MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Common tangent is drawn to $y^2 = 16x$ and $x^2 + y^2 = 8$. Find square of distance between point of contact of common tangent on both the curves.

(1) 78 (2) 72

(3) 42 (4) 76

Answer (2)

Sol. $y = mx + 4/m$

Perpendicular from (0,0)

$$\left|\frac{\frac{4}{m}}{\sqrt{m^2+1}}\right| = \sqrt{8}$$

$\Rightarrow$ $m = \pm 1$

Point of contact on parabola $= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

= (4, ±8)

Point of contact in circle = (–2, 2)

OR (2, –2)

Distance between (4, 8) & (–2, 2)

$= \sqrt{72}$

2. Let $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1 & x = 0 \end{cases}$

$g(x) = \begin{cases} \frac{\sin(x+1)}{x+1} & x \neq -1 \\ 1 & x = -1 \end{cases}$

$h(x) = 2[x] + f(x)$

([·] denotes greatest integer function)

then $\lim\limits_{x\to 1} g(h(x-1))$ is

(1) $\frac{\sin 1}{1}$ (2) $\frac{\sin 2}{2}$

(3) –1 (4) 2

Answer (2)

Sol. $h(x - 1) = 2[x - 1] + f(x - 1)$

$\text{lt}_{x\to 1^+} h(x-1) = 2.0 + f(0^+)$

$= 1$

$\text{lt}_{x\to 1^-} h(x-1) = 2(-1) + f(0^-)$

$= -3$

RHL

$\text{lt}_{x\to 1^+} g(h(x-1)) = \frac{\sin 2}{2}$

LHL

$\text{lt}_{x\to 1^-} g(h(x-1)) = \frac{\sin 2}{2}$

LHL = RHL

$\therefore \lim\limits_{x\to 1} g(h(x-1)) = \frac{\sin 2}{2}$

3. If $|\vec{a}| = 1, |\vec{b}| = 2, \vec{a}\cdot\vec{b} = 4, \vec{c} = 2(\vec{a}\times\vec{b}) - 3\vec{b}$ then $\vec{b}\cdot\vec{c}$ equals

(1) – 48 (2) – 12

(3) 12 (4) 48

Answer (2)

Sol. $\vec{c} = 2(\vec{a}\times\vec{b}) - 3\vec{b}$

$\vec{b}\cdot\vec{c} = -3\vec{b}\cdot\vec{b}$

$= -3|\vec{b}|^2$

$= -12$

4. $\lim\limits_{n\to\infty} \frac{3}{n}\left[4 + \left(2+\frac{1}{n}\right)^2 + \left(2+\frac{2}{n}\right)^2 + \ldots + \left(3-\frac{1}{n}\right)^2\right]$ is

(1) 19 (2) 21

(3) –19 (4) 0

Answer (1)

Sol. $\lim\limits_{n\to\infty} \frac{3}{n}\sum\limits_{r=0}^{n-1}\left(2+\frac{r}{n}\right)^2$

$= 3\int_0^1 (2+x)^2 dx$

$= 3\frac{(2+x)^3}{3}\Big|_0^1$

$= 27 - 8$

$= 19$

5. Let $f(x)=\sqrt{3-x}+\sqrt{x+2}$. The range of $f(x)$ is

(1) $[2\sqrt{2}, \sqrt{10}]$ (2) $[\sqrt{5}, \sqrt{10}]$

(3) $[\sqrt{2}, \sqrt{7}]$ (4) $[\sqrt{7}, \sqrt{10}]$

Answer (2)

Sol. $y=\sqrt{3-x}+\sqrt{x+2}$

$$y'=\frac{1}{2\sqrt{3-x}}(-1)+\frac{1}{2\sqrt{x+2}}=0$$

$\Rightarrow \sqrt{x+2}=\sqrt{3-x}$

$\Rightarrow x=\frac{1}{2}$

$\Rightarrow y\left(\frac{1}{2}\right)=\sqrt{\frac{5}{2}}+\sqrt{\frac{5}{2}}$

$y_{max}=\sqrt{10}$

y_{min} at $x=-2$ or $x=3$ is $\sqrt{5}$

$\therefore\ y\in\left[\sqrt{5}, \sqrt{10}\right]$

6. The value of $\tan^{-1}\left(\frac{1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{1}{1+a_2a_3}\right)+....+\tan^{-1}\left(\frac{1}{1+a_{2021}a_{2022}}\right)$

if $a_1 = 1$ and a_i are consecutive natural numbers

(1) $\frac{\pi}{4}-\cot^{-1}(2021)$

(2) $\frac{\pi}{4}-\cot^{-1}(2022)$

(3) $\frac{\pi}{4}-\tan^{-1}(2021)$

(4) $\frac{\pi}{4}-\tan^{-1}(2022)$

Answer (2)

Sol. $\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)+....+\tan^{-1}\left(\frac{a_{2022}-a_{2021}}{1+a_{2021}a_{2022}}\right)$

$$=\left(\tan^{-1}a_2-\tan^{-1}a_1\right)+\left(\tan^{-1}a_3-\tan^{-1}a_2\right)+...+\left(\tan^{-1}a_{2022}-\tan^{-1}a_{2021}\right)$$

$=\tan^{-1}a_{2022}-\tan^{-1}a_1$

$\therefore\ a_1=1, a_2=2....a_{2022}=2022$

$=\tan^{-1}2022-\tan^{-1}1$

$=\tan^{-1}2022-\frac{\pi}{4}$

$=\frac{\pi}{2}-\cot^{-1}2022-\frac{\pi}{4}$

$=\frac{\pi}{4}-\cot^{-1}2022$

7. Let $P=\left(8\sqrt{3}+13\right)^{13}, Q=\left(6\sqrt{2}+9\right)^{9}$ then (where [ ] represents greatest integer function)

(1) [P] = Odd, [Q] = Even

(2) [P] = Even, [Q] = Odd

(3) [P] = Odd, [Q] = Odd

(4) [P] + [Q] = Even

Answer (4)

Sol. Let $P=I_1+f_1, f_1'=\left(8\sqrt{3}-13\right)^{13}$

$$I_1+f_1-f_1'=\left(8\sqrt{3}+13\right)^{13}-\left(8\sqrt{3}-13\right)^{13}$$

$$=2\left({}^{13}C_1\left(8\sqrt{3}\right)^{12}(13)^1+{}^{13}C_3\left(8\sqrt{3}\right)^{10}(13)^3+{}^{13}C_5\left(8\sqrt{3}\right)^{8}(13)^5+...+{}^{13}C_{13}\left(8\sqrt{3}\right)^{0}(13)^{13}\right)$$

$f_1-f'=0$

So, I_1 is even

Let $Q=I_2+f_2, f_2'=\left(9-6\sqrt{2}\right)^{9}$

$$I_2+f_2-f_2'=\left(9+6\sqrt{2}\right)^{9}-\left(9-6\sqrt{2}\right)^{9}$$

$$=2\left[{}^{9}C_0 9^9+{}^{9}C_2 9^7\left(6\sqrt{2}\right)^2+...\right]$$

Again $f_2-f_2'=0$

I_2 = even

8. Let p : I am well.

q : I will not take rest

r : I will not sleep properly, then

"If I am not well then I will not take rest and I will not sleep properly" is logically equivalent to

(1) $(\sim p\to q)\vee r$ (2) $\sim p\to(q\wedge r)$

(3) $(\sim p\wedge q)\to r$ (4) $(\sim p\vee q)\to r$

Answer (2)

Sol. ~ p : I am not well

q : I will not take rest

r : I will not sleep properly

I will not take rest and I will not sleep properly $\equiv q \wedge r$

If I am not well then I will not take rest and I will not sleep properly $\equiv \sim p \rightarrow (q \wedge r)$

9. q is maximum value of p lying in interval [0, 10], roots of $x^2 - px + \frac{5p}{d} = 0$ are having rational roots.

Find area of region

$S : \{0 \le y \le (x - q)^2\}$

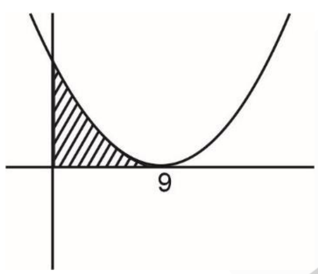
(1) 243 (2) 723

(3) 81 (4) 3

Answer (1)

Sol. $D = p^2 - 5p$ must be a perfect square i.e. possible when $p = 9$

Region for $0 \le y \le (x-9)^2$, in 1st quadrant



$$A = \int_0^9 (x-9)^2 dx$$

$$= \left.\frac{(x-9)^3}{3}\right|_0^9 = 0 + \frac{9^3}{3}$$

= 243 sq. unit

10. If $\frac{dy}{dx} = -\frac{3x^2 + y^2}{3y^2 + x^2}$, $y(1) = 0$, then $f(x)$ is

(1) $\log(x+y) + \frac{2xy}{(x+y)^2} = 0$

(2) $\log(x+y) - \frac{2xy}{(x+y)^2} = 0$

(3) $3 = (3y^2 - 2xy + 3x^2)(x+y)^2$

(4) $3 = (3y^2 - 2xy + 3x^2)(x+y)$

Answer (3)

Sol. $\frac{dy}{dx} = -\frac{3x^2 + y^2}{3y^2 + x^2} = -\frac{3 + \left(\frac{y}{x}\right)^2}{3\left(\frac{y}{x}\right)^2 + 1}$

Let, $\frac{y}{x} = u$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{-(3+u^2)}{3u^2+1}$$

$$x\frac{du}{dx} = \frac{-(3+u^2) - u(3u^2+1)}{3u^2+1}$$

$$x\frac{du}{dx} = \frac{-[3u^3 + u^2 + u + 3]}{(3u^2+1)}$$

$$x\frac{du}{dx} = \frac{-(u+1)(3u^2 - 2u + 3)}{3u^2+1}$$

$$\int \frac{3u^2+1}{(u+1)(3u^2-2u+3)} du = -\int \frac{dx}{x}$$

$$\int \frac{\frac{1}{2}}{u+1} + \frac{\frac{1}{4}(6u-2)}{3u^2 - 2u + 3} du = -\int \frac{dx}{x}$$

$$\frac{1}{2}\ln|(u+1)| + \frac{1}{4}\ln|3u^2 - 2u + 3| = -\ln x + C$$

$$\frac{1}{2}\ln(x+y) - \frac{1}{2}\ln x + \frac{1}{4}\ln(3y^2 - 2xy + 3x^2)$$

$$-\frac{1}{4} \times 2\ln x = -\ln x + C$$

$$\ln(x+y)^2 + \ln(3y^2 - 2xy + 3x^2) = C$$

$$(x+y)^2(3x^2 - 2xy + 3y^2) = C$$

$y(1) = 0$

$\Rightarrow$ $C = 3$

$$\boxed{(x+y)^2(3x^2 - 2xy + 3y^2) = 3}$$

11. A bag contains 3 same balls and 3 different balls of three different colours. Two balls are drawn randomly with replacement. The probability they have same colour is m. Again four balls are drawn one by one with replacement, then probability of getting three same balls is n. The value of $m.\ n$ is

(1) $\frac{3}{49}$ (2) $\frac{6}{49}$

(3) $\frac{43}{147}$ (4) $\frac{8}{81}$

Answer (4)

Sol. For m

both balls is one of different colours $=\left(\frac{1}{6}\times\frac{1}{6}\right)\cdot 3$

both balls is from the same balls $=\frac{1}{2}\times\frac{1}{2}$

$\therefore \quad m=\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$

For n

Same ball is from the different coloured balls

$$=3\left(4\left(\frac{1}{6}\right)^3\cdot\frac{5}{6}\right)$$

Or same ball is from the 3 same balls

$$= =\left(4\left(\frac{1}{2}\right)^3\cdot\frac{1}{2}\right)$$

$\therefore \quad n=\frac{10}{6^3}+\frac{1}{4}=\frac{8}{27}$

$\therefore \quad m\cdot n=\frac{8}{81}$

12.

13.

14.

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

21. Two A.P.'s are given as under

3, 7, 11, ……

1, 6, 11, 16,….

Find 8th common term that is appearing in both the series

Answer (151)

Sol. First common term is 11 and common terms will appear in an A.P. having common difference as LCM of (4, 5) = 20

$T_8 = 11 + (8 – 1)\ 20$

$= 151$

22. Using 1, 2, 2, 2, 3, 3, 5 find number of 7-digit odd numbers that can be formed

Answer (240)

Sol. - - - - - - 1 $\rightarrow \frac{6!}{2!3!}=60$

- - - - - - 3 $\rightarrow \frac{6!}{3!}=120$

- - - - - - 5 $\rightarrow \frac{6!}{3!2!}=60$

Total = 240

23. 50th root of x is 12

50th root of y is 18

Remainder when $x + y$ is divided by 25.

Answer (23)

Sol. $12^{50} + 18^{50} = 144^{25} + 324^{25}$

$= (25K_1 – 6)^{25} + (25K_2 – 1)^{25}$

$= 25\lambda – 6^{25} – 1$

$6^{25} + 1 = (6^5)^5 + 1$

$= (7776)^5 + 1$

$= (25\lambda_1 + 1)^5 + 1 = 25p + 2$

$\Rightarrow \quad 12^{50} + 18^{50} = 25\lambda – (25p + 2)$

$\Rightarrow$ Remainder = 23

24. Let $a = \{1, 3, 5, \ldots 99\}$

and $b = \{2, 4, 6, \ldots 100\}$

The number of ordered pair (a, b) such that $a + b$ when divided by 23 leaves remainder 2 is

Answer (108)

Sol. $a + b = 23\lambda + 2$

$\lambda = 0, 1, 2, \ldots.$

But λ can't be even

$\therefore$ if $\lambda = 1$ $(a, b) \rightarrow$ 12 pairs

$\lambda = 3$ $(a, b) \rightarrow$ 35 pairs

$\lambda = 5$ $(a, b) \rightarrow$ 42 pairs

$\lambda = 7$ $(a, b) \rightarrow$ 19 pairs

$\lambda = 9$ $(a, b) \rightarrow$ 0 pairs

$\vdots$

Total = 12 + 35 + 42 + 19 = 108 ordered paris

25. Let a line parallel to $x + 3y - 2z - 2 = 0 = x - y + 2z$ and passes through (2, 3, 1). If distance of point (5, 3, 8) from the line is α, then $3\alpha^2$ is

Answer (158)

Sol. Let $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$

$\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$

Line will be parallel to $\vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i}(4) - \hat{j}(4) + \hat{k}(-4)$$

$\Rightarrow \quad \vec{n} = \hat{i} - \hat{j} - \hat{k}$

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{n}|}{|\vec{n}|}$$

where $\vec{a}_2 = 5\hat{i} + 3\hat{j} + 8\hat{k}$, $\vec{a}_1 = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{a}_2 - \vec{a}_1 = 3\hat{i} + 7\hat{k}$

$$(\vec{a}_2 - \vec{a}_1) \times \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 7 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \hat{i}(7) - \hat{j}(-10) + \hat{k}(-3)$$

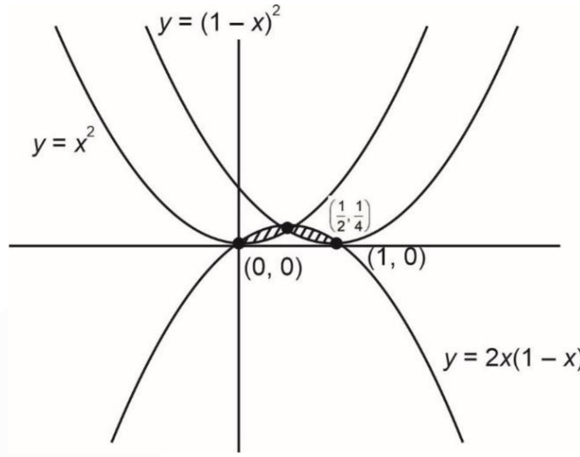
$$d = \frac{\sqrt{100 + 49 + 9}}{\sqrt{3}} = \frac{\sqrt{158}}{\sqrt{3}} = \alpha$$

$3\alpha^2 = 158$

26. If area of the region bounded by the curves $y = x^2$, $y = (1 - x)^2$ and $y = 2x(1 - x)$ is A, then find the value of $540A$,

Answer (135)

Sol. $$A = \int_0^1 2x(1-x)dx - \int_0^{1/2} x^2 dx - \int_{1/2}^1 (1-x)^2 dx$$



$$= x^2 - \frac{2x^3}{3}\Bigg]_0^1 - \frac{x^3}{3}\Bigg]_0^1 + \frac{(1-x)^3}{3}\Bigg]_{1/2}^1$$

$$= \frac{1}{4}$$

$540A = 135$

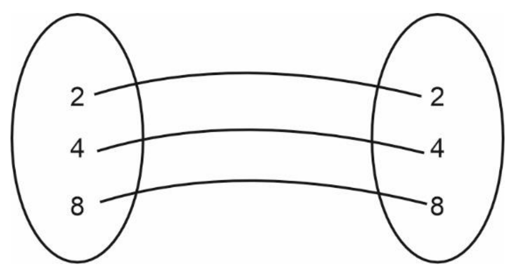
27. $A = \{2, 4, 6, 8, 10\}$

Find total no. of functions defined on A such that $f(m \cdot n) = f(m) \cdot f(n)$, $m, n \in A$

Answer (25)

Sol. $f(4) = (f(2))^2 = 4$

$f(8) = (f(2))^3 = 8$



For 6 and 10 we have 5 options

Total functions = 5 × 5 = 25

28.

29.

30.

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