MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

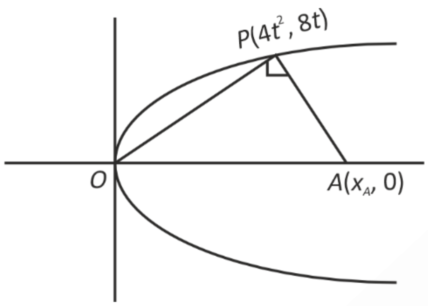
1. Let O be the vertex of the parabola $y^2 = 16x$. The locus of centroid of ΔOPA when P lies on parabola and A lies on x-axis and $\angle OPA = 90°$

(1) $y^2 = 8(3x – 16)$ (2) $9y^2 = 8(3x – 16)$

(3) $y^2 = 8(3x + 16)$ (4) $9y^2 = 8(3x + 16)$

Answer (2)

Sol.



$m_{OP} \cdot m_{PA} = –1$

$$\frac{2}{t}\cdot\frac{8t}{4t^2 - x_A} = -1$$

$–16 = 4t^2 – x_A$

$x_A = 4t^2 + 16$

$$h = \frac{4t^2 + 4t^2 + 16}{3} \qquad k = \frac{8t}{3}$$

$$h = \frac{8t^2 + 16}{3} \qquad t = \frac{3k}{8}$$

$$3h - 16 = 8\left[\frac{3k}{8}\right]^2$$

Replace (h, k) with (x, y)

$$3x - 16 = \frac{9y^2}{8}$$

$$\boxed{9y^2 = 8(3x - 16)}$$

2. If the product

$$\left(\frac{1}{15C_0} + \frac{1}{15C_1}\right)\left(\frac{1}{15C_1} + \frac{1}{15C_2}\right)\cdots\left(\frac{1}{15C_{12}} + \frac{1}{15C_{13}}\right)$$

$$= \frac{\alpha^{13}}{14C_0 \cdot {}^{14}C_1 \cdot {}^{14}C_2 \ldots {}^{14}C_{12}}$$, then 30α is equal to

(1) 16 (2) 32

(3) 15 (4) 28

Answer (2)

Sol. Notice that

$$\frac{1}{{}^nC_r} + \frac{1}{{}^nC_{r+1}} = \frac{{}^nC_{r+1} + {}^nC_r}{{}^nC_r \cdot {}^nC_{r+1}} = \frac{{}^{n+1}C_{r+1}}{{}^nC_r \cdot {}^nC_{r+1}}$$

$$= \frac{\frac{n+1}{r+1} + {}^nC_r}{{}^nC_r \cdot {}^nC_{r+1}} = \frac{(n+1)}{(r+1)\cdot\frac{n}{r+1}\cdot {}^{n-1}C_r} = \frac{n+1}{n \cdot {}^{n+1}C_r}$$

$$\therefore \quad \left(\frac{1}{{}^{15}C_0} + \frac{1}{{}^{15}C_1}\right)\left(\frac{1}{{}^{15}C_1} + \frac{1}{{}^{15}C_2}\right)\cdots\left(\frac{1}{{}^{15}C_{12}} + \frac{1}{{}^{15}C_{13}}\right)$$

$$= \frac{16}{15\cdot {}^{14}C_0}\cdot\frac{16}{15\cdot {}^{14}C_1}\cdots\frac{16}{15\cdot {}^{14}C_{12}}$$

$$= \frac{\left(\frac{16}{15}\right)^{13}}{{}^{14}C_0 \cdot {}^{14}C_1 \cdot {}^{14}C_2 \ldots {}^{14}C_{12}}$$

$$\therefore \quad \alpha = \frac{16}{15}$$

$\therefore 30\alpha = 32.$

3. If probability distribution is given by,

x	0	1	2	3
$p(n)$	$\frac{8a-1}{30}$	$\frac{4a-1}{30}$	$\frac{2a+1}{30}$	b

. If it is given that $\sigma^2+\mu^2=2,$ where σ is standard derivation and μ is mean of distribution than $\frac{a}{b}$ is

(1) $\frac{22}{71}$ (2) $\frac{110}{71}$

(3) $\frac{220}{71}$ (4) $\frac{1110}{71}$

Answer (4)

Sol.

x	0	1	2	3
$p(n)$	$\frac{8a-1}{30}$	$\frac{4a-1}{30}$	$\frac{2a+1}{30}$	b

$\mu=\Sigma xP(n)$

$=\frac{(4a-1)}{30}+2\left(\frac{2a+1}{30}\right)+36 \quad =\frac{16a+90b}{30}$

$\sigma^2=\Sigma x^2P(n)-\mu^2$

$\sigma^2+\mu^2=\Sigma x^2P(n)=2$

$=\left(\frac{4a-1}{30}\right)(1)+\left(\frac{2a+1}{30}\right)4+9b=2$

= 4*a* – 1 + 8*a* + 4 + 270*b* = 60

= 12*a* + 270*b* = 57

= 4*a* + 90*b* = 19

$\sum P(n)=1\Rightarrow 14a-1+30b=30$

14*a* + 90*b* = 31

Solving we get $a=\frac{37}{19}, b=\frac{71}{570}$

$\frac{a}{b}=\frac{1110}{71}$

4. $\int_0^1 \cot^{-1}\left(x^2+x+1\right)dx$ is equal to

(1) $\int_0^1 \tan^{-1}(x+1)dx-\int_0^1 \tan^{-1}x dx$

(2) $\int_0^1 \left(\tan^{-1}(x+1)+\tan^{-1}x\right)dx$

(3) $\int_0^1 4\tan^{-1}x dx$

(4) $3\int_0^1 \tan^{-1}(x+1)dx$

Answer (1)

Sol. $I=\int_0^1 \cot^{-1}\left(x^2+x+1\right)dx$

$=\int_0^1 \tan^{-1}\left(\frac{1}{1+x(x+1)}\right)dx$

$=\int_0^1 \tan^{-1}\left(\frac{(x+1)-x}{1+(x+1)\cdot x}\right)dx$

$=\int_0^1 \left(\tan^{-1}(1+x)-\tan^{-1}x\right)dx$

$=\int_0^1 \tan^{-1}(1+x)dx-\int_0^1 \left(\tan^{-1}x\right)dx$

5. Let $f(x)=\lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{k=1}^{n}\left[\frac{k^2}{3^x}\right]\right)$, where [.] denotes the greatest integer function, then $12\sum_{j=1}^{\infty}f(j)$ is equal to

(1) 2 (2) 3

(3) 4 (4) 1

Answer (1)

Sol. $f(x)=\lim\limits_{n\to\infty}\left(\frac{1}{n^3}\cdot\sum\limits_{k=1}^{n}\left[\frac{k^2}{3^x}\right]\right)$

$$=\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\left(\frac{k^2}{3^x}\right)-\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\left\{\frac{k^2}{3^x}\right\}$$

$$f(x)=\lim_{n\to\infty}\frac{1}{n^3}\times\frac{n(n+1)(2n+1)}{6\times 3^x}-0$$

$$\Rightarrow\ f(x)=\frac{1}{3^{x+1}}$$

$$12\sum_{j=1}^{\infty}f(j)=12\left(\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\ldots\right)$$

$$=12\left(\frac{\frac{1}{9}}{1-\frac{1}{3}}\right)=\frac{12}{9-3}=2$$

6. The number of solutions of equation tan3x = cot$x$ in $x\in[0,2\pi]$ is

(1) 4 (2) 6

(3) 2 (4) 8

Answer (4)

Sol. $\tan 3x=\cot=\tan\left(\frac{\pi}{2}-x\right)$

$$\Rightarrow 3x=\frac{\pi}{2}-x+n\pi,\ n\in I$$

$$\Rightarrow 4x=\frac{\pi}{2}+n\pi,\ n\in I$$

$$\Rightarrow x=\frac{\pi}{8}+\frac{n\pi}{4},\ n\in I$$

$$\Rightarrow\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8},\frac{9\pi}{8},\frac{11\pi}{8},\frac{13\pi}{8},\frac{15\pi}{8}$$

= 8

7. If $(\sec x)\frac{dy}{dx}-2y=2+3\sin x$ and $y(0)=-\frac{7}{4}$ then $y\left(\frac{\pi}{6}\right)$ is

(1) $\frac{3}{4}$ (2) $\frac{4}{3}$

(3) $\frac{5}{2}$ (4) $-\frac{5}{2}$

Answer (4)

Sol. $(\sec x)\frac{dy}{dx}-2y=2+3\sin x$

$$\Rightarrow\frac{dy}{dx}-2\cos xy=2\cos x+3\sin x\cdot\cos x$$

IF = $e^{-2\sin x}$

$$y\cdot e^{-2\sin x}=\int e^{-2\sin x}\cdot\cos x(2+3\sin x)dx$$

sin$x$ = t $\Rightarrow$ cos $x dx$ = dt

$$y\cdot e^{-2\sin x}=\int e^{-2t}(2+3t)dt$$

$$=2\frac{e^{-2t}}{-2}+3\int t\cdot e^{-2t}\,dt$$

$$=-e^{-2t}+3\left(\frac{t\cdot e^{-2t}}{-2}+\frac{1}{2}\int e^{-2t}\,dt\right)$$

$$-e^{-2t}+3\left(-\frac{te^{-2t}}{2}-\frac{1}{4}e^{-2t}\right)+c$$

$$y=-1-\frac{3}{2}(\sin x)-\frac{3}{4}+ce^{2\sin x}$$

$$=-\frac{7}{4}-\frac{3}{2}\sin x+c\cdot e^{2\sin x}$$

$$y(0)=-\frac{7}{4}\Rightarrow c=0$$

$$y=-\frac{7}{4}-\frac{3}{2}\sin x$$

$$y(\pi/6)=-\frac{7}{4}-\frac{3}{4}=-\frac{10}{4}=-\frac{5}{2}$$

8. The area bounded by

$2 - 4x \le y \le x^2 + 4$ and $x = \frac{1}{2}$

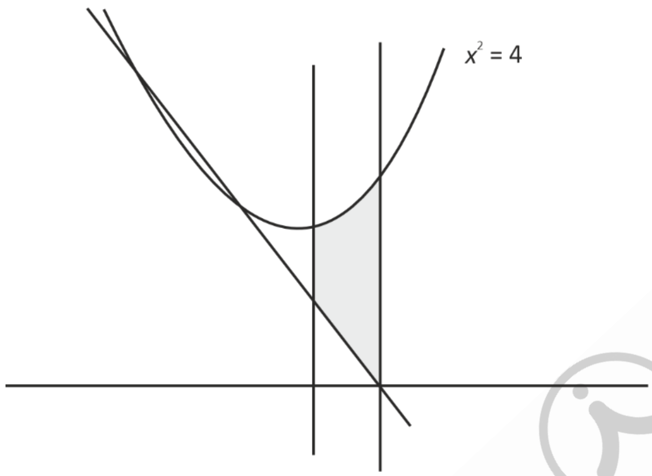
$x \ge 0$, $y \ge 0$ (in square unit) is

(1) $\frac{25}{37}$ sq. unit (2) $\frac{24}{37}$ sq. unit

(3) $\frac{37}{25}$ sq. unit (4) $\frac{37}{24}$ sq. unit

Answer (4)

Sol.



$x = \frac{1}{2}$

Area = $\int_0^{\frac{1}{2}} (x^2+4)-(2-4x)dx$

$= \frac{x^3}{3} + 2x + 2x^2 \Big]_0^{\frac{1}{2}} 1$

$\Rightarrow \frac{37}{24}$ sq unit

9. Let A = {2, 3, 5, 7, 11} and a relation R is defined as $R = \{(x,y) : x, y \in A, 2x \le 3y\}$. Then minimum number of elements are to be added to relation R such that R becomes symmetric relation is

(1) 4 (2) 8

(3) 7 (4)6

Answer (2)

Sol. $R = \{(x, y) : 2x \le 3y\}$

{(2, 2), (3, 2)

(3, 3), (2, 3)

(5, 5), (2, 5), (3, 5), (7, 5)

(7, 7), (2, 7), (3, 7), (5, 7)

(11, 12), (2, 11), (3, 11), (5, 11), (7, 11)}

Since we want the relation to be symmetric relation,

We need to add

(5, 2), (5, 3), (7, 2), (7, 3), (11, 2), (11, 3), (11, 5), (11, 7)

$\Rightarrow$ 8 elements need to be added.

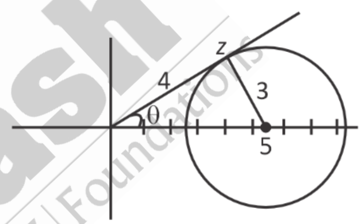
10. Let z be the complex number satisfying $|z - 5| \le 3$ and having maximum possible positive argument, then the value of $34\left|\frac{5z-12}{5iz+16}\right|^2$ is equal to

(1) 20 (2) 17

(3) 7 (4) 21

Answer (1)

Sol.



$\Rightarrow \arg(z) = \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right).$

$\Rightarrow z = |z|^{i\theta}$

$= 4 \times (\cos\theta + i\sin\theta)$

$= 4 \times \left(\frac{4}{5} + i\frac{3}{5}\right)$

$= \frac{16}{5} + \frac{12i}{5}$ $\Rightarrow 5z = 16 + 12i$

$5zi = 16i - 12$

$\left(\frac{5z-12}{5zi+16}\right) = \frac{(4+12i)}{(4+16i)} = \frac{(1+3i)}{(1+4i)}.$

$$\Rightarrow \left|\frac{5z-12}{5zi+16}\right| = \frac{\sqrt{10}}{\sqrt{17}}.$$

$$34\left|\frac{5z-12}{5zi+16}\right|^2 = 34\times\frac{10}{17} = 20.$$

11. Let $f(x) = x^3 + x^2 f'(1) + 2xf''(2) + f'''(3)\ \forall x \in R$ then the value of $f'(5)$ is

(1) $\frac{109}{5}$ (2) $\frac{117}{5}$

(3) $\frac{119}{5}$ (4) $\frac{118}{5}$

Answer (2)

Sol. Let $f'(1) = a$

$f''(2) = b$

$f''(3) = c$

$$f(x) = x^3 + ax^2 + bx + c$$

$$f'(x) = 3x^2 + 2ax + b$$

$$f'(1) = a = 3 + 2a + b \Rightarrow a + b = 3 \ldots(1)$$

$$f''(x) = 6x + 2a$$

$$\Rightarrow f''(2) = 12 + 2a = \frac{b}{2} \Rightarrow 4a - b = -24$$

$$\Rightarrow f''(x) = 6$$

$$\Rightarrow f''(3) = c = 6$$

$$\Rightarrow a = \frac{-27}{5}, b = \frac{12}{5}$$

$$f'(5) = 75 + 10a + b$$

$$= 75 - 54 + \frac{12}{5}$$

$$= 21 + \frac{12}{5} = \frac{117}{5}$$

12.

13.

14.

15.

16.

17.

18.

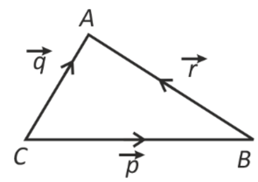
19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. If three vectors are given as shown.

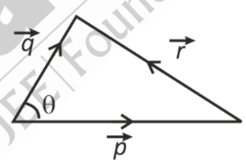


If angle between vectors $\vec{p}$ and $\vec{q}$ is θ where $\cos\theta = \frac{1}{\sqrt{3}}$ and $|\vec{p}| = 2\sqrt{3}, |\vec{q}| = 2$.

Then the value of $|\vec{p}\times(\vec{q}-3\vec{r})|^2 - 3|\vec{r}|^2$ is

Answer (104)

Sol.



$$\vec{p}+\vec{r}-\vec{q} = 0 \Rightarrow \vec{r} = \vec{q}-\vec{p}$$

$$\cos\theta = \frac{|\vec{p}|^2 + |\vec{q}|^2 - |\vec{r}|^2}{2|\vec{p}||\vec{q}|} = \frac{12 + 4 - |\vec{r}|^2}{2\times2\times2\sqrt{3}}$$

$$\frac{16-|\vec{r}|^2}{8\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow |\vec{r}|^2 = 8$$

$$\left|\vec{p}\times(\vec{q}-3(\vec{q}-\vec{p}))\right|^2 - 3|\vec{r}|^2$$

$= |-2\vec{p}\times\vec{q}|^2 - 3|\vec{r}|^2$

$= 4\times|\vec{p}||\vec{q}|^2 \sin^2\theta - 3\times 8$

$= 4\times 12\times 4\times\left(\frac{2}{3}\right) - 24$

= 128 – 24 = 104

22. The largest value of $n\in N$ such that 7^n divides (101)! is _______.

Answer (16)

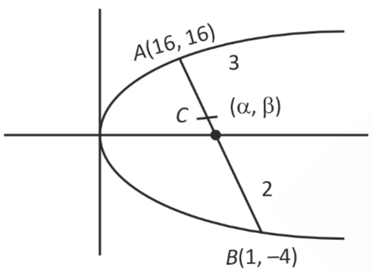
Sol. Exponent of 7 in $101! = \left[\frac{101}{7}\right] + \left[\frac{101}{7^2}\right] + ...$ (where [·] is G.I.F.)

= 14 + 2 = 16

23. Let $y^2 = 16x$, from point (16, 16) a focal chord is passing. Point (α, β) divides the focal chord in ratio 2 : 3, then minimum value of $\alpha + \beta$ is

Answer (11)

Sol. $y^2 = 16x$



$\therefore$ AB is focal chord $\Rightarrow t_1t_2 = -1$

$t_1 = 2$

$t_2 = -\frac{1}{2}$

$\Rightarrow B(1, -4)$

C can be $\left(\frac{2+48}{5}, \frac{-8+48}{5}\right)$ or $\left(\frac{3+32}{5}, \frac{-12+32}{5}\right)$

(10, 8) or (7, 4)

$\Rightarrow (\alpha+\beta)_{\min} = 11$

24. The minimum value of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$ in $x\in\left[-\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right]$ is $\frac{a\pi^2}{b}$, then $a + b$ is

Answer (9)

Sol. $(\sin^{-1})^2 + (\cos^{-1}x)^2$

$E = (\sin^{-1}x + \cos^2 x)^2 - 2(\sin^{-1}x)(\cos^{-1}x)$

$= \left(\frac{\pi}{2}\right)^2 - 2(\sin^{-1}x)\left(\frac{\pi}{2} - \sin^{-1}x\right)$

$= \frac{\pi^2}{4} - (\sin^{-1}x)\pi + 2(\sin^{-1}x)^2$

$E = 2\left\{\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{8} - \frac{\pi^2}{16}\right\}$

$= 2\left\{\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right\}$

For min $\sin^{-1}x = \frac{\pi}{4}$ $i.e.\ x = \frac{1}{\sqrt{2}}$

$E_{\min} = \frac{\pi^2}{8}$

25. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 23 & 49 \\ 45 & 21 \end{bmatrix}$ and if $\left(A^{15} + B\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 36 \end{bmatrix}$ then $(x + 2y)$ is equal to

Answer (7)

Sol. $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 23 & 49 \\ 45 & 21 \end{bmatrix}$

Charactertic equation of A,

$(3 - \lambda)(-1 - \lambda) + 4 = 0$

$(\lambda - 3)(\lambda + 1) + 4 = \lambda^2 - 2\lambda + = 0$

$\Rightarrow$ $A^2 - 2A + I = 0$

$A^2 = 2A - I$, $A^3 = 2A^2 - A = 2(2A - I) - A = 3A - 2I$

$A^4 = 4A^2 + I - 4A = 4(2A - I) - 4A + I$

$= 4(2A - I) - 3A = 5A - 4I$

$\left(A^5\right)^3 = \left(5A - 4I\right)^3$

$= 125A^3 - 3 \times 25A^2(4) + 3(5A)(4^2) - 64I$

$= 125(3A - 2I) - 300(2A - I) + 240A - 64I$

$= A(375 - 600 + 240) + I(-250 + 300 - 64)$

$= 15A - 14I$

$\Rightarrow A^{15} + B = 15A - 14I + B$

$= 4A - 3I$

$A^5 = 4A^2 - 3A$

$= \begin{bmatrix} 45 & -60 \\ 15 & -15 \end{bmatrix} + \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix} + \begin{bmatrix} 23 & 49 \\ 45 & 21 \end{bmatrix}$

$= \begin{bmatrix} 54 & -11 \\ 60 & -8 \end{bmatrix} \Rightarrow \left(A^{15} + B\right)\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \quad 54x - 11y = 21 \qquad \Rightarrow \quad x = 1, y = 3$

$60x - 8y = 36$

$\Rightarrow \quad x + 2y = 7$

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