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Time: 3 hrs.   **Mock Test Paper for Class-XII**   **Max. Marks: 75**

MATHEMATICS
Paper - II(B)

Answers & Solutions

1. Given equation of the circle is $\sqrt{1-m^2}\left(x^2+y^2\right)-2cx-2mcy=0$

$$\Rightarrow x^2+y^2-\frac{2c}{\sqrt{1+m^2}}x-\frac{2mc}{\sqrt{1+m^2}}y=0 \underline{\hspace{2cm}}(1)$$

By comparing eq. (1) with $x^2+y^2+2gx+2fy+c=0$, we get

$$2g=-\frac{2c}{\sqrt{1+m^2}}\Rightarrow g=-\frac{c}{\sqrt{1+m^2}};2f=-\frac{2mc}{\sqrt{1+m^2}}\Rightarrow f=-\frac{mc}{\sqrt{1+m^2}};c=0.$$

$\therefore$ Centre, $C=\left(-g,-f\right)=\left(\dfrac{c}{1+m^2},\dfrac{mc}{\sqrt{1+m^2}}\right)$;

Radius, $r=\sqrt{g^2+f^2-c}=\sqrt{\dfrac{c^2}{\sqrt{1+m^2}}+\dfrac{m^2c^2}{\sqrt{1+m^2}}=0}=\sqrt{\dfrac{c^2\left(1+m^2\right)}{1+m^2}}=c$ units.

2. Given equation of the circle is $x^2+y^2-5x+4y+k=0 \underline{\hspace{1cm}}(1)$

By comparting equation (1) with $x^2+y^2+2gx+2fy+c=0$, we get $2g=-5\Rightarrow g=\dfrac{-5}{2};2f=4\Rightarrow f=2;c=k$

Let the given point is $P\left(x_1,y_1\right)=\left(2,5\right)$ and given length of the tangent is $\sqrt{S_{11}}=\sqrt{37}$

$\therefore$ The length of the tangent $=\sqrt{S_{11}}=\sqrt{2^2+5^2-5\left(2\right)+4\left(5\right)+k}$

[$\because$ The length of tangent from $P\left(x_1,y_1\right)$ to the circle $S=x^2+y^2+2gx+2fy+c=0$

$\sqrt{S_{11}}=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c=0}]\Rightarrow 4+25-10+20+k=37\Rightarrow 39+k=37\Rightarrow k-37-39=-2.$

3. Given equation of the circles are $x^2 + y^2 - 12x - 6y + 41 = 0$ and $x^2 + y^2 + 4x + 6y - 59 = 0$

Here, $g = -6, f = -3, c = 41$; $g' = 2, f' = 3, c' = -59$.

If 'θ' is the angle between the circles, then

$$\cos\theta = \frac{41 - 59 - 2(-6)(2) - 2(-3)(3)}{2\sqrt{36 + 9 - 41}\sqrt{4 + 9 + 59}} = \frac{41 - 59 + 24 + 19}{2\sqrt{4}\sqrt{72}} = \frac{24}{2.2.6\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

[$\because$ If 'θ' is the angle between two intersecting circles $S' = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$S'' = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ then $\cos\theta = \dfrac{c_1 + c_2 - 2(2g_1g_2 + 2f_1f_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}$.

4. Given equation of the parabola is $y^2 = 8x \Rightarrow 4x = 8 \Rightarrow a = 2$

Let $P(x_1, y_1)$ be any point on the parabola $y^2 = 8x$

Given focal distance $= 10 \Rightarrow x_1 + a = 10$

[$\because$ Focal distance of a point $P(x_1, y_1)$ on the parabola $y^2 = 4ax$ is $|x_1 + a|$]

$\Rightarrow x_1 + 2 = 10 \Rightarrow x_1 = 8$

$\therefore y_1^2 = 8x_1 \Rightarrow y_1^2 = 8(8) \Rightarrow y_1 = \pm 8$.

$\therefore$ Required points are $(8,8)$ and $(8,-8)$

5. Given eccentricity (e) $= 5/4$. Since if e and e_1 the eccentricity of hyperbola and its conjugate hyperbola, then

$$\frac{1}{e^2} + \frac{1}{e_1^2} = 1.$$

$$\Rightarrow \frac{16}{25} + \frac{1}{e_1^2} = 1 \Rightarrow \frac{1}{e_1^2} = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow e_1^2 = \frac{25}{9} \Rightarrow e_1 = \frac{5}{3}.$$

$\therefore$ Eccentricity of conjugate hyperbola is $e_1 = \dfrac{5}{3}$

6. Put $x^9 = t \Rightarrow 9x^8 dx = dt \Rightarrow x^8 dx = \dfrac{dt}{9}$ $\qquad \left[\because \dfrac{d}{dx}(x^n) = nx^{n-1} \right]$

Now, $\displaystyle\int \frac{x^8}{1 + x^{18}} dx = \int \frac{x^8}{1 + (x^9)^2} dx = \frac{1}{9}\int \frac{dt}{1 + t^2} = \frac{1}{9}\tan^{-1} t + C = \frac{1}{9}\tan^{-1}(x^9) + C.$

$$\left[\because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + c \right]$$

7. Put $xe^x = t \Rightarrow e^x(1 + x)dx = dt$ $\qquad \left[\because (uv)' = uv' + vu', \dfrac{d}{dx}(e^x) = e^x, \dfrac{d}{dx}(x) = 1 \right]$

Now, $\displaystyle\int \frac{e^x(1 + x)}{\cos^2(xe^x)} dx = \int \frac{dt}{\cos^2 t} dx = \int \sec^2 t\, dt = \tan t + C = \tan(xe^x) + C.$

$$\left[\because \int \sec^2 x\, dx = \tan x + C \right]$$

8. We have

$|1-x| = 1-x$ for $1-x \geq 0 \Rightarrow x-1 \leq 0 \Rightarrow x \leq 1$

And $|1-x| = -(1-x) = x-1$ for $1-x < 0 \Rightarrow x-1 > 0 \Rightarrow x > 1$

Now, $\int\limits_0^2 |1-x| dx = \int\limits_0^1 (1-x) dx + \int\limits_1^2 (x-1) dx = \left(x - \frac{x^2}{2}\right)_0^1 = \left(1 - \frac{1}{2} - 0\right) + \left(\left(\frac{4}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right) = 0$

9. Given $\int\limits_0^{\pi/2} \sin^4 x . \cos^5 x\, dx$, here $m = 4, n = 5 (\text{odd})$

$\int\limits_0^{\pi/2} \sin^4 x . \cos^5 x\, dx = \frac{5-1}{4+5} . \frac{5-3}{4+5-2} . \frac{1}{4+1} = \frac{4}{9} . \frac{2}{7} . \frac{1}{5} = \frac{8}{315}$

$\left[\because \int\limits_0^{\pi/2} \sin^m x . \cos^n x\, dx = \frac{n-1}{m+n} \frac{n-3}{m+n-2} ...... \frac{1}{n+1}, \text{if m is even, n is odd}\right]$

10. Given differential equation is $\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\right]^6 = 6y \Rightarrow \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = (6y)^{5/6}$

Here, the highest derivative is $\left(\frac{d^2y}{dx^2}\right)^1$.

$\therefore$ for the given differential equation, order $=2$ and degree$=$exponent $=1$.

11. Let $P = (x_1, y_1)$

$S = x^2 + y^2 - 4x - 6y - 12$

$S^1 = x^2 + y^2 + 6x + 18y + 26$

$PA = \sqrt{S_{11}} = \sqrt{x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12}$

$PB = \sqrt{S_{11}^1} = \sqrt{x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26}$

Given $\frac{PA}{PB} = \frac{2}{3} \Rightarrow 3PA = 2PB$

$\Rightarrow 9PA^2 = 4PB^2$

$9\left(x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12\right) = \left(x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26\right)$

$5x_1^2 + 5y_1^2 - 60x_1 - 13y_1 - 212 = 0$

Locus of P is $5x^2 + 5y^2 - 60x - 136y - 212 = 0$

12. The equation of the given circle is $S \equiv x^2 + y^2 - 2x + 4x - 8 = 0$ and the line is $L \equiv x + y - 3 = 0$

The equation of any circle passing through the point of intersection of $S = 0, L = 0$ is $S + \lambda L = 0$

$\Rightarrow \left(x^2 + y^2 - 2x + 4y - 8\right) + \lambda(x + y - 3) = 0 \Rightarrow x^2 + y^2 - (2-\lambda)x + (4+\lambda)y - (3\lambda + 8) = 0$ _____(1)

$\therefore$ Centre $C = (-g, -f) = \left(\dfrac{2-\lambda}{2}, -\dfrac{4+\lambda}{2} \right)$

The line $L = 0$ become diameter of the circle, if the above centre lies on $L = 0$

$\Rightarrow \left(\dfrac{2-\lambda}{2} \right) - \left(\dfrac{4+\lambda}{2} \right) - 3 = 0 \Rightarrow 2 - \lambda - 4 - \lambda - 6 = 0 \Rightarrow 2\lambda = -8 \Rightarrow \lambda = -4$

$\therefore$ By substituting $\lambda = -4$ in equation (1), we get the equation of the circle as

$x^2 + y^2 - (2+4)x + (4-4)y - (-12+8) = 0 \Rightarrow x^2 + y^2 - 6x + 4 = 0$

13. Given equation of the ellipse is $9x^2 + 16y^2 = 144 \Rightarrow \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1 \Rightarrow a^2 = 16, b^2 = 9$ (a>b).

Eccentricity is $e = \sqrt{\dfrac{a^2 - b^2}{a}} = \dfrac{\sqrt{16-9}}{4} = \dfrac{\sqrt{7}}{4}$, Centre =(0,0).

$\text{Foci} = \left(\pm 4 \left(\dfrac{\sqrt{7}}{4} \right), 0 \right) = (\pm\sqrt{7}, 0)$, Length of major axis =2x=2(4)=8,

Equations of the directrices are $x = \pm \dfrac{a}{e} \Rightarrow x = \pm \dfrac{4}{(\sqrt{7}/4)} \Rightarrow x = \pm \dfrac{16}{\sqrt{7}} \Rightarrow \sqrt{7}x = \pm 16$.

14. Let $L = \left(ae, \dfrac{b^2}{a} \right)$ be one end of latus rectum of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Then equation of the normal at 'L' is $\dfrac{a^2 x}{x_1} - \dfrac{b^2 y}{y_1} = a^2 - b^2$

$\Rightarrow \dfrac{a^2 x}{ae} - \dfrac{b^2 y}{(b^2/a)} = a^2 - b^2 \Rightarrow \dfrac{ax}{e} - ay = a^2 - b^2$ _____(1)

From given (1) passes through $B' = (0, -b)$

$\Rightarrow \dfrac{a(0)}{e} - a(-b) = a^2 - b^2 \Rightarrow ab = a^2 - b^2 \Rightarrow ab = a^2 - a^2(1 - e^2) \Rightarrow ab = a^2 e^2$

$\Rightarrow e^2 = \dfrac{b}{a} \Rightarrow e^4 = \dfrac{b^2}{a^2} = \dfrac{a^2(1-e^2)}{a^2} \Rightarrow e^4 = 1 - e^2$.

15. Given equation of the hyperbola is

$3x^2 - 4y^2 = 12 \Rightarrow \dfrac{3x^2}{12} - \dfrac{4y^2}{12} = \dfrac{12}{12} \Rightarrow \dfrac{x^4}{4} - \dfrac{y^2}{3} = 1 \Rightarrow a = 2, b = \sqrt{3}$.

Given equation of the line is $y = x - 7$ _____(1)

Slope of line=1

(i) **Parallel Tangents:** Slope of parallel tangents =m=Slope of (1) $\Rightarrow m = 1$

Equation of parallel tangents are $y = mx \pm \sqrt{a^2 m^2 - b^2}$

$\Rightarrow y = (1)x \pm \sqrt{4(1) - 3} \Rightarrow y = x \pm \sqrt{4-3} \Rightarrow y = x \pm 1 \Rightarrow x - y \pm 1 = 0$

(ii)   **Perpendicular Tangents:** Slope of perpendicular tangents $= m = \dfrac{-1}{\text{Slope of } (1)}$.

Since, slope of perpendicular line $= \dfrac{-1}{m}$.

Equation of perpendicular tangents are $y = mx \pm \sqrt{a^2 m^2 - b^2}$

$\Rightarrow y = (-1)x \pm \sqrt{4(1) - 3} \Rightarrow y = -x \pm 1 \Rightarrow x + y \pm 1 = 0$

16. Put $\tan\dfrac{x}{2} = t$, then $\cos x = \dfrac{1 - t^2}{1 + t^2}$ and $dx = \dfrac{2dt}{1 + t^2}$. Also $x = 0 \Rightarrow t = 0, x = \dfrac{\pi}{2} \Rightarrow \tan\dfrac{\pi}{2} = 1$

Now, $\displaystyle\int_0^{\pi/2} \dfrac{dx}{4 + 5\cos x} = \int_0^1 \dfrac{\left[(2dt)/(1 + t^2)\right]}{4 + 5\left[(1 - t^2)/(1 + t^2)\right]} = \int_0^1 \dfrac{2t\,dt}{4\left[(1 + t^2) + 5(1 - t^2)\right]} = \int_0^1 \dfrac{2t\,dt}{4 + 4t^2 + 5 - 5t^2}$

$= \displaystyle\int_0^1 \dfrac{2dt}{9 - t^2} = 2\int_0^1 \dfrac{dt}{3^2 - t^2} = 2 \cdot \dfrac{1}{2(3)} \log\left[\dfrac{3 + t}{3 - t}\right]_0^1$ $\left[\because \displaystyle\int \dfrac{1}{a^2 - x^2}dx = \dfrac{1}{2a}\log\left|\dfrac{a + x}{a - x}\right| + c\right]$

$= \dfrac{1}{3}\left[\log\dfrac{4}{2} - \log\left(\dfrac{3}{3}\right)\right] = \dfrac{1}{3}[\log 2 - \log 1] = (1/3)\log 2$ $\left[\because \log a - \log b = \log\left(\dfrac{a}{b}\right), \log 1 = 0\right]$

17. Given differential equation is $(1 + y^2)dx = \left(\text{Tan}^{-1} y - x\right)dy$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{\tan^{-1} y - x}{1 + y^2} \Rightarrow \dfrac{dx}{dy} + \dfrac{x}{1 + y^2} = \dfrac{\tan^{-1} y}{1 + y^2}.$

Which is in the form of $\dfrac{dx}{dy} + P(y).x = Q(y).x = Q(y)$, where $P(y) = \dfrac{1}{1 + y^2}$ & $Q(y) = \dfrac{\tan^{-1} y}{1 + y^2}$.

Hence it is a Linear differential equation in 'x'.

Now, I.F $= e^{\int P(y)dy} = e^{\int \frac{1}{1 + y^2}dy} = e^{\tan^{-1} y}.$ $\left[\because \displaystyle\int \dfrac{1}{1 + x^2}dx = \text{Tan}^{-1} x + c\right]$

18. Let the equation of the required circle is $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ _____(1)

Given that, centre of the circle $C(-g, -f)$ lies on the x-axis $\Rightarrow t = 0 \Rightarrow$ Centre$= C(-g, 0)$.

Let the circle passes through the points $(-2, 3)$ & $(4, 5)$.

Since, eq (1) passes through $(-2, 3) \Rightarrow 4 + 9 - 4g + c = 0 \Rightarrow -4g + c + 13 = 0$ ____(2)

Since, eq (3) $-$eq. (2) $\Rightarrow 12g + 28 = 0 \Rightarrow g = -7/3$

By substituting the value of 'g' in equation (2) we get $-4(-7/3) + c + 13 = 0 \Rightarrow c = \dfrac{-28 - 39}{3} = \dfrac{-67}{3}$.

By substituting the value of g, f and c in equation (1) we get

$x^2 + y^2 + 2\left(\dfrac{-7}{3}\right)x + 2(0) - \dfrac{67}{3} = 0 \Rightarrow 3(x^2 + y^2) - 14x - 67 = 0$

$\Rightarrow 3(x^2 + y^2) - 14x - 67 = 0$

Which is required equation of the circle

19. For the circle $x^2 + y^2 - 4x - 10y + 28 = 0$, Centre $C_1 = (-g, -f) = (2, 5)$,

Radius $r_1 = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 25 - 28} = 1$

For the circle x $x^2 + y^2 + 4x + 6y + 4 = 0$, Centre $C_2 = (-g, -f) = (-2, 3)$

Radius $r_2 = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 9 - 4} = 3$. Now $C_1C_2 = \sqrt{(-2, -2)^2 + (3 - 5)^2} = [\sqrt{16 + 4} = \sqrt{20}$

[$\because$ The distance between the two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Also, $r_1 + r_2 = 1 + 3 = 4$

$\therefore C_1C_2 > r_1r_2$. Hence the internal centre of similitude I divides $\overline{C_1C_2}$ in the ratio 1:3 internally, when $C_1 = (2, 5)$, $C_2 = (-2, 3)$.

$\Rightarrow I = \left(\dfrac{r_1x_2 + r_2x_1}{r_1 + r_2}, \dfrac{r_1y_1}{r_1 + r_2} \right) = \left(\dfrac{1(-2) + 3(2)}{1 + 3}, \dfrac{1(3) + 3(5)}{1 + 3} \right) = \left(\dfrac{-2 + 6}{4}, \dfrac{3 + 15}{4} \right) = \left(1, \dfrac{9}{2} \right)$

Let 'm' be the slope of transverse common tangent

$\therefore$ Equation of traverse common tangent is $(y - y_1) = m(x - x_1)$

$\Rightarrow y\left(y - \dfrac{9}{2} \right) = m(x - 1) \Rightarrow 2y - 9 = 2mx - 2m \Rightarrow 2mx - 2y - 2m + 9 = 0$ _____(1)

(1) is tangent to the circles S=0.

$\therefore$ 'p' is perpendicular distance from c(2,5) to the line 2mx-2y-m+9=0

$\Rightarrow \dfrac{|4m - 10 - 2m + 9|}{\sqrt{4m^2 + 4}} = 1 \Rightarrow (2m - 1) = \sqrt{4m^2} + 4$.

[$\because$ The perpendicular distance (p) from the point $((x_1, y_1)$ to the line $ax + by + c = 0$ is $\dfrac{|ax_1 + by_1 + c = 0|}{\sqrt{a^2 + b^2}}$]

By squaring on both sides, we get $4m^2 - 4m + 1 = 4m^2 + 4$

$\Rightarrow 1 - 4 = 4m \Rightarrow 4m = -3$ (or) $m = \dfrac{1}{2}$(undefined) $\Rightarrow m = \dfrac{-3}{4}(or)m = \dfrac{1}{0}$

Case-I: If $m = \dfrac{-3}{4}$, then the (1) is $\left(y - \dfrac{9}{2} \right) = \dfrac{-3}{4}(x - 1) \Rightarrow 4y - 18 = -3x + 3 \Rightarrow 3x + 4y - 21 = 0$.

Case-II: If $m = \dfrac{1}{0}$, then the equation (1) is $\left(y - \dfrac{9}{2} \right) = \dfrac{1}{0}(x - 1) \Rightarrow 0 = x - 1 \Rightarrow x - 1 = 0$

$\therefore$ Equations of transverse common tangents are $3x + 4y - 21 = 0$ and $x - 1 = 0$

20. Given equation of the parabola is $y^2 = 8ax$ and equation of the circle is $x^2 + y^2 = 2a^2$

$\therefore$ Centre $C = (0, 0)$ and radius $r = \sqrt{2}a$

Let, the equation of the any tangent to the parabola $y^2 = 8ax$ is $y = mx + \dfrac{2a}{m}$

$\Rightarrow my = m^2x + 2a \Rightarrow m^2x - my + 2a = 0$ _____(1)

Equation (1) Become tangent to the circle $x^2 + y^2 = 2a^2$ if radius of the circle is equal to perpendicular distance from (1) to the centre of the circle .i.e r=d

$$\Rightarrow \sqrt{2a} = \frac{|2a|}{\sqrt{\left(m^2\right)^2 + m^2}} \Rightarrow 2a^2 = \frac{4a^2}{m^4 + m^2}$$

$$\Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow m^4 + 2m^2 - m^2 - 2 = 0 \Rightarrow \left(m^2 + 2\right)\left(m^2 - 1\right) = 0 \Rightarrow m^2 = 1 \Rightarrow m \pm 1$$

$\therefore$ Required equation of common tangents $y = \left(\pm 1\right)x + \dfrac{2a}{\pm 1} \Rightarrow y = \pm\left(x + 2a\right)$.

21. $2\sin x + 3\cos x + 4 = A\dfrac{d}{dx}\left(3\sin x + 4\cos x + 5\right) + B\left(3\sin x + 4\cos x + 5\right) + C$

$2\sin x + 3\cos x + 4 = A\left(3\cos x - 4\sin x\right) + B\left(3\sin x + 4\cos x + 5\right) + C$ _____(1)

Equating co-efficient of $\sin x$ on both sides $\Rightarrow -4A + 3B = 2$ _____(1)

Equating co-efficient of $\cos x$ on both sides $\Rightarrow 3A + 4B = 3$ _____(2)

Equating co-efficient of constant terms on both sides $\Rightarrow 5b + C = 4$ _____(3)

$3 \times (1) \Rightarrow -12A + 9B = 6$ _____(4);

$4 \times (2) \Rightarrow 12A + 16B = 12$ _____(5)

$(4) + (5) \Rightarrow 25B = 18 \Rightarrow B = \dfrac{18}{25}$.

From (2), $3A = 3 - 4\left(\dfrac{18}{25}\right) = \dfrac{75 - 72}{25} = \dfrac{3}{25} \Rightarrow 3A = \dfrac{3}{25} \Rightarrow A = \dfrac{1}{25}$.

From (3) $\Rightarrow 5B + C = 4 \Rightarrow 5\left(\dfrac{18}{25}\right) + C = 4 \Rightarrow C = 4 - \dfrac{18}{5} \Rightarrow C = \dfrac{20 - 18}{5} = \dfrac{2}{5}$.

By substituting A, B and C values in (I) , We get

$2\sin x + 3\cos x + 4 = \dfrac{1}{25}\left(3\cos x - 4\sin x\right) + \dfrac{18}{25}\left(3\sin x + 4\cos x + 5\right) + \dfrac{2}{5}$

Now, $\displaystyle\int \dfrac{2\sin x - 3\cos x + 4}{3\sin x + 4\cos x + 5}dx =$

$\dfrac{1}{25}\displaystyle\int \dfrac{3\cos x - 4\sin x + 4}{3\sin x + 4\cos x + 5}dx = \dfrac{18}{25}\displaystyle\int \dfrac{3\sin x + 4\cos x + 5}{3\sin x + 4\cos x + 5}dx + \dfrac{2}{5}\displaystyle\int \dfrac{dx}{3\sin x + 4\cos x + 5} =$

$= \dfrac{1}{2}\log\left|3\sin x + 4\cos x + 5\right| + \dfrac{18}{25}\displaystyle\int 1dx + \dfrac{2}{5}\displaystyle\int \dfrac{dx}{3\sin x + 4\cos x + 5} + C$ $\left[\because \displaystyle\int \dfrac{f^1 x}{f(x)}dx = \log\left|f(x)\right| + C\right]$

Put $\tan\left(\dfrac{x}{2}\right) = t \Rightarrow dx = \dfrac{2dt}{1 + t^2}$

$\mathrm{Sin}\,x = \dfrac{2t}{1 + t^2}; \cos x = \dfrac{1 - t^2}{1 + t^2}$ $\left[\because \sin x = \dfrac{2\tan\dfrac{x}{2}}{1 + \tan^2\dfrac{x}{2}}\right]\left[\because \cos x = \dfrac{1 - \tan^2\dfrac{x}{2}}{1 + \tan^2\dfrac{x}{2}}\right]$

$$= \frac{1}{25}\log|3\sin x + 4\cos x + 5| + \frac{18}{25}x + \frac{2}{5}\int \frac{\frac{2dt}{1+t^2}}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) + 5} + C$$

$$= \frac{1}{25}\log|3\sin x + 4\cos x + 5| + \frac{18}{25}x + \frac{2}{5}\int \frac{2dt}{t^2 + 6t + 9} + C$$

$$= \frac{1}{25}\log|3\sin x + 4\cos x + 5| + \frac{18}{25}x + \frac{2}{5}\int \frac{2dt}{(1+3)^2} + C$$

$$= \frac{1}{25}\log|3\sin x + 4\cos x + 5| + \frac{18}{25}x + \frac{4}{5}\int \frac{dt}{(1+3)^2} + C$$

$$= \frac{1}{25}\log|3\sin x + 4\cos x + 5| + \frac{18}{25}x + \frac{4}{5}\left(\frac{-1}{t+3}\right) + C$$

22. Let $I_n = \int \tan^n x\, dx \Rightarrow I_n = \int \tan^{n-2} x \tan^2 x\, dx$

$$= \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^{n-2} x \sec^2 x - \int \tan^{n-2} dx = \frac{(\tan x)^{n-2+1}}{n-2+1} - I_{n-2} \Rightarrow I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

If 'n' is even, after successive reduction we get, $I_0 = \int \tan^0 x\, dx = \int dx = x + c$

If 'n' is odd, after successive reduction we get, $I_1 = \log|\sec x| + c$

$$I_6 = \frac{\tan^{6-1} x}{6-1} - I_{6-2} = \frac{1}{5}\tan x - \left[\frac{\tan^{4-1} x}{4-1} - I_{4-2}\right] = \frac{1}{5}\tan^5 x - \frac{\tan^3 x}{3} + I_2$$

$$= \frac{1}{5}\tan^5 x - \frac{\tan^3 x}{3} + \left[\frac{\tan^{2-1} x}{2-1} - I_{2-1}\right] = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan x}{1} + I_0(-1) = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$$

23. $I \displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\, dx$ _____(1)

$$\Rightarrow I = \int_0^{\pi/2} \frac{[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]}\, dx = \int_0^{\pi/2} \frac{[(\pi/2) - x]}{\sin x + \cos x}\, dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{(\pi/2)dx}{\sin x + \cos x}\, dx - \int_0^{\pi/2} \frac{x}{\sin x + \cos x}\, dx = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx - I \quad \text{(from (1)}$$

$$\Rightarrow 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx. \text{ Now put, } \tan\left(\frac{x}{2}\right) = t$$

$$\Rightarrow \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2} \text{ and } dx = \frac{2dt}{1+t^2}$$

Also when $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$$\therefore I = \frac{\pi}{4} \int_0^1 \frac{2\left[dt/\left(1+t^2\right)\right]}{\left[(2t)/\left(1+t^2\right)\right]+\left[\left(1-t^2\right)/\left(1+t^2\right)\right]} = \frac{\pi}{2}\int_0^1 \frac{dt}{2t+1-t^2} = \frac{\pi}{2}\int_0^1 \frac{dt}{\left(\sqrt{2}\right)^2-(t-1)^2}$$

$$\therefore \frac{\pi}{2}\left[\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right)\right]_0^1 = \frac{\pi}{4\sqrt{2}}\left(\log\left(\frac{\sqrt{2}+1-1}{\sqrt{2}-1+1}\right)-\log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$= \frac{-\pi}{4\sqrt{2}}\left(\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) = \frac{\pi}{4\sqrt{2}}\log\left(\sqrt{2}+1\right) = \frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

24. Given differential eq. is $\dfrac{dy}{dx} = \dfrac{y^2-2xy}{x^2-xy}$

Which is a homogeneous differential equation

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ The given eq. reduces to , $v + x\dfrac{dv}{dx} = \dfrac{v^2x^2-2x(vx)}{x^2-x(vx)}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2\left(v^2-2v\right)}{x^2(1-v)} \Rightarrow x\frac{dv}{dx} = \frac{\left(v^2-2v\right)}{(1-v)}-v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-2v-v+v^2}{1-v} \Rightarrow x\frac{dv}{dx} = \frac{2v^2-3v}{1-v} \Rightarrow \frac{1-v}{2v^2-3v}dv = \frac{dx}{x}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-2v-v+v^2}{1-v} \Rightarrow x\frac{dv}{dx} = \frac{2v^2-3v}{1-v} \Rightarrow \frac{1-v}{2v^2-3v}dv = \frac{dx}{x}$$

By integrating both sides, we get

$$\int \frac{1-v}{2v^2-3v}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{1-v}{v(2v-3)}dv = \log x + \log c$$

Let $\dfrac{1-v}{v(2v-3)} = \dfrac{A}{v}+\dfrac{B}{2v-3} \Rightarrow 1-v = a(2v-3)+bv$

By comparting the coefficients of like terms, $1=3a$ & $-1=2a+b \Rightarrow a = \dfrac{-1}{3}$ & $b = \dfrac{-1}{3}$

$$\therefore \int\left(\frac{(-1/3)}{v}+\frac{(-1/3)}{2v-3}\right)dv = \log x + \log c \Rightarrow \frac{-1}{3}\int\left(\frac{1}{v}+\frac{1}{2v-3}\right)dv = \log x + \log c$$

$$\Rightarrow \frac{-1}{3}\left[\log v + \frac{1}{2}\log(2v-3)\right] = \log x + \log c$$

$$\Rightarrow \frac{-1}{3}\left[\log v\sqrt{2v-3}\right] \Rightarrow \frac{-1}{3}\left[\log v\sqrt{2v-3}\right] = \log x + \log c \Rightarrow \log\left(v\sqrt{2v-3}\right) = 3\left(\log x + \log c\right)$$

By replacing $v = y/x, x^3\left(\dfrac{y}{x}\sqrt{2\left(\dfrac{y}{x}\right)-3}\right) = c^3$

$$\Rightarrow x^2 y\sqrt{\frac{2y}{x}-3} = c^3 \Rightarrow xy\sqrt{2xy-3x^2} = c^3$$