 \text{ and } \frac{2b+1}{3c+1}=1$$

$$\frac{a+1}{2b+1}=1$$

$\Rightarrow \quad (a + 1) = (2b + 1) = (3(+1)) = k$

$$\Rightarrow \quad \frac{1}{k}+\frac{1}{k}+\frac{1}{k}=2 \Rightarrow \frac{3}{k}=2 \Rightarrow k=\frac{3}{2}$$

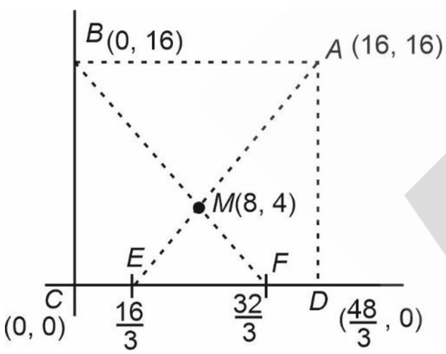
$$\Rightarrow \quad a=\frac{1}{2}, b=\frac{1}{4}, c=\frac{1}{6}$$

$$\Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2+4+6=12$$

12. Consider a square $ABCD$ of side length 16. Let E, F be points on CD such that $CE = EF = FD$. Let the line BF and AE meet in M. The area of ΔMAB is:

Answer (96)

Sol. Eq of line AE: $2y - 3x + 16 = 0$



Eq of line BF: $2y + 3x - 32 = 0$

$\Rightarrow M \equiv (8, 4)$

Area of triangle AMB

$$\left|\frac{1}{2}\begin{vmatrix}0 & 16 & 1\\ 16 & 16 & 1\\ 8 & 4 & 1\end{vmatrix}\right| = \frac{1}{2}\big(-16(8) + 1(64 - 8\times 16)\big)$$

$$= \left|\frac{1}{2}(-16\times 16 + 64)\right|$$

$$= \left|\frac{16}{2}(4 - 16)\right| = 8\times 12$$

$$= 96 \text{ sq.units}$$

13. Three positive integers a, b, c with $a > c$ satisfy the following equations:

$ac + b + c = bc + a + 66$, $a + b + c = 32$.

Find the value of a.

Answer (19)

Sol. $a > c$

$A, b, c\ I^+$

$ac + b + c = bc + a + 66, \quad a + b + c = 32$

$\Rightarrow \quad c(a - b) + (b - a) + c = 66$

$\Rightarrow \quad (a - b)(c - 1) + (c - 1) = 66 - 1$

$\Rightarrow \quad (c - 1)(a - b + 1) = 65 = 1 \times 65 = 65 \times 1$

$= 5 \times 13 = 13 \times 5$

Case-(ii)

$c - 1 = 65, \quad a - b + 1 = 1$

$\Rightarrow \quad c = 66$, but $a + b + c = 32$

$\Rightarrow$ not possible

Case(ii)

$c - 1 = 1, \quad a - b + 1 = 65$

$c = 2, \quad a - b = 64$

$a + b + c = 32 \Rightarrow a + b = 30$

$\Rightarrow 2a = 64 \Rightarrow a = 47, \ b = -17$ not possible

Case(iii)

$c - 1 = 5, \quad a - b + 1 = 13$

$c = 6, \quad a + i = 26$

$a - b = 12$

$\Rightarrow a = 19, \ b = 7$

Case(iv)

$c - 1 = 13, a - b + 1 = 5$

$\Rightarrow c - 1 = 13, \quad a - b + 1 = 5$

$\Rightarrow c = 14, \ a - b = 4$

$a = b = 18$

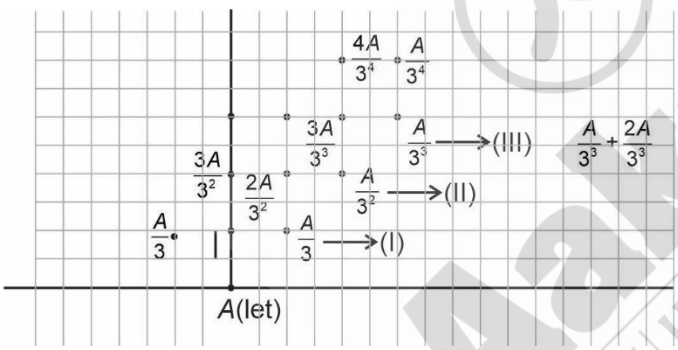
$\Rightarrow a = 11$ but $a < c$

$\Rightarrow$ only $a = 19$

14. Initially, there are 3^{80} particles at the origin (0, 0). At each step the particles are moved to points above the x-axis as follows: if there are n particles at any point (x, y), then $\left[\frac{n}{3}\right]$ of them are moved to $(x+1, y+1)$, $\left[\frac{n}{3}\right]$ are moved to (x, y + 1) and the remaining to (x – 1, y + 1). For example, after the first step, there are 3^{79} particles each at (1, 1), (0, 1) and (–1, 1). After the second step, there are 3^{78} particles each at (–2, 2) and (2, 2), 2 × 3^{78} particles each at (–1, 2) and (1, 2), and 3^{79} particle at (0, 2). After 80 steps, the number of particles at (79, 80) is:

Answer (80)

Sol.



At n^{th} step particle at (n – 1, n) is $\frac{nA}{3^n}$

$\therefore$ At 80^{th} step particles at (79, 80) is $\frac{80A}{3^{80}} = 80$

(where A = 380)

$\therefore$ 80

15. Let X be the set consisting of twenty positive integers n, n + 2,…., n + 38. The smallest value of n for which any three numbers $a, b, c \in X$, not necessarily distinct, form the sides of an acute-angled triangle is:

Answer (92)

Sol. $X = \{n, n+2, ..., n+38\}$

$a, b, c \in X$

For any a, b, c

(i) Triangle should be formed

(ii) Triangle should be acute

$\rightarrow$ only one angle can be obtuse at max

(i) let $a \le b \le c$

$\Rightarrow$ for triangle

$a + b > c$ for all possible combination

$\Rightarrow$ even if a, b are smallest $a = b = n$

$\Rightarrow n + n > n + 38$

$\Rightarrow n > 38 \Rightarrow$ triangle will from

(ii) now using cosine formula largest side longest angle

$\Rightarrow \cos c = \dfrac{a^2 + b^2 - c^2}{2ab} > 0$ for acute Δ

$\Rightarrow \quad a^2 + b^2 - c^2 > 0$ for acute Δ $\forall a, b, c \in x$

$n^2 + (n)^2 - (n+38)^2 > 0$

$\Rightarrow \quad n^2 - 76n - 38^2 > 0$

$\Rightarrow \quad n > 91.74$

$\Rightarrow \quad n = 92$

16. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the relation $4f(3-x) + 3f(x) = x^2$ for any real x. Find the value of $f(27) - f(25)$ to the nearest integer. (Here $\mathbb{R}$ denotes the set of real numbers.)

Answer (8)

Sol. $4f(3-x) + 3f(x) = x^2 \qquad \forall x \in R$

$4f(3-(3-x)) + 3f(3-x) = (3-x)^2$

$= 4f(x) + 3f(3-x) = (x-3)2$

$\Rightarrow \quad 12f(3-x) + 9f(x) = 3x^2$

$12f(3-x) + 16f(x) = 4(x-3)^2$

$\Rightarrow \quad 7f(x) = 4(x-3)^2 = 3x^2$

$= (4(24)^2 - 3.27^2) - (4(22)^2 - 3.25^2)$

$= 4(24^2 - 22^2) + 3(25^2 - 27^2)$

$= 4(46)(2) + 3(52)(-2)$

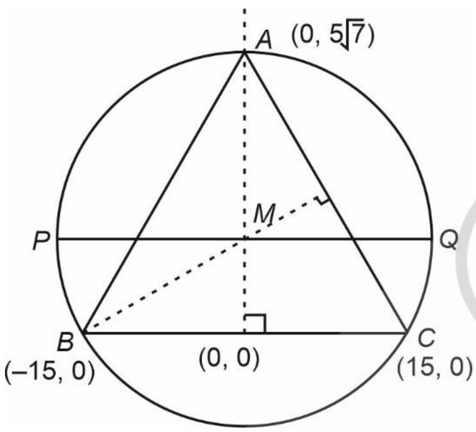
$= 8 \times 46 \times 6 \times 52$

$$\Rightarrow \quad f(27) - f(25) = \frac{1}{7}(46 \times 8 - 6 \times 52) = \left(\frac{56}{7}\right) = 8$$

17. Consider an isosceles triangle ABC with sides $BC = 30$, $CA = AB = 20$. Let D be the foot of the perpendicular from A to BC, and let M be the midpoint of AD. Let PQ be a chord of the circumcircle of triangle ABC, such that M lies on PQ is parallel to BC. The length of PQ is:

Answer (25)

Sol. Eq. of PQ : $y = \dfrac{5\sqrt{7}}{2}$



Perpendicular bisector of AC:

$$\left(y - \frac{5\sqrt{7}}{2}\right) = \frac{-1}{\left(\dfrac{5\sqrt{7} - 0}{0 - 15}\right)}\left(x - \frac{15}{2}\right)$$

$$\Rightarrow \quad \left(y - \frac{5\sqrt{7}}{2}\right) = \frac{15}{5\sqrt{7}}\left(x - \frac{15}{2}\right)$$

Centre $\equiv$ intersection of perpendicular bisector

$$\equiv \left(0, \frac{-5}{\sqrt{7}}\right) \equiv \left(0, \frac{-5\sqrt{7}}{7}\right)$$

$\Rightarrow$ Eq^n of circumcircle

$$(x-0)^2 + \left(y + \frac{5\sqrt{7}}{7}\right)^2 = \left(5\sqrt{7} + \frac{5\sqrt{7}}{7}\right)^2$$

$$x^2 + \left(y + \frac{5}{\sqrt{7}}\right)^2 = 25 \times 7 \times \frac{64}{49} = \frac{25 \times 64}{7}$$

Intersection with $PQ : y = \dfrac{5\sqrt{7}}{2}$

$$x^2 = \frac{25 \times 64}{7} - \left(\frac{5\sqrt{7}}{2} + \frac{5}{\sqrt{7}}\right)^2 = \frac{25 \times 64}{7} - \frac{1}{4 \times 7}\left(45^2\right)$$

$$x^2 = \frac{25 \times 64 \times 4 - 45^2}{28} = \frac{5^2}{2^2}\left(\frac{64 \times 4 - 81}{7}\right) = \frac{5^2}{2^2} \times 25$$

$$\Rightarrow \quad x = \pm\frac{25}{2} \ \Rightarrow \text{distance} = \left|\frac{25}{2} - \left(-\frac{25}{2}\right)\right| = 25$$

18. Let p, q be two-digit numbers neither of which are divisible by 10. Let r be the four-digit number by putting the digits of p followed by the digits of q (in order). As p, q vary, a computer prints r on the screen if $gcd(p, q) = 1$ and $p + q$ divides r. Suppose that the largest number that is printed by the computer is N. Determine the number formed by the last two digits of N (in the same order).

Answer (13)

Sol. $r = 100r + q$

$r + q \mid r = 100\, r + q$

$\Rightarrow \quad r + q \mid 99r$

But gcd($r + q$, r) = 1

$\Rightarrow \quad r + q \mid 99$

$\therefore \quad r + q = 33$ or 99

For N to be maximum $r + q = 99$

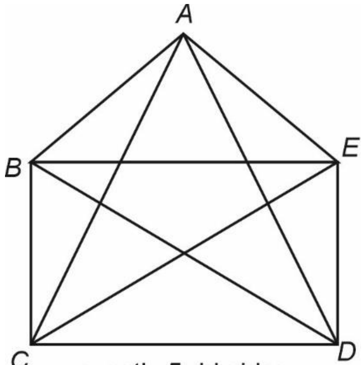
Where $r = 89$, $q = 13$

Answer 13.

19. Consider five points in the plane, with no three of them collinear. Every pair of points among them is joined by a line. In how many ways can we color these lines by red or blue, so that no three of the points form a triangle with lines of the same color.

Answer (12)

Sol. Case-I

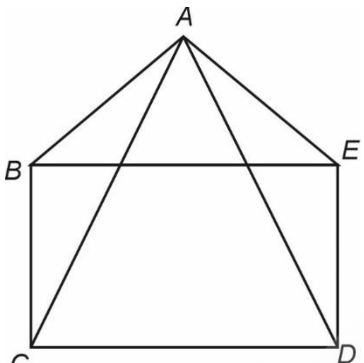


exactly 5 rid sides

2 ways (corresponding to each color)

As color of other sides got fixed

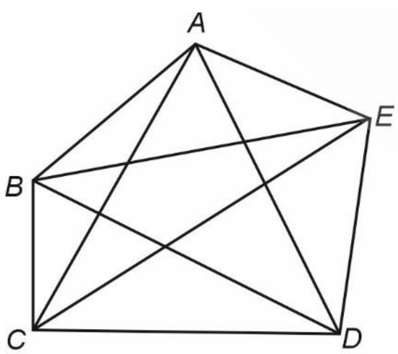
Case-II



exactly 4 rid sides

Not possible $\therefore$ 0 ways as color of other sides got fixed and there will be one triangle which will have all sides rud on block

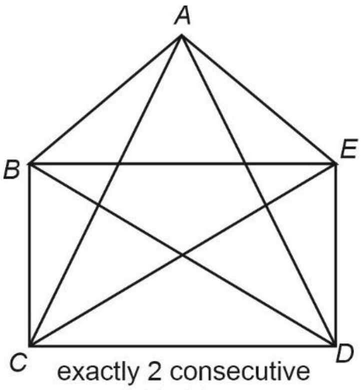
Case-III



exactly 3 rid sides consecutive

$\therefore$ not possible $\therefore$ 0 ways (same reason as above)

Case-IV



exactly 2 consecutive rid sides

Only one way of colouring as color of other sides got fixed

$\therefore$ 2 ways (corresponding to each color for shown figure)

$\therefore$ total number of ways

$= 2 + {}^5c_1 \times 2 = 12$ ways

${}^5c_1$ ways of choosing 2 will active rides

20. On a natural number n you are allowed two operations: (1) multiply n by 2 or (2) subtract 3 from n. For example starting with 8 you can reach 13 as follows: $8 \to 16 \to 13$. You need two steps and you cannot do in less than two steps. Starting from 11, what is the least number of steps required to reach 121?

Answer (10)

Sol. $11 \to$ ……. Some steps $\to$ 121 minimum steps

Notice that for each step the number is a natural number

$\Rightarrow$ Reverse the process

Start from 121 and now condition will be

$$n \begin{cases} \nearrow n/2 \\ \quad \text{or} \\ \searrow n+3 \end{cases}$$

Now, 121 cannot be divide by 2 as result will not be natural number and for minimum step it is better to keep dividing till we get on odd number

$$\Rightarrow \quad 121 \xrightarrow{1} (121+3=124) \xrightarrow{2} \left(\frac{124}{2}=62\right) \xrightarrow{3} \left(\frac{62}{2}=31\right) \xrightarrow{4}$$

$$(31+3=34) \xrightarrow{5} \left(\frac{34}{2}\right) \xrightarrow{6} (17+3=20) \xrightarrow{7} \left(\frac{20}{2}=10\right) \xrightarrow{8}$$

$$\left(\frac{10}{2}=5\right) \xrightarrow{9} (5+3=8) \xrightarrow{10} (8+3)$$ at it doesn't make sense to go half as 1 move to reach 11

$\Rightarrow$ minimum 10 moves

21. An integer n is such that $\left[\dfrac{n}{9}\right]$ is a three digit number with equal digits, and $\left[\dfrac{n-172}{4}\right]$ is a 4 digit number with the digits 2, 0, 2, 4 in some order. What is the remainder when n is divided by 100?

Answer (91)

Sol. Let $\left[\dfrac{n-172}{4}\right] = k,\ k \in I$

$\Rightarrow \quad \dfrac{n-172}{4} \in [k, k+1)$

$n - 172 \in [4k, 4k + 4)$

$n \in [4k + 172, 4k + 176)$

$\dfrac{n}{9} \in \left[\dfrac{4k+172}{9}, \dfrac{4k+176}{9}\right)$

Now, $\left[\dfrac{n}{9}\right] \in \{111, 222, 333, ..., 999\}$

$\Rightarrow \quad \dfrac{4k+176}{9} > \overline{aaa} > \dfrac{4k+172}{9}$, for $a \in \{1, ..., 9\}$

For $a = 9$

$k \in (2203.75, 2204.75)$

$\Rightarrow \quad k = 2204$

$\left[\dfrac{n-172}{4}\right] = 2204 \quad$ ... (i)

And $\left[\dfrac{n}{9}\right] = 999 \quad$ ... (ii)

From (i)

$\dfrac{n-172}{4} \in [2204, 2205)$

$\Rightarrow \quad n \in [8988, 8992)$

From (ii)

$\dfrac{n}{9} \in [999, 1000)$

$\Rightarrow \quad n \in [8991, 9000)$

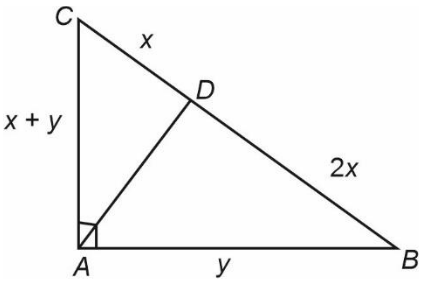
$\Rightarrow \quad n = 8991$

$\Rightarrow \quad 91$

22. In a triangle ABC, $\angle BAC = 90°$. Let D be the point on BC such that $AB + BD = AC + CD$. Suppose $BD : DC = 2 : 1$. If $\frac{AC}{AB} = \frac{m + \sqrt{p}}{n}$, where m, n are relatively prime positive integers and p is a prime number, determine the value of $m + n + p$.

Answer (34)

Sol. $AB + BD = AC + CD$



$y + 2x = AC + x$

$\Rightarrow AC = x + y$

$\Rightarrow \quad (x + y)^2 + y^2 = (3x)^2$

$\Rightarrow \quad x^2 + 2y^2 + 2xy = 9x^2$

$\Rightarrow \quad 8x^2 - 2y^2 - 2xy = 0$

$\Rightarrow \quad 4a^2 - a - 1 = 0$

$\left[\therefore a = \frac{x}{y}\right]$

$\Rightarrow \quad a = \frac{1 \pm \sqrt{1 + 16}}{8}$

$\Rightarrow \quad a = \frac{1 \pm \sqrt{17}}{8}$

$\Rightarrow \quad \frac{AC}{AB} = \frac{x}{y} + 1 = a + 1$

$\Rightarrow \quad \frac{AC}{AB} = \frac{1 + \sqrt{17}}{8} + 1$

$\Rightarrow \quad \frac{AC}{AB} = \frac{9 + \sqrt{17}}{8}$

$\Rightarrow \quad m + n + p = 34$

23. Consider the fourteen numbers, 1^4, 2^4 ,….,14^4. The smallest natural number n such that they leave distinct remainders when divided by n is:

Answer (31)

Sol. $1^4, 2^4 \ldots\ldots, 14^4$

$x^4 \equiv a \pmod{n}$

$y^4 \equiv b \pmod{n}$ such that $a \neq b$ for $x \neq y$ and $x, y \in \{1, 2, ...14\}$

$(x^4 - y^4) \equiv (a - b) \pmod{n}$

$\Rightarrow \quad (x - y)(x + y)(x^2 + y^2) \equiv (a - b) \pmod{n}$

$\Rightarrow \quad n \nmid (x - y)(x + y)(x^2 + y^2)$ …(i)

We have to find minimum n with condition (i)

Clearly, $n > 27$ as $(x + y) \in \{3, \ldots 27\}$

Now $n = 28$, $x = 6$, $y = 8$ works

$n = 29$, $x = 5$, $y = 2$ works

$n = 30$, $x = 8$, $y = 2$ works

for $x = 31$, there are no such x, y,

$31 \mid \underbrace{(x - y)(x + y)(x^2 + y^2)}_{\text{Must be prime factor}}$

$31 \nmid (x^2 + y^2)$ and $31 \nmid (x - y)(x + y)$

$\Rightarrow$ 31 will be the answer

24. Consider the set F of all polynomials whose coefficients are in the set of {0, 1}. Let $q(x) = x^3 + x + 1$. The number of polynomials $p(x)$ in F of degree 14 such that the product $p(x)q(x)$ is also in F is:

Answer (50)

Sol. $p(x)\, q(x) = (x^{14} + \ldots)(x^3 + x + 1)$

$p(x) = x^{14} \rightarrow$ 1 case

$p(x) = x^{14} + x^2$

$\Rightarrow \quad \alpha = 10, 9, 8, \ldots, 10 \rightarrow$ 11 case

$p(x) = x^{14} + x^{\alpha} + x^{\beta}$

$$\left.\begin{array}{ll}\alpha=10, & \beta=6,5,4,3,2,1,0\\ \alpha=9, & \beta=5,4,3,2,1,0\\ \alpha=8, & \beta=4,3,2,1,0\\ \alpha=7, & \beta=3,2,1,0\\ \alpha=6, & \beta=2,1,0\\ \alpha=5, & \beta=1,0\\ \alpha=4, & \beta=0\end{array}\right\}\text{25 cases}$$

$p(x) = x^{14} + x^{\alpha} + x^{\beta} + x^{r}$

$$\left.\begin{array}{lll}\alpha=10, & \beta=6, & r=2,1,0\\ \alpha=10, & \beta=5, & r=1,0\\ \alpha=10, & \beta=4, & r=0\end{array}\right\}\text{6 cases}$$

$$\alpha=9,\ \left.\begin{array}{ll}\beta=5, & r=1,0\\ \beta=4, & r=0\end{array}\right\}\text{3 cases}$$

$$\alpha=8,\ \beta=4,\ r=0\}\ \text{1 cases}$$

Hence, total case = 1 + 11 + 28 + 6 + 3 + 1

= 50 cases.

25. A finite set M of positive integers consists of distinct perfect squares and the number 92. The average of the numbers in M is 85. If we remove 92 from M, the average drops to 84. If N^2 is the largest possible square in M, what is the value of N?

Answer (22)

Sol. $\dfrac{a_1^2+a_2^2+a_3^2+...+a_0^2+92}{n+1}=85$ ...(i)

If we remove 92, then

$\dfrac{a_1^2+a_2^2+a_3^2+...+a_n^2}{n}=84$ ...(ii)

From (i) and (2)

$84n + 92 \times 85n + 85$

$\Rightarrow n = 7$

Now, $a_1^2+a_2^2+a_3^2+...+a_6^2+a_7^2=84\times 7=588$

If $a_1=1, a_2=2, a_3=3, a_4=4, a_5=5, a_6=6$

Then $1^2+2^2+3^2+4^2+5^2+6^2+a_7^2=588$

$\Rightarrow \quad a_7^2=588-91$

$\Rightarrow \quad a_7^2 = 497$ (not possible)

If $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 5, a_6 = 7$

Then $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 7^2 + a_7^2 = 588$

$\Rightarrow \quad a_7^2 = 484$

$\Rightarrow \quad a_7 = 22$

$\Rightarrow \quad N = 22$

26. The sum of $[x]$ for all real numbers x satisfying the equation $16 + 15x + 15x^2 = [x]^3$ is:

Answer (33)

Sol. $[x]^3 = 15x^2 + 15x + 16$

$[x] \in (x - 1, x]$

$[x]^3 \in ((x - 1)^3, x^3]$

$\Rightarrow \quad 15x^2 + 15x + 16 \in ((x - 1)^3, x^3]$

$x^3 \geq 15x^2 + 15x + 16$

$\Rightarrow \quad x^3 - 15x^2 - 15x - 16 \geq 0$

$(x - 16)(x^2 + x + 1) \geq 0$

$\Rightarrow \quad x \geq 16$

Also, $15x^2 + 15x + 16 > x^3 - 3x^2 + 3x - 1$

$\Rightarrow \quad x^3 - 18x^2 - 12x - 17 < 0$

Let $f(x) = x^3 - 18x^2 - 12x - 17$

$f'(x) = 3x^2 - 36x - 12$

$= 3(x^2 - 12x - 4) \begin{smallmatrix} \nearrow \alpha \\ \searrow \beta \end{smallmatrix} \qquad \alpha < \beta$

$f(\alpha) < 0$

$\Rightarrow \quad f(x)$ has only one real root

$\Rightarrow \quad f(18) = -233$

$f(19) = 116$

$\Rightarrow \quad f(\alpha) = 0$

$\Rightarrow \quad \alpha \in (18, 19)$

For this equation

$x \in [16, 19)$

$\Rightarrow \quad [x] = 16, 17, 18$

(i) $[x] = 16$

$\Rightarrow \quad 15x^2 + 15x + 16 = 16^3$

$(x - 16)(x + 17) = 0$

$\Rightarrow \quad x = 16$ satisfies

(ii) $[x] = 17$

$\Rightarrow \quad 15x^2 + 15x + 16 = 17^3$

$$\Rightarrow \quad x = \frac{-15 \pm \sqrt{15^2 - 4.16.15}}{30}$$

$\Rightarrow \quad [x] = 17$ satisfies

(iii) $[x] = 18$

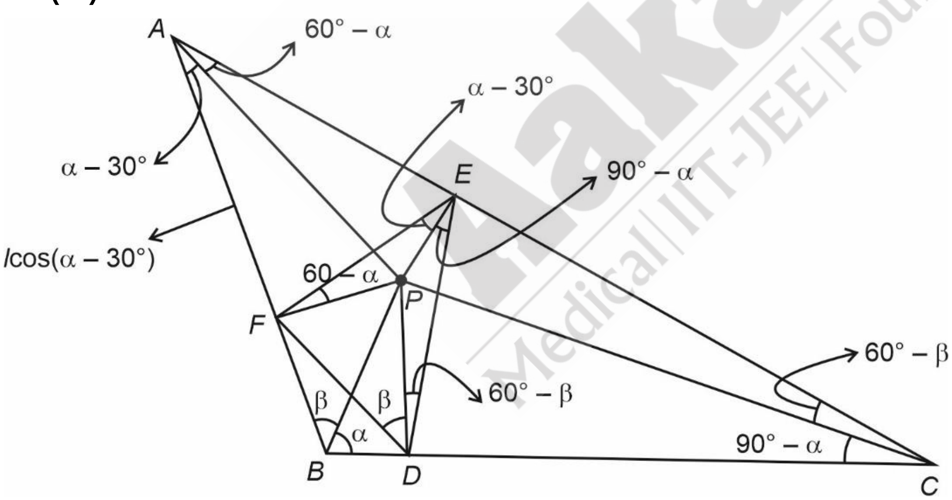
$\Rightarrow \quad x > 19$ and $x < -20$

$\Rightarrow$ No received value of x

$\Rightarrow$ Sum of $[x] = 16 + 17 = 33$.

27. In a triangle ABC, a point P in the interior of ABC is such that $\angle BPC - \angle BAC = \angle CPA - \angle CBA = \angle APB - \angle ACB$. Suppose $\angle BPC = 30°$ and $AP = 12$. Let D, E, F be the feet of perpendiculars form P on to BC, CA, AB respectively. If $m\sqrt{n}$ in the area of the triangle DEF where m, n are integers with n prime, then what is the value of the product mn?

Answer (27)



Sol.

$$EF^2 = l^2\left[\cos^2(\alpha - 30°) + \cos^2(60° - \alpha) - \frac{2\sqrt{3}}{2}\cos(\alpha - 30°)\cos(\alpha - 60°)\right]$$

$$= l^2\left[\left(\frac{\sqrt{3}}{2}\cos\alpha+\frac{1}{2}\sin\alpha\right)^2+\left(\frac{1}{2}\cos\alpha+\frac{\sqrt{3}}{2}\sin\alpha\right)^2-\frac{\sqrt{3}}{2}\left(\cos(2\alpha-90^\circ)+\frac{\sqrt{3}}{2}\right)\right]$$

$$= l^2\left[\cos^2\alpha+\sin^2\alpha+\frac{\sqrt{3}}{2}\sin\alpha\cos\alpha-\frac{3}{4}\right]$$

$EF^2=\frac{l^2}{4}$, area = $\frac{\sqrt{3}}{4}.EF^2=\frac{\sqrt{3}}{4}\frac{144}{4}=9\sqrt{3}$

28. Find the largest positive integer n < 30 such that $\frac{1}{2}\left(n^8+3n^4-4\right)$ is not divisible by the square of any prime number.

Answer (20)

Sol. Let $f(n)=\frac{1}{2}\left(n^8+3n^4-4\right)$

$$= \frac{1}{2}(n-1)(n+1)\left(n^2+1\right)\left(\left(n^2+1\right)+1\right)\left((n-1)^2+1\right)$$

For $n = 2k + 1$

$\Rightarrow \quad f(2k+1)=\frac{1}{2}(2k)2(k+1)\left(4k^2+4k+2\right)...$

Clearly 4 | $f(2k + 1)$

$\Rightarrow$ Only even cases

$\Rightarrow \quad n = 28,\ f(28)=\frac{1}{2}\times 27\times 29\times\left(28^2+1\right)\left(27^2+1\right)\left(29^2+1\right)$

Clearly 3^2 | $f(28)$

$n = 26,\ f(26)=\frac{1}{2}\times 25\times 27\times\left(25^2+1\right)\left(26^2+1\right)\left(27^2+1\right)$

Clearly 5^2 | $f(26)$

$n = 24,\ \frac{1}{2}\cdot 23\times 25\cdot\left(24^2+1\right)\left(25^2+1\right)\left(26^2+1\right)$

Again 5^2 | $f(26)$

$n = 22,\ f(22)=\frac{1}{2}\times 21\times 23\times\left(22^2+1\right)\left(21^2+1\right)\left(23^2+1\right)$

5 | 22^2 + 1, 5 | 23^2 + 1

$\Rightarrow$ 52 | $f(22)$

$n=20,\ f(20)=\frac{1}{2}(19\times 21)\left(20^2+1\right)\left(21^2+1\right)\left(19^2+1\right)$

Not divisible by any prime square

$\Rightarrow \quad n = 20$

29. Let $n = 2^{19}3^{12}$. Let M denote the number of positive divisors of n^2 which are less than n but would not divide n. What is the number formed by taking the last two digits of M (in the same order)?

Answer (28)

Sol. $n^2 = 2^{38} \cdot 3^{24}$

$\therefore$ Required number of divisors

$$= \frac{(38+1)(24+1)+1}{2} - 20 \times 13$$

$= 228$

$\therefore$ 28

30. Let ABC be a right-angled triangle with $\angle B = 90°$. Let the length of the altitude BD be equal to 12. What is the minimum possible length of AC, given that AC and the perimeter of triangle ABC are integers?

Answer (25)

Sol. $ac = 12\,b$

Let $a + c = l$ $\quad l \in$ integer

$a^2 + c^2 = l^2 - 24b = b^2$

$l^2 = lx = 12b \left\langle \begin{matrix} a \\ c \end{matrix} \right.$

$D = \geq 0$

$l^2 - 48b \geq 0$

$b^2 + 24\,l \geq 0 \quad b \geq 24$

Let $b = k - 12 \quad k \geq 36$

$l^2 = k^2 - 144$

$144 = (k - l)(k + l)$

For $k = 37$, $l = 35$

$l = 25$ is minimum value

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