MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. If y = f(x) is the solution of the differential equation $(1+\sin x)\frac{dy}{dx}+\cos x=0$, such that $f(0) = 0$ then $f\left(\frac{\pi}{2}\right)$ is equal to

   (1) ln2 (2) –ln2

   (3) ln3 (4) ln4

Answer (2)

Sol. $\frac{dy}{dx}=\frac{-\cos x}{1+\sin x}$

$\Rightarrow \int dy=\int\left(\frac{-\cos x}{1+\sin x}\right)dx$

$\Rightarrow y = -\ln|1 + \sin x| + c$

Since, $f(0) = 0$

$\Rightarrow 0 = 0 + c \Rightarrow c = 0$

$f(x)=\ln\left|\frac{1}{1+\sin x}\right|$

$f\left(\frac{\pi}{2}\right)=\ln\left|\frac{1}{2}\right|=-\ln 2$

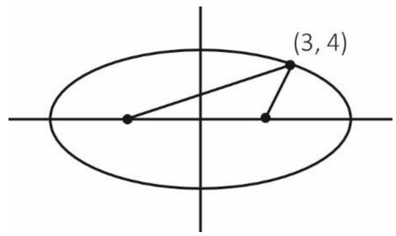
2. If eccentricity of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which passes through point (3, 4) is $\frac{\sqrt{5}}{3}$, then length of latus rectum of ellipse is

   (1) $\frac{4\sqrt{5}}{3}$ (2) $\frac{8\sqrt{5}}{3}$

   (3) $\frac{4\sqrt{7}}{3}$ (4) $\frac{8\sqrt{7}}{3}$

Answer (2)

Sol.



$e^2=\frac{5}{9},\quad \frac{9}{a^2}+\frac{16}{b^2}=1$

$1-\frac{b^2}{a^2}=\frac{5}{9}\Rightarrow \boxed{\frac{b^2}{a^2}=\frac{4}{9}}$

$\frac{9}{a^2}+\frac{16}{\frac{4a^2}{9}}=1\ \Rightarrow \frac{9}{a^2}+4\times\frac{9}{a^2}=1$

$\Rightarrow a^2 = 45$

$\Rightarrow b^2 = =\frac{4a^2}{9}=20$

$\Rightarrow$ length of latus rectum = $\frac{2b^2}{a}$

$=\frac{8\sqrt{5}}{3}$

3. Let $x\in[-\pi, \pi]$.

   $S = \{x : \sin x(\sin x + \cos x) = a, a\in I\}$

   Then number of elements in set S is equal to

   (1) 5

   (2) 10

   (3) 9

   (4) 4

Answer (3)

Sol. $x \in [-\pi, \pi]$

Let $E = \sin^2 x + \sin x \cdot \cos x$

$$\Rightarrow \frac{1}{2}(2\sin^2 x + 2\sin x \cdot \cos x) = E$$

$$\Rightarrow \frac{1}{2}(1 - \cos 2x + \sin 2x) = E$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2}[\sin 2x - \cos 2x] = E$$

$$\sin 2x - \cos 2x \in [-\sqrt{2}, \sqrt{2}]$$

$$\frac{1}{2} + \frac{1}{2}(\sin 2x - \cos 2x) \in \left[\frac{1}{2} - \frac{1}{\sqrt{2}}, \frac{1}{2} + \frac{1}{\sqrt{2}}\right]$$

a = 0 and 1

For a = 0

$\sin x(\sin x + \cos x) = 0 \Rightarrow \sin x = 0, \tan x = -1$

$$\Rightarrow x = -\pi, -\frac{\pi}{4}, 0, \frac{3\pi}{4}, \pi$$

For a = 1

$$\sin x \cdot \cos x = \cos^2 x$$

$$\Rightarrow \cos x(\sin x - \cos x) = 0$$

$$\Rightarrow \cos x = 0, \tan x = 1$$

$$x = -\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}$$

$\therefore$ Number of elements = 9

4. Number of seven-digit number formed by using the digits 1, 2, 3, 4, 5 at least once are

(1) 16800 (2) 13200

(3) 15200 (4) 15800

Answer (1)

Sol. Case 1 : One digit repeat 3 times and the other repeat once (x, x, x, a, b, c, d)

Total Number $^5C_1 \times \frac{7!}{3!}$ = 5 × 840 = 4200

Case 2 : Two digits repeat twice each and other repeated once (x, x, a, a, b, c, d)

Total Number $^5C_2 \times \frac{7!}{2!2!}$ = 10 × 1260 = 12600

Total numbers 4200 + 12600 = 16800

5. The value of $\int_0^3 \frac{e^x + e^{-x}}{([x])!} dx$ equals (Here [.] denotes the greatest integer function)

(1) $\frac{1}{2}\left(e^2 + e^3 - e^{-2} - e^{-3}\right)$

(2) $e^2 - e^3 + e^{-2} - e^{-3}$

(3) $\frac{1}{4}\left(e^2 + e^3 - e^{-2} - e^{-3}\right)$

(4) $\frac{1}{2}\left(e^2 + e - e^{-1} - e^{-2}\right)$

Answer (1)

Sol. $\int_0^3 \frac{e^x + e^{-x}}{([x])!} dx$

$$= \int_0^1 \frac{\left(e^x + e^{-x}\right)}{0!} dx + \int_1^2 \frac{e^x + e^{-x}}{1!} + \int_2^3 \frac{e^x + e^{-x}}{2!} dx$$

$$= \left(e^x - e^{-x}\right)\Big|_0^1 + \left(e^x - e^{-x}\right)\Big|_0^2 + \frac{\left(e^x - e^{-x}\right)}{2}\Bigg|_2^3$$

$$= \left(e - \frac{1}{e}\right) - (1 - 1) + \left(e^2 - \frac{1}{e^2}\right) - \left(e - \frac{1}{e}\right) + \frac{1}{2}\left(e^3 - \frac{1}{e^3}\right) - \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right)$$

$$= \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) + \frac{1}{2}\left(e^3 - \frac{1}{e^3}\right)$$

$$= \frac{1}{2}\left(e^2 + e^3 - e^{-2} - e^{-3}\right)$$

6. The value of $\sin\frac{\pi}{18}\cdot\sin\frac{5\pi}{18}\cdot\sin\frac{7\pi}{18}$ is equal to

(1) $\frac{1}{8}$ (2) $\frac{1}{16}$

(3) $\frac{1}{4}$ (4) $\frac{1}{32}$

Answer (1)

Sol. We know that $\sin\theta\cdot\sin\left(\frac{\pi}{3}-\theta\right)\cdot\sin\left(\frac{\pi}{3}+\theta\right)=\frac{1}{4}\sin 3\theta$

$\therefore\quad \sin\frac{\pi}{18}\cdot\sin\frac{5\pi}{18}\cdot\sin\frac{7\pi}{18}=\sin\frac{\pi}{18}\cdot\sin\left(\frac{\pi}{3}-\frac{\pi}{18}\right)\cdot\sin\left(\frac{\pi}{3}+\frac{\pi}{18}\right)$

$=\frac{1}{4}\sin\left(3\cdot\frac{\pi}{18}\right)$

$=\frac{1}{8}$

7. Let $f(x)=\begin{cases}\left|\frac{\sin x}{x}\right| & , \quad x\neq 0\\ 1 & , \quad x=0\end{cases}$. Then total number of critical point(s) in $x \in (-2\pi, 2\pi)$ is

(1) 3 (2) 5

(3) 9 (4) 1

Answer (2)

Sol. Critical points: Point at which derivative is zero or function is not differentiable

$$f(x)=\begin{cases}\left|\frac{\sin x}{x}\right| & , \quad x-\{0\}\\ 1 & , \quad x=0\end{cases}$$

$$f'(0^+)=\lim_{h\to 0}\left|\frac{\frac{\sinh}{h}-1}{h}\right|=\lim_{h\to 0}\frac{\sinh-h}{h^2}=\lim_{h\to 0}\frac{\cosh-1}{2h}=0$$

Similarly, $f'(0^-)=0$

$f'(0)=0$ and differentiable but $f'(0)\Rightarrow$ critical point

$$f'(\pi^+)=\frac{\left|\frac{\sin(\pi+h)}{\pi+h}\right|-0}{h}=\lim_{h\to 0}\frac{-\sin(\pi+h)}{(\pi+h)h}=\lim_{h\to 0}\frac{-\cos(\pi+h)}{2h+\pi}=\frac{1}{\pi}$$

Similarly, $f'(\pi^-)=-\frac{1}{\pi}$

$\Rightarrow f(x)$ is not differentiable at $x=\pi$ and $-\pi$

And $f'(x)=\frac{x\cos x-\sin x}{x^2}\Rightarrow \tan x = x$

$\Rightarrow\quad x \simeq 4.49$

$\Rightarrow \tan x = x$ has 2 solutions in $x\in(-2\pi,2\pi)$

$\Rightarrow 2$ critical points

$\Rightarrow$ Total 5 points

8. Consider two A.P.s such that A.P.$_1$ has 1st term 1, common difference 5 and total terms 101 and A.P.$_2$ has 1st term 9, common difference 7 and total terms 71 then the number of common terms which appear in both these A.P.s is

(1) 11 (2) 14

(3) 12 (4) 13

Answer (2)

Sol. $(\text{A.P.})_1 : a_1 = 1, d_1 = 5, n_1 = 101$

$\Rightarrow\quad A_n = 5n - 4$, last term = 501

$(\text{A.P.})_2 : a_2 = 9, d_2 = 7, n_2 = 71$

$\Rightarrow$ Last term $= 499$

$B_n = 7m + 2$

$\Rightarrow\quad 5n - 4 = 7m + 2$

$\Rightarrow\quad 5n = 7m + 6$

$\Rightarrow 5/7m+6 \Rightarrow m_{\min}=2$,

$\Rightarrow$ 1st common term is 7(2) + 2 = 16

Common difference = Lcm (5, 7) = 35

$\Rightarrow$ Common A.P. is

16, 51, 86, … (≤499)

$\Rightarrow\quad 16 + (N - 1)\,35 \le 499$

$\Rightarrow\quad N-1\le\frac{483}{35}\simeq 13.8$

$\Rightarrow\quad N \le 14.8$

$\Rightarrow\quad N = 14$

9. If ${}^{36}C_{r+1} = \dfrac{6.{}^{35}C_r}{k^2-3}$ such that $S = \{(r, k)\}$ then number of elements in set S is

(1) 9 (2) 4

(3) 13 (4) 7

Answer (2)

Sol. ${}^{36}C_{r+1} = \dfrac{6.{}^{35}C_r}{k^2-3}$

$\Rightarrow \dfrac{36}{r+1}\left({}^{35}C_r\right) = \dfrac{6.{}^{35}C_r}{(k^2-3)}$

since ${}^{35}C_r > 0$

$\Rightarrow \dfrac{6}{r+1} = \dfrac{1}{k^2-3}$

$\Rightarrow 6(k^2-3) = r+1$

$\Rightarrow 6 | r+1$

$\Rightarrow r + 1 \in \{0, 6, 12, 18, 24, 30, 36\}$

$\Rightarrow$

$r+1$	0	6	12	18	24	30	36
k^2-3	0	1	2	3	4	5	6
k^2	3	(4)	5	6	7	8	(9)

$k = \pm 2, \pm 3$

$r = 5 \Rightarrow k \pm 2$

$r = 35 \Rightarrow k \pm 3$

$S = \{(5, 2), (5, -2), (35, 3), (35, -3)\}$

$\Rightarrow n(S) = 4$

10. If the system of equations

$x + 2y + z = 5$

$2x + y + \alpha z = 5$

$8x + y + z = 18$ has no solution, then α is equal to

(1) 3 (2) $\dfrac{1}{3}$

(3) 4 (4) $\dfrac{9}{15}$

Answer (4)

Sol. $D = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & \alpha \\ 8 & 1 & 1 \end{vmatrix} = 0$

$\Rightarrow 15\alpha - 9 = 0$

$\Rightarrow \alpha = \dfrac{9}{15}$

$D_x = \begin{vmatrix} 1 & 5 & 5 \\ 2 & 1 & 5 \\ 8 & 1 & 18 \end{vmatrix} \neq 0$

11. The domain of $f(x) = \sqrt{\log_{0.6}\left(\dfrac{|2x-5|}{|x^2-4|}\right)}$ is

(1) $\left(-\infty, -1-\sqrt{10}\right) \cup \left(-1+\sqrt{10}, \dfrac{5}{2}\right) \cup \left(\dfrac{5}{2}, \infty\right)$

(2) $\left(-\infty, -1-\sqrt{10}\right) \cup \left(\sqrt{10}, \infty\right)$

(3) $\left(-1-\sqrt{10}, \dfrac{5}{2}\right)$

(4) $\left(-\infty, -1-\sqrt{10}\right) \cup \left(\sqrt{10}-1, \infty\right)$

Answer (1)

Sol. $\dfrac{|2x-5|}{|x^2-4|} \neq 0$

$x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$ and $x \neq \dfrac{5}{2}$

$\log_{0.6}\left(\dfrac{|2x-5|}{|x^2-4|}\right) \geq 0$

$\dfrac{|2x-5|}{|x^2-4|} \leq 1$ $\{\because$ base $< 1\}$

$|2x-5| \leq |x^2-4|$

$\Rightarrow (2x-5)^2 - (x^2-4)^2 \leq 0$

$\Rightarrow x^4 - 12x^2 + 20x - 9 \geq 0$

$\Rightarrow (x-1)^2 (x^2 + 2x - 9) \geq 0$

$(x-1)^2 \geq 0$ (always)

$x^2 + 2x - 9 \geq 0$

$\left(x+1-\sqrt{10}\right)\left(x+1+\sqrt{10}\right) \geq 0$

$\Rightarrow x \in (-\infty, -1-\sqrt{10}\,] \cup \left[-1+\sqrt{10}, \infty\right) - \left\{\frac{5}{2}\right\}$

12. If $L_1 : \frac{x-a}{2} = \frac{y-2}{3} = \frac{z-b}{6}$ and $L_2 : \frac{x-b}{3} = \frac{y-7}{6} = \frac{z-1}{3}$ intersect in xy plane. Then, the value of $|a+b|$ is

(1) 15 (2) 10

(3) 14 (4) 11

Answer (1)

Sol. General point on L_1 is $(2k+a, 3k+2, 6k+b)$

General point on L_3 is $(3\lambda+b, 6\lambda+7, 3\lambda+1)$

$2K + a = 3\lambda + b$ ...(1)

$3k + 2 = 6\lambda + 7$ ...(2)

Now Lines Intersect in xy plane

$\Rightarrow 6K + b = 0$ and $3\lambda + 1 = 0$

and $\boxed{b = -6K}$ $\Rightarrow \lambda = \frac{-1}{3}$

Putting $\lambda = \frac{-1}{3}$ in eq^n (2)

$3K + 2 = -2 + 7$

$\Rightarrow 3K + 2 = 5$

$\Rightarrow \boxed{K = 1}$

$\Rightarrow \boxed{b = -6}$

Now putting value of b, k and λ in (1)

$2(1) + a = 3\left(\frac{-1}{3}\right) - 6$

$2 + a = -1 - 6$

$2 + a = -7$

$\boxed{a = -9}$

$|a+b| = |-9-6| = 15$

13. If $\frac{dy}{dx} = (2 + \ln x)$ has solution $y = y(x)$ such that $y(1) = e$ then $y(e)$ is equal to

(1) $2e - 1$ (2) $3e - 1$

(3) $2e + 1$ (4) $3e + 1$

Answer (2)

Sol. $\frac{dy}{dx} = (2 + \ln x)$

$\int dy = \int (2 + \ln x)dx$

$y = 2x + x\ln x - x + C$

$y = x + x\ln x + C$

$y(1) = e = 1 + C \Rightarrow C = e - 1$

$y(e) = e + e\ln e + e - 1$

$= 3e - 1$

14. If $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and the angle between $\vec{a}$ and $\vec{b}$ is 60°, then $3|3\vec{a}+4\vec{b}| + 4|3\vec{a}-4\vec{b}|$ is equal to____.

(1) $9\sqrt{7} + 12\sqrt{3}$ (2) $18\sqrt{7} + 24\sqrt{3}$

(3) $18\sqrt{7} - 6\sqrt{3}$ (4) $12\sqrt{7} - 6\sqrt{3}$

Answer (2)

Sol. $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos 60° = 2 \times 3 \times \frac{1}{2} = 3$

$|13\vec{a} + 4\vec{b}|^2 = |3\vec{a}|^2 + |4\vec{b}|^2 + 2(3\vec{a}.4\vec{b})$

$= 9|\vec{a}|^2 + 16|\vec{b}|^2 + 24 \times 3$

$= 9(4) + 16(9) + 72$

$= 252$

$|3\vec{a} + 4\vec{b}| = 6\sqrt{7}$

$|3\vec{a} - 4\vec{b}|^2 = |3\vec{a}|^2 + |3\vec{b}|^2 - 2(3\vec{a}.4\vec{b}) = 108$

$|3\vec{a} - 4\vec{b}| = 6\sqrt{3}$

$3|3\vec{a} + 4\vec{b}| + 4|3\vec{a} - 4\vec{b}| = 18\sqrt{7} + 24\sqrt{3}$

15. The value of $\int_{2}^{4}\sqrt{2^x-4}\,dx+\int_{0}^{2\sqrt{3}}\log_2(x^2+4)dx$ is equal to

(1) $4\sqrt{3}$ (2) $2\sqrt{3}$

(3) $8\sqrt{3}$ (4) $16\sqrt{3}$

Answer (3)

Sol. $y^2 = 2^x - 4$

$\Rightarrow y^2 + 4 = 2^x$

$\Rightarrow \log_2(y^2 + 4) = x$

$\sqrt{2^x-4}$ and $\log_2(x^2 + 4)$ are inverse of each other

$\int_{a}^{b} f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = b.f(b) - a.f(a)$

$\therefore$ $a = 2$, $b = 4$, $f(a) = 0$, $f(b) = 2\sqrt{3}$

$\therefore$ $4\cdot 2\sqrt{3} - 0 = 8\sqrt{3}$

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. If the roots of the equation $x(x + 2) + (x + 1)(x + 3) + ... + (x + (n - 1)) + (x + n + 1) = 4n$ are α and $\alpha + 2$. Then, the value of $|2\alpha + n|$ is

Answer (3)

Sol. Put $x = \alpha$

$\alpha(\alpha + 2) + (\alpha + 1)(\alpha + 3) + (\alpha + 2)(\alpha + 4)+....+(\alpha + (n -1)) + (\alpha +(n + 1))) = 4n$ ....(i)

Put $x = \alpha + 2$

$(\alpha + 2)(\alpha + 4) + (\alpha + 3)(\alpha + 5)+....+(\alpha + (n -1)) + (\alpha +(n + 1)) + (\alpha + n) + (\alpha + (n + 2)) + (\alpha + n + 1)(\alpha + n + 3) = 4n$....(ii)

Subtracting equation (ii) from (i)

$\alpha(\alpha + 2) + (\alpha + 1)(\alpha + 3) = (\alpha + n)(\alpha +(n+2)) + (\alpha + n + 1)(\alpha +(n + 3))$

$\Rightarrow \alpha^2 + 2\alpha + \alpha^2 + 4\alpha + 3 = \alpha^2 + (2n + 2)\alpha + n(\alpha + 2) + \alpha^2 + (2n + 4)\alpha + (n + 1)(n + 3)$

$\Rightarrow 6\alpha + 3 = 2n\alpha + 2\alpha + n^2 + 2n + 2n\alpha + 4\alpha + n^2 + 4n + 3$

$\Rightarrow 4n\alpha + 2n^2 + 6n = 0$

$\Rightarrow n(4\alpha + 2n + 6) = 0$

$n \neq 0$

$4\alpha + 2n = -6$

$2\alpha + n = -3$

$|2\alpha + n| = 3$

22.

23.

24.

25.

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