# Chapter 10

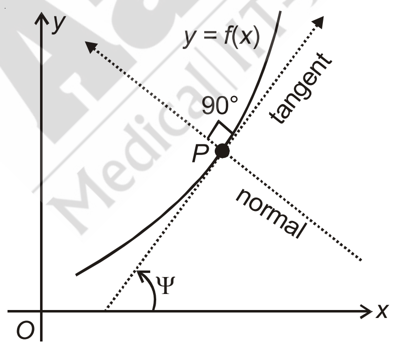
# Application of Derivatives

The notion of derivative was introduced to solve problems of physics concerning motion of an object. The velocity of an object is a measure of the rate of change of distance with respect to time. Acceleration is a measure of the rate of change of velocity with respect to time. This rate of change is precisely given by the notion of derivative. The derivative is also used to find the equation of tangent to a curve at a specific point.

The concept of derivative has been applied to other fields as well. A biologist uses it to determine the rate of growth of bacteria in a culture. An economist uses it to solve optimization problems of a company. An electrical engineer uses it to describe the change in current in an electric circut in it.

## TANGENT AND NORMAL

Let $y = f(x)$ be the equation of a curve. At a point $P$, on the curve, the value of $\frac{dy}{dx}$ (if defined) gives the slope of the tangent ($= \tan \Psi$) to the curve at the point $P$. This is also said to give the slope of the curve at the point $P$.




(i) Equation of tangent at point $P(x_1, y_1)$ to the curve $y = f(x)$ is given by the equation:

$$\boxed{y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)}$$

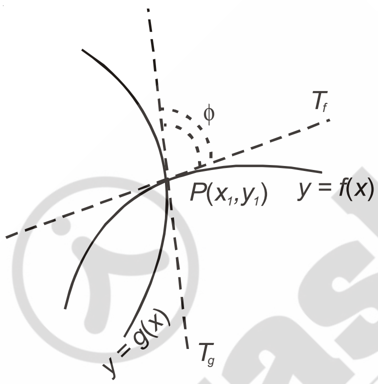
where $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ denotes value of $\frac{dy}{dx}$ at the point $(x_1, y_1)$.

(ii) Equation of normal at point $P(x_1, y_1)$ (by definition, normal is a line $\perp$ to tangent at $P$) is given by

$$y - y_1 = \left(-1\Big/\left(\frac{dy}{dx}\right)_{(x_1,y_1)}\right)(x - x_1) \qquad \left(\text{here, } \left(\frac{dy}{dx}\right)_p \neq 0\right)$$

or $$\boxed{(x - x_1) + \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(y - y_1) = 0}$$

**(iii) Angle between two curves:** Let $y = f(x)$ and $y = g(x)$ be the equation of two curves intersecting at point $P(x_1, y_1)$. Then the angle between curves $y = f(x)$ and $y = g(x)$ at their point of intersection $P$ is defined as the angle $\phi$ between the tangents $T_f$ and $T_g$ to the curves $y = f(x)$ and $y = g(x)$ respectively at their point of intersection.




Slope of $T_f \equiv \left(\frac{df(x)}{dx}\right)_{(x_1,y_1)} = f'(x_1)$

and slope of $T_g \equiv \left(\frac{dg(x)}{dx}\right)_{(x_1,y_1)} = g'(x_1)$

Then $$\boxed{\phi = \tan^{-1}\left|\frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)}\right|}$$

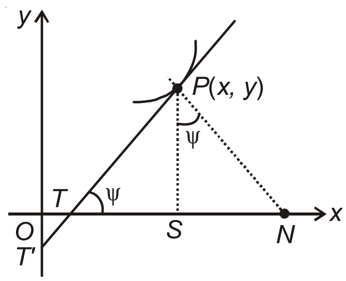
**Remarks:**

1. Curves $y = f(x)$ and $y = g(x)$ touch each other if $\phi = 0$ *i.e.*, $\boxed{f'(x_1) = g'(x_1)}$

2. Curves cut orthogonally (angle between $T_f$ and $T_g$ is 90°, $T_f \perp T_g$) if $\boxed{f'(x_1) \cdot g'(x_1) = -1}$

## LENGTH OF CARTESIAN TANGENT, NORMAL, SUB TANGENT AND SUBNORMAL

Let $P(x, y)$ be any point on the curve $y = f(x)$. Suppose the tangent and normal at $P$ on the curve meet $x$ axis in $P$ and $N$ respectively and $PS$ is the ordinate of the point $P$. $ST$ is called the length of sub tangent and $NS$ the length of subnormal at point $P$ of the curve.




(i) Length of tangent = $PT = y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

(ii) Length of sub tangent = $TS = y\frac{dx}{dy}$

(iii) Length of normal = $PN = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(iv) Length of sub normal = $SN = y\frac{dy}{dx}$

(v) Intercept cut off by tangent on $x$-axis = $x-\frac{ydx}{dy}$

(vi) Intercept cut off by tangent on $y$-axis = $x-\frac{xdy}{dx}$

## Monotonicity (Increasing & Decreasing functions)

***Algebraic Definition :*** Let $y = f(x)$ be a real valued function defined in an interval $I$, then $f(x)$ is said to be monotonic increasing (non-decreasing) in $I$ if for $x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ and strictly increasing or increasing in $I$ if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

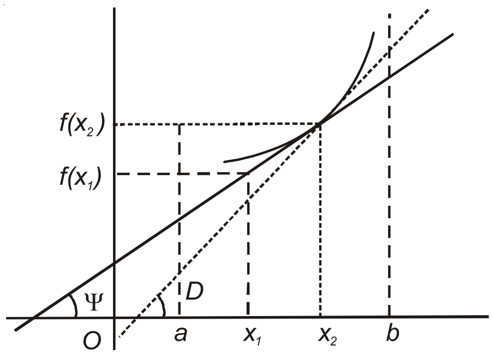
Similarly, it is said to be monotonic decreasing (non-increasing) if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ and strictly decreasing or decreasing, if

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

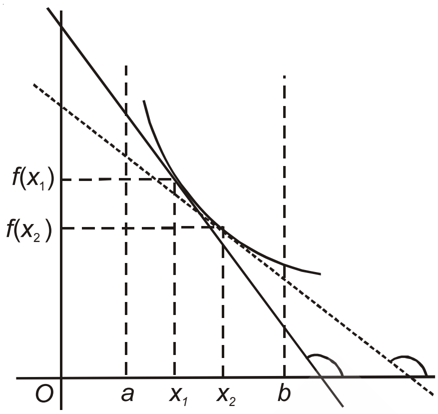
## Testing a Function for Monotonicity : (Geometrical)

Let a continuous function $f(x)$ be defined on the interval $I \equiv [a, b]$ and have a finite derivative inside this interval. Then,

(a) $f(x)$ is non-decreasing (monotonic increasing) on $[a, b]$ if and only if $f'(x) \ge 0,\ \forall\ x \in (a, b)$. It is increasing (strictly increasing) if and only if $f'(x) > 0,\ \forall\ x \in (a, b)$



(b) $f(x)$ is non-increasing (monotonic decreasing) on $[a, b]$ if and only if $f'(x) \le 0$, $\forall x \in (a, b)$. It is decreasing (strictly decreasing) if and only if $f'(x) < 0, \forall x \in (a, b)$




**Remark :**

1. A function $f(x)$ is said to be monotonic in the interval $[a,b]$ if either $f(x)$ is monotonic increasing or monotonic decreasing in $[a,b]$
2. If $f$ be monotonically increasing on $(a,b)$. Then $f(x+)$ and $f(x-)$ exist at every point $x$ of $(a,b)$. If $a < x < y < b$ then $f(x+) \le f(y-)$
3. It should be noted that $f(x)$ has no discontinuities of second kind.
4. A strictly increasing or strictly decreasing continuous function is one-one but converse is not true.
5. Let $f(x)$ be monotonic in $[a,b]$ then $f(x)$ has at most one zero in $[a,b]$.

## MAXIMA AND MINIMA

If a function $y = f(x)$ is defined on the interval $I$, then an interior point $x_0$ of this interval is called the point of maximum of the function $f(x)$ if there exists a neighborhood $U$ of point $x_0$ such that inequality $f(x) \le f(x_0)$ holds true within it or equivalently for small $h$ (positive) $f(x) \le f(x_0) \forall x \in (x_0 - h, x_0 + h)$. Similarly, $x_0$ is called the point of minimum if $f(x) \ge f(x_0)$ $\forall x \in (x_0 - h, x_0 + h)$.

***Note :*** *1 The generic terms used for points of maximum or minimum of a function are the points of extremum.*

*2. Maxima and minima are local properties (therefore we also use the terms local maxima/minima or relative maxima/minima)*

## The Necessary Condition for Existence of an Extremum

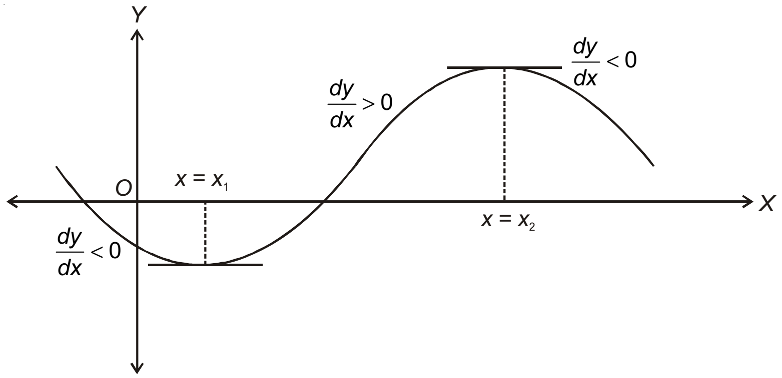
Let $f(x)$ be defined in the domain $D$. Then $x_0 \in D$ is point of extremum if (i) $f'(x_0) = 0$ or (ii) $f'(x_0)$ does not exist but is continuous at $x_0$.

***Stationary point :*** Point $x_0$ is called stationary point if $f'(x) = 0$.

***Critical Point:*** The points $x_0$ at which $f'(x_0) = 0$ or does not exist are called critical points. So maximum and minimum can exist only at critical points.

### Sufficient conditions for existence of an extremum :

1. **First derivative condition:** Let the function $f(x)$ be continuous at some neighborhood of $x_1$ and $x_2$ (critical points.)




**Case I:** If $f'(x) < 0$ for $x < x_1$ and $f'(x) > 0$ for $x > x_1$ $(x \in (x_1 - h, x_1 + h))$ (If moving from left to right through the point $x_1$ the derivative changes sign from – ve to +ve), then at the point $x_1$ the function attains a minimum.

**Case II:** If $f'(x) > 0$ for $x < x_2$ and $f'(x) < 0$ for $x > x_2$ (If moving from left to right through the point $x_2$ the derivative changes from +ve to –ve), then at the point $x_2$ the function attains a maximum

**Case III:** If the derivative does not change sign in moving through the critical point, then there is no extremum.

2. **Second derivative condition:** Let the function be twice differentiable at the critical point, $x_0$, $f'(x_0) = 0$: If $f''(x_0) < 0$ then at $x_0$ the function has a maximum, if $f''(x_0) > 0$ then at $x_0$ the function has a minimum. But if $f''(x_0) = 0$, then existence of an extremum at this point remains open (i.e. cannot be decided by this condition)

3. **Higher order derivatives condition:**

Let $n \geq 2$ and $f'(x_0) = f''(x_0) = 0 = f'''(x_0) = \ldots\ldots\ldots\ldots = f^{(n-1)}(x_0) = 0$, *but* $f^{(n)}(x_0) \neq 0$ (where $f^{(n)}(x)$ denotes $n$th derivative).

If $n$ is even $\Rightarrow \begin{cases} f^{(n)}(x_0) < 0, & x_0 \text{ point of maximum} \\ f^{(n)}(x_0) > 0, & x_0 \text{ is point of minimum} \end{cases}$

If $n$ is odd, then there is no extremum at the point $x_0$.

**Note :** *It should be noted that maxima/minima are local properties of the function, therefore function may have many points of local maximum and local minimum and local maximum values may be less than local minimum values of function.*

## Maxima and Minima of Functions of Parametric Forms

Let a function $y = f(x)$ be represented parametrically

$$x = \phi(t),\ y = \Psi(t)$$

Where $\phi(t)$ and $\Psi(t)$ have derivatives upto second order within a certain interval of change of the argument (parameter $t$) and $\phi'(t) \neq 0$. If at a point $t = t_0$, $\Psi'(t_0) = 0$

Then,

(i) If $\Psi''(t_0) < 0$, the function $y = f(x)$ has a maximum at $t = (t_0)$ or $x = \phi(t_0)$

(ii) If $\Psi''(t_0) > 0$, the function $y = f(x)$ has a minimum at $t = (t_0)$ or $x = \phi(t_0)$

(iii) If $\Psi''(t_0) = 0$, then existence of maxima/minima remains open (cannot be decided).

Remember the points where $\phi'(t) = 0$ needs special considerations.

**Greatest and least values of a function (Absolute maximum and absolute minimum):** The greatest (or least) value of a continuous function on [*a, b*] is attained either at the critical point or at either of the end-points *a* and *b*.

**To find abs. maximum or abs. minimum :**

(i) First locate the critical points of *f* in (*a,b*)

(ii) Then find the value of *f* at each of these critical points and at the two ends points *a* and *b*

Now The greatest value of $f(x) \equiv \max\{f(a), f(b), \text{values at critical points}\}$

The least value of $f(x) \equiv \min\{f(a), f(b), \text{values at critical points}\}$

## BASIC THEOREMS ON DIFFERENTIABLE FUNCTIONS

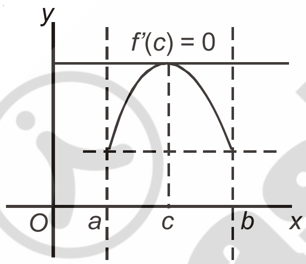
### Rolle's Theorem

Let a real valued function $f(x)$ be defined on [*a, b*] such that

(i) $f(x)$ is continuous in [*a, b*]

(ii) $f(x)$ has finite derivative at all point of (*a, b*) and

(iii) $f(a) = f(b)$



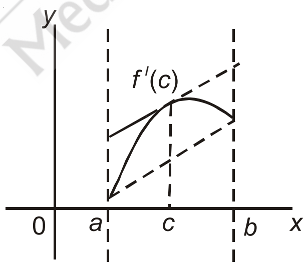

then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Geometrically speaking, at point $(c, f(c))$ on the graph of function $f(x)$, the tangent is parallel to *x*-axis.

### Lagrange's Mean Value Theorem

Let a real valued function $f(x)$ be defined on [*a, b*] be such that it is continuous in [*a, b*] and differentiable in (*a, b*), then there exists $c \in (a, b)$

such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$




Geometrically speaking, at point $(c, f(c))$ on the graph of the function $f(x)$ tangent is parallel to the chord joining points $(a, f(a))$ and $(b, f(b))$

### Test for the constancy of a function

If at all points of certain interval *I*, $f'(x) = 0$, then the function $f(x)$ preserves a constant value within this interval.

i.e., $f'(x) = 0,\ \forall x \in I \Rightarrow f(x) = \text{constant},\ \forall x \in I$.

## Rate of Change of Quantities

Let $y = f(x)$ be a function of $x$ and $x$ is a function (of time), then $y$ is also a function of $t$.

$$\text{Now} \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \boxed{\frac{\text{rate of change of } y}{\text{rate of change of } x}}$$

$\frac{dy}{dx}$ is equal to the ratio of the rate of change of $y$ and rate of change of $x$. So $\frac{dy}{dx}$ represents instantaneous rate of change in $y$ with respect to $x$.

## VELOCITY AND ACCELERATION

$$\boxed{\text{Velocity} = \frac{ds}{dt} = \lim_{\delta t \to 0} \frac{\delta s}{\delta t}} \quad \text{and} \quad \boxed{\text{acceleration} = \frac{d^2 s}{dt^2}}$$

where $s$ is distance covered in time $t$ and $s + \delta s$ is the distance covered in time $t + \delta t$

Approximate change in the value of the function $y = f(x)$ is $\boxed{\delta y = \frac{dy}{dx}\delta x}$

where $\delta y$ = approximate change in $y$.

Approximate value of function $y = f(x)$ at $x = a$ is $\boxed{f(a+h) = f(a) + hf'(a)}$; where $h \to 0$

## Problem Based on Approximate Value

**Working Rule:**

1. Take suitable function according to the given quantity
2. Then find $f'(x)$ and put in $f(a + h) = f(a) + h\, f'(a)$
3. Put number in place of a (in case of decimal value $a$ is taken as nearest integer) and $h$ such that $f(a + h)$ becomes equal to the quantity whose approximate value is to be calculated.
4. If approximate change in the value of function $y = f(x)$ is to be found use $\delta y = \frac{dy}{dx}\delta x$.

## Useful Formulae of Mensuration to Remember

1. Volume of a cuboid $= \ell bh$
2. Surface area of cuboid $= 2(\ell b + bh + h\ell)$
3. Volume of cube = $a^3$
4. Surface area of cube = $6a^2$
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$
6. Curved surface area of cone $= \pi r\ell (\ell =$ slant height)
7. Curved surface area of a cylinder = $2\pi rh$
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$

9. Volume of a sphere = $\frac{4}{3}\pi r^3$
10. Surface area of a sphere = $4\pi r^2$
11. Area of a circular sector = $\frac{1}{2}r^2\theta$, when $\theta$ is in radians.
12. Volume of a prism = (area of the base) × (height)
13. Lateral surface area of a prism = (perimeter of the base) × (height)
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
    (Note that lateral surfaces of a prism are all rectangle)
15. Volume of a pyramid = $\frac{1}{3}$(area of the base) × (height)
16. Curved surface area of a pyramid = $\frac{1}{2}$(perimeter of the base) × (slant height).
    (Note that slant surfaces of a pyramid are triangles)

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