

Chapter 6

Binomial Theorem and its Applications

BINOMIAL EXPRESSION

An algebraic expression containing only two terms is called a binomial expression. Such as $a + x$, $2x + 5y$,

$2x + \frac{1}{y}$, $a + \frac{b}{x}$ etc. all are binomial expressions.

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

If $x, y \in R$, then $\forall n \in N$

$$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_n y^n$$

Here all nC_r are called binomial coefficient

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$${}^nC_0 = 1 = {}^nC_n$$
$${}^nC_1 = n$$

$${}^nC_r = {}^nC_{n-r}; 0 \leq r \leq n$$

Note : (i) n is the exponent or index of binomial

(ii) Total number of terms is $n + 1$

(iii) The $(r + 1)^{\text{th}}$ term $= T_{r+1} = {}^nC_r x^{n-r} y^r$

(iv) Sum of exponents of x and y in any term $= n$

(v) Powers of x decreasing by 1 and powers of y increasing by 1 in subsequent terms

(vi) In any term suffix of binomial coefficient is power of y

(a) Replacing y with $-y$ in the expansion of $(x + y)^n$, we get

$$(x - y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 - \dots + (-1)^n {}^nC_n y^n$$

(b) Replacing x by 1 and y with x , we get

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$$

$$(c) (1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - \dots + (-1)^n {}^nC_n x^n$$

$$(d) (x+y)^n + (x-y)^n = 2[{}^nC_0x^n + {}^nC_2x^{n-2}y^2 + \dots]$$

$$= 2[\text{Sum of terms at odd places}]$$

$$\text{last term } {}^nC_n y^n \text{ if } n \text{ is even and } {}^nC_{n-1}xy^{n-1} \text{ if } n \text{ is odd}$$

$$(e) (x+y)^n - (x-y)^n = 2[\text{Sum of terms at even places}]$$

$$\text{last term } {}^nC_{n-1}xy^{n-1} \text{ if } n \text{ is even and } {}^nC_n y^n \text{ if } n \text{ is odd}$$

General term in the expansion of $(x+y)^n$

The $(r+1)^{\text{th}}$ term = T_{r+1} is called the general term.

$$T_{r+1} = {}^nC_r x^{n-r} y^r$$

Middle term in the expansion of $(x+y)^n$

Case 1.

$$n = 2m; m \in N$$

Then, number of terms in the expansion = $n+1 = 2m+1$; Thus middle term is T_{m+1}

$$\text{i.e. } \left(\frac{n}{2} + 1\right)^{\text{th}} \text{ term}$$

$$T_{m+1} = {}^nC_{\frac{n}{2}} x^{\frac{n}{2}} \cdot y^{\frac{n}{2}}$$

Case 2.

$$n = 2m+1; m \in N$$

Then, number of terms in the expansion is = $n+1 = 2m+2$. There are two middle terms, viz. T_{m+1} and T_{m+2} .

$$\text{i.e. } \left(\frac{n+1}{2}\right)^{\text{th}} \text{ and } \left(\frac{n+3}{2}\right)^{\text{th}} \text{ term}$$

$$T_{m+1} = {}^nC_{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}} \cdot y^{\frac{n-1}{2}}$$

$$T_{m+2} = {}^nC_{\frac{n+1}{2}} \cdot x^{\frac{n-1}{2}} \cdot y^{\frac{n+1}{2}}$$

p^{th} term from the end in the expansion of $(x+y)^n$

p^{th} term from end = $(n+2-p)^{\text{th}}$ term from beginning

$$= {}^nC_{n-p+1} x^{p-1} y^{(n-p+1)}$$

Numerically Greatest Term in $(1 + x)^n$

The method for finding the greatest term is as below :

1. Find the value of $K = \frac{(n+1)|x|}{1+|x|}$.
2. If K is an integer, then T_K and T_{K+1} both are equal and greatest, term in the given expansion.
3. If K is not an integer then $T_{[K]+1}$ is the greatest term where $[K]$ is the greatest integer less than or equal to K .
4. To find the greatest term in $(x + y)^n = x^n \left(1 + \frac{y}{x}\right)^n$, find the greatest term in $\left(1 + \frac{y}{x}\right)^n$ and then multiply it with x^n .

PROPERTIES OF BINOMIAL COEFFICIENTS

In the expansion of $(1 + x)^n$, the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are denoted by $C_0, C_1, C_2, \dots, C_n$, then

1. If n is even, then the greatest coefficient is ${}^nC_{n/2}$.
2. If n is odd, then the greatest coefficients are ${}^nC_{(n-1)/2}$ or ${}^nC_{(n+1)/2}$.
3. $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$, $0 < r \leq n$.
4. $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$
5. ${}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1}$
6. $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$
7. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$
8. ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$
9. ${}^nC_r = {}^nC_{n-r}$

Note : (1) The number of terms in the expansion of $(x + y + z)^n$ is $\frac{(n+1)(n+2)}{2}$ where $n \in N$.

(2) Sum of all coefficients in the expansion of $(x_1 + x_2 + \dots + x_k)^n$ is obtained by keeping each $x_i = 1$, so it is k^n .

Some Relation Involving Binomial Coefficients

$$(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n \quad \dots(1)$$

Relation (1) is important. Many properties of binomial coefficients are obtained by

- (i) Giving different values to x in (1)
- (ii) Differentiating (1) with respect to x and then putting convenient values of x , and/or
- (iii) Integrating (1) with respect to x and then putting convenient values of x .

Some Important Relations are :

$$1. \sum_{r=0}^n {}^nC_r = C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$$2. C_0 + C_2 + C_4 + \dots = C_1 + C_3 + \dots = 2^{n-1}$$

$$3. \sum_{r=1 \text{ or } 0}^n r \cdot {}^nC_r = C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1}$$

$$4. \sum_{r=0}^n \frac{{}^nC_r}{r+1} = \frac{2^{n+1} - 1}{n+1}$$

$$5. C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n = 0$$

$$6. C_0^2 + C_1^2 + \dots + C_n^2 = \sum_{r=0}^n ({}^nC_r)^2 = {}^{2n}C_n = \frac{2n!}{(n!)^2}$$

$$7. C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots + (-1)^n C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$$

$$8. C_0C_1 + C_1C_2 + \dots + C_{n-1}C_n = {}^{2n}C_{n-1}$$

□ □ □