Date: 09/03/2024

Medical | IIT-JEE | Foundations

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Time: 3 hrs. Max. Marks: 80

MATHEMATICS

Class-XII

(CBSE 2023-24)

Answers & Solutions

GENERAL INSTRUCTIONS

Read the following instructions very carefully and follow them :

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) Question paper is divided into **FIVE** sections – Section **A**, **B**, **C**, **D** and **E**.

(iii) **In Section – A :** Questions Number **1** to **18** are Multiple Choice Questions (MCQs) type and Questions Number 19 & 20 are Assertion-Reason based questions of **1** mark each.

(iv) **In Section – B :** Questions Number **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.

(v) **In Section – C :** Questions Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.

(vi) **In Section – D :** Questions Number **32** to **35** are Long Answer (LA) type questions, carrying **5** marks each.

(vii) **In Section – E :** Questions Number **36** to **38** are case study based questions, carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section – B, 3 questions in Section – C, 2 questions in Section – D and 2 questions in Section – E.

(ix) Use of calculators is **NOT** allowed.

SECTION - A

This section has **20** multiple choice questions of **1 mark** each.

1. Derivative of $e^{\sin^2 x}$ with respect to $\cos x$ is

(A) $\sin x\, e^{\sin^2 x}$ (B) $\cos x\, e^{\sin^2 x}$

(C) $-2\cos x\, e^{\sin^2 x}$ (D) $-2\sin^2 x \cos x\, e^{\sin^2 x}$

Answer (C)

Sol. Let $P = e^{\sin^2} x$ ...(1)

$Q = \cos x$ ...(2)

Differentiating equation (1) w.r.t. 'x'

We get $\frac{dP}{dx} = \frac{d}{dx}\left(e^{\sin^2 x}\right)$

$$\frac{dP}{dx} = e^{\sin^2 x}\frac{d}{dx}(\sin^2 x)$$

$$\frac{dP}{dx} = e^{\sin^2 x}(2\sin x \cos x)$$

$$\frac{dP}{dx} = e^{\sin^2 x}(\sin 2x) \quad \text{...(3)}$$

Differentiating equation (2) w.r.t. 'x'

$$\frac{dQ}{dx} = \frac{d}{dx}(\cos x)$$

$$\frac{d(Q)}{dx} = -\sin x \quad \text{...(4)}$$

Dividing equation (3) and (4)

$$\frac{\frac{dP}{dx}}{\frac{dQ}{dx}} = \frac{e^{\sin^2 x}(\sin 2x)}{-\sin x}$$

$$\frac{\frac{dP}{dx}}{\frac{dQ}{dx}} = \frac{e^{\sin^2 x}(2\sin x \cos x)}{-\sin x}$$

$$\frac{dP}{dQ} = -2e^{\sin^2 x}\cos x$$

2. If A is a square matrix of order 2 and |A| = –2, then value of |5A′| is

(A) –50 (B) –10

(C) 10 (D) 50

Answer (A)

Sol. Given A is square matrix

|A| = –2

NOTE: We know if A is n × n matrix

Then $|kA| = k^n|A|$ where k is scalar

$\therefore\ |5A'| = 5^2\,|A'|$

$|5A'| = 5^2\,|A|\quad [\because\ |A'| = |A|]$

$|5A'| = 25|A|$

Given $|A| = -2$

$\therefore\ |5A'| = 25(-2)$

$= -50$

3. The function $f(x)=\dfrac{x}{2}+\dfrac{2}{x}$ has a local minima at x equal to

(A) 2

(B) 1

(C) 0

(D) –2

Answer (A)

Sol. Given $f(x)=\dfrac{x}{2}+\dfrac{2}{x}$

Differentiate both sides w.r.t. ‘x’

$$f'(x)=\frac{d}{dx}\left(\frac{x}{2}+\frac{2}{x}\right)$$

$$f'(x)=\frac{d}{dx}\left(\frac{x}{2}\right)+\frac{d}{dx}\left(\frac{2}{x}\right)$$

$$f'(x)=\frac{1}{2}-\frac{2}{x^2} \qquad \ldots(1)$$

We know function has local minima at ‘x’

If $f'(x) = 0$ and $f''(x) > 0$

$\therefore$ from (1) put $f'(x) = 0$

$$\frac{1}{2}-\frac{2}{x^2}=0$$

$$\frac{1}{2}=\frac{2}{x^2}$$

$\Rightarrow\ x^2 = 4$

$\Rightarrow\ x = \pm 2$

Differentiating (1) w.r.t. ‘x’

$$f''(x)=\frac{d}{dx}\left(\frac{1}{2}-\frac{2}{x^2}\right)$$

$$f''(x)=-\left(\frac{2(-2)}{x^3}\right)$$

$$f''(x)=\frac{4}{x^3}$$

At $x = -2$

$$f''(-2) = \frac{4}{(-2)^3}$$

$$f''(-2) = \frac{-4}{8} < 0$$

$\therefore$ We get local maxima at x = –2

At x = 2

$$f''(2) = \frac{4}{2^3}$$

$$= \frac{4}{8}$$

$$f''(2) = \frac{1}{2} > 0$$

$\therefore$ We get local minima at x = 2

4. Given a curve $y = 7x - x^3$ and *x* increases at the rate of 2 units per second. The rate at which the slope of the curve is changing, when x = 5 is

(A) –60 units/sec
(B) 60 units/sec
(C) –70 units/sec
(D) –140 units/sec

Answer (A)

Sol. Given $y = 7x - x^3$

Differentiating both sides w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(7x - x^3)$$

$$\frac{dy}{dx} = \frac{d}{dx}(7x) - \frac{d}{dx}(x^3)$$

$$m = \frac{dy}{dx} = 7 - 3x^2 \quad \ldots(1)$$

Where m be the slope of $y = 7x - x^3$

$\therefore$ $m = 7 - 3x^2$

Given that slope is changing

$\therefore$ Differentiating (1) w.r.t 't'

$$\frac{dm}{dt} = -6x\frac{dx}{dt} \quad \ldots(2)$$

As $\frac{dx}{dt}$ = 2 units/sec. and x = 5

$\therefore$ from (2) $\frac{dm}{dt} = -6(5)(2)$

$$\frac{dm}{dt} = -60$$

5. The product of matrix P and Q is equal to a diagonal matrix. If the order of matrix Q is 3 × 2, then order of matrix P is

(A) 2 × 2 (B) 3 × 3

(C) 2 × 3 (D) 3 × 2

Answer (C)

Sol. Let P is m × n matrix

Given PQ is diagonal matrix

We know diagonal matrix is always square matrix.

$\Rightarrow$ PQ is square matrix

$\Rightarrow$ PQ is defined

Given Q is 3 × 2 matrix

$\therefore$ For PQ has to be square matrix

m has to be 2 and n has to be 3

i.e. m = 2

n = 3

6. A function $f : R \to R$ defined as $f(x) = x^2 - 4x + 5$ is

(A) injective but not surjective. (B) surjective but not injective.

(C) both injective and surjective. (D) neither injective nor surjective.

Answer (D)

Sol. Given $f : R \to R$ $f(x) = x^2 - 4x + 5$

For one-one

We known if $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

Where $x_1, x_2 \in R$

$\therefore f(x_1) = x_1^2 - 4x_1 + 5$ ...(1)

$f(x_2) = x_2^2 - 4x_2 + 5$ ...(2)

equating (1) and (2)

$x_1^2 - 4x_1 + 5 = x_2^2 - 4x_2 + 5$

$x_1^2 - x_2^2 - 4x_1 + 4x_2 = 0$

$(x_1 - x_2)(x_1 + x_2) - 4(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(x_1 + x_2 - 4) = 0$

$\Rightarrow x_1 + x_2 - 4 = 0$

$\Rightarrow x_1 = 4 - x_2$

$\therefore x_1 \neq x_2$

$\therefore$ f(x) in not one-one.

$y = x^2 - 4x + 5$

$y = (x - 2)^2 + 5 - 4$

$y = (x - 2)^2 + 1$

As $(x - 2)^2 \geq 0$

$\Rightarrow \quad y - 1 \geq 0$

$\Rightarrow \quad y \geq 1$

$\therefore$ Range (f) $\in [1, \infty)$

And codomain $\in$ R

$\therefore$ Range $\neq$ Codomain

$\therefore$ f(x) is not onto

7. If sin(xy) = 1, then $\frac{dy}{dx}$ is equal to

(A) $\frac{x}{y}$

(B) $-\frac{x}{y}$

(C) $\frac{y}{x}$

(D) $-\frac{y}{x}$

Answer (D)

Sol. Given sin(xy) = 1

Differentiating both sides w.r.t 'x'

$$\cos(xy)\frac{d}{dx}(xy) = 0$$

$$\cos xy\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} = -y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$$

8. If inverse of matrix $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then value of λ is

(A) –4

(B) 1

(C) 3

(D) 4

Answer (D)

Sol. A = $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

And $\quad A^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{bmatrix}$

We know $A A^{-1} = I$

$|A A^{-1}| = |I|$

$|A| \cdot |A^{-1}| = 1$ $[\because |AB| = |A|\,|B|$ ...(1)

Now $A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$|A| = 7(1) + 3(-1) - 3(+1)$

$|A| = 7 - 3 - 3$

$|A| = 1$

$\therefore$ $|A^{-1}| = \begin{bmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$= 1(4\lambda - 9) - 3(4 - 3) + 3(3 - \lambda)$

$= 4\lambda - 9 - 3(1) + 3(3 - \lambda)$

$= 4\lambda - 9 - 3 + 9 - 3\lambda$

$\boxed{|A^{-1}| = \lambda - 3}$

$\therefore$ from (1)

$(\lambda - 3) = 1$

$\lambda - 3 = 1$

$\boxed{\lambda = 4}$

9. Find the matrix A^2, where $A = [a_{ij}]$ is a 2×2 matrix whose elements are given by a_{ij} = maximum (i, j) – minimum (i, j)

(A) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Answer (C)

Sol. Given $A = [a_{ij}]$ is 2×2 matrix

and a_{ij} = maximum (i, j) – minimum (i, j)

$\therefore$ a_{11} = max (1, 1) – min (1, 1)

$a_{11} = 1 - 1 = 0$

$\boxed{a_{11} = 0}$

a_{12} = max (1, 2) – min (1, 2)

$a_{12} = 2 - 1$

$\boxed{a_{12} = 1}$

a_{21} = max(2, 1) – min(2, 1)

$a_{21} = 2 - 1$

$\boxed{a_{21} = 1}$

$a_{22} = \max(2, 2) - \min(2, 2)$

$a_{22} = 2 - 2$

$\boxed{a_{22} = 0}$

$\therefore \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

10. If A is a square matrix of order 3 such that the value of $|adj \cdot A| = 8$, then the value of $|A^T|$ is

(A) $\sqrt{2}$

(B) $-\sqrt{2}$

(C) 8

(D) $2\sqrt{2}$

Answer (D)

Sol. If A is n × n matrix then

$|adj\ A| = |A|^{n-1}$ ...(1)

Given A is 3 × 3 matrix, $|adj\ A| = 8$

$\therefore$ from (1) $8 = |A|^{n-1}$

Put n = 3

$8 = |A|^2$

$|A| = 2\sqrt{2}$

Also $|A'| = |A|$

$\therefore \quad \boxed{|A'| = 2\sqrt{2}}$

11. The value of $\int_{\pi/4}^{\pi/2} \cot\theta \operatorname{cosec}^2\theta \, d\theta$ is :

(A) $\frac{1}{2}$

(B) $-\frac{1}{2}$

(C) 0

(D) $-\frac{\pi}{8}$

Answer (A)

Sol. $\int_{\pi/4}^{\pi/2} \cot\theta \operatorname{cosec}^2\theta \, d\theta$ ...(i)

Let $\cot\theta = t$

Differentiating both sides,

$-\operatorname{cosec}^2\theta d\theta = dt$

$\Rightarrow \quad \operatorname{cosec}^2\theta \, d\theta = -dt$

When $\theta = \frac{\pi}{4}$ $\Rightarrow$ $t = \cot\frac{\pi}{4} = 1$

When $\theta = \frac{\pi}{2}$ $\Rightarrow$ $t = \cot\frac{\pi}{2} = 0$

Hence, equation (i) become

$$-\int_1^0 t\,dt = \int_0^1 t\,dt \quad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)dx\right]$$

$$= \left[\frac{t^2}{2}\right]_0^1$$

$$= \left(\frac{1}{2} - 0\right)$$

$$= \frac{1}{2}$$

So, option (A) is correct.

12. The integral $\int \frac{dx}{\sqrt{9-4x^2}}$ is equal to :

(A) $\frac{1}{6}\sin^{-1}\left(\frac{2x}{3}\right) + c$

(B) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$

(C) $\sin^{-1}\left(\frac{2x}{3}\right) + c$

(D) $\frac{3}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$

Answer (B)

Sol. $\int \frac{1}{\sqrt{9-4x^2}}dx = \int \frac{1}{\sqrt{9\left(1-\frac{4x^2}{3}\right)}}dx$

$$= \int \frac{1}{3\sqrt{1-\left(\frac{2}{3}x\right)^2}}dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{2}{3}x\right)^2}}dx \qquad \text{...(i)}$$

Let $\frac{2}{3}x = \sin t$

Differentiating both sides

$\frac{2}{3}dx = \cos t\,dt$

$\Rightarrow$ $dx = \frac{3}{2}\cos t\,dt$

Hence, equation (i) becomes

$$=\frac{1}{3}\int\frac{1}{\sqrt{1-\sin^2 t}}\cdot\frac{3}{2}\cos t\,dt$$

$$=\frac{1}{3}\times\frac{3}{2}\int\frac{\cos t}{\sqrt{\cos^2 t}}\,dt\ [\sin^2 t+\cos^2 t=1]$$

$$=\frac{1}{2}\int 1\,dt=\frac{1}{2}t+c$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+c$$

Hence, option (B) is correct.

13. The area of the region bounded by the curve $y^2 = 4x$ and $x = 1$ is :

(A) $\frac{4}{3}$

(B) $\frac{8}{3}$

(C) $\frac{64}{3}$

(D) $\frac{32}{3}$

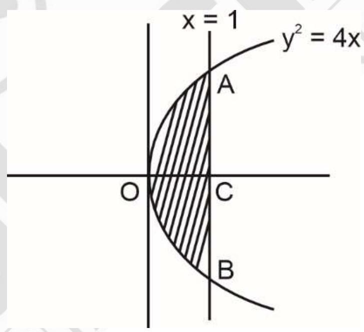
Answer (B)

Sol. Let AB represents the line $x = 1$

and AOB represents the curve $y^2 = 4x$

Area of AOBC = 2 × [Area of AOC]

$$=2\times\int_0^1 y\,dx$$



We know that

$y^2 = 4x$

$y=\pm\sqrt{4x}$

$y=\pm 2\sqrt{x}$

As AOC is in 1st quadrant

$y=2\sqrt{x}$

$$\therefore\quad \text{Area of AOBC}=2\times\int_0^1 y\,dx$$

$$=2\times\int_0^1 2\sqrt{x}\,dx$$

$$=4\int_0^1\sqrt{x}\,dx$$

$$=4\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^1$$

$$= 4 \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1$$

$$= \frac{8}{3}[1-0]$$

$$= \frac{8}{3}$$

Hence, option (B) is correct.

14. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is :

(A) $e^x + e^{-y} = c$

(B) $e^{-x} + e^{-y} = c$

(C) $e^{x+y} = c$

(D) $2e^{x+y} = c$

Answer (A)

Sol. $\frac{dy}{dx} = e^{x+y}$

$\Rightarrow \frac{dy}{dx} = e^x \cdot e^y$

$\Rightarrow \frac{dy}{ey} = e^x dx$

$\Rightarrow e^{-y}\, dy = e^x\, dx$

Integrating both sides, we get

$\Rightarrow -e^{-y} = e^x + k$

$\Rightarrow e^x + e^{-y} = c$

Hence, option (A) is correct.

15. The angle which the line $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$ makes with the positive direction of Y-axis is :

(A) $\frac{5\pi}{6}$

(B) $\frac{3\pi}{4}$

(C) $\frac{5\pi}{4}$

(D) $\frac{7\pi}{4}$

Answer (B)

Sol. Given, $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$

Direction ratio of y-axis is (0, 1, 0) and direction ratio of the given line is (1, –1, 0)

$$\therefore \quad \cos\theta = \frac{(0)(1)+(1)(-1)+(0)(0)}{\sqrt{0^2+1^2+0^2}\sqrt{1^2+(-1)^2+0^2}}$$

$$\cos\theta = \frac{-1}{\sqrt{2}}$$

$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$\Rightarrow \quad \theta = \frac{3\pi}{4}$

Hence, option (B) is correct.

16. The Cartesian equation of the line passing through the point (1, –3, 2) and parallel to the line :

$\vec{r} = (2+\lambda)\hat{i} + \lambda\hat{j} + (2\lambda - 1)\hat{k}$ is

(A) $\frac{x-1}{2} = \frac{y+3}{0} = \frac{z-2}{-1}$

(B) $\frac{x+1}{1} = \frac{y-3}{1} = \frac{z+2}{2}$

(C) $\frac{x+1}{2} = \frac{y-3}{0} = \frac{z+2}{-1}$

(D) $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}$

Answer (D)

Sol. Line passes through the point A(1, –3, 2)

$\therefore$ Position vector of the point is $\vec{a} = \hat{i} - 3\hat{j} + 2\hat{k}$

Also, the required line is parallel to the line

$\vec{r} = (2+\lambda)\hat{i} + \lambda\hat{j} + (2\lambda - 1)\hat{k}$

$\vec{r} = 2\hat{i} - \hat{k} + \lambda\left(\hat{i} + \hat{j} + 2\hat{k}\right)$

$\therefore$ It is parallel to the vector

$\vec{b} = \hat{i} + \hat{j} + 2\hat{k}$

The vector equation of the line passing through $A(\vec{a})$ and parallel to $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a scalar.

$\therefore$ The required vector equation of the line is

$\vec{r} = \left(\hat{i} - 3\hat{j} + 2\hat{k}\right) + \lambda\left(\hat{i} + \hat{j} + 2\hat{k}\right)$

and required cartesian equation of the above line is

$\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}$

Hence, option (D) is correct.

17. If A and B are events such that P(A/B) = P(B/A) $\neq$ 0, then :

(A) $A \subset B$, but $A \neq B$

(B) A = B

(C) $A \cap B = \phi$

(D) P(A) = P(B)

Answer (D)

Sol. $P\left(\frac{A}{B}\right) = P\left(\frac{B}{A}\right) \neq 0$

$\Rightarrow \quad \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} \neq 0$

$\Rightarrow$ P(A) = P(B) $\neq$ 0

Hence, option (D) is correct.

18. The position vectors of points P and Q are $\vec{p}$ and $\vec{q}$ respectively. The point R divides line segment PQ in the ratio 3 : 1 and S is the mid-point of line segment PR. The position vector of S is :

(A) $\dfrac{\vec{p}+3\vec{q}}{4}$

(B) $\dfrac{\vec{p}+3\vec{q}}{8}$

(C) $\dfrac{5\vec{p}+3\vec{q}}{4}$

(D) $\dfrac{5\vec{p}+3\vec{q}}{8}$

Answer (D)

Sol. Given, Position vector of point P, $\overrightarrow{OP}=\vec{p}$

Position vector of point Q, $\overrightarrow{OQ}=\vec{q}$

$\because$ Point R divides line segment PQ in the ratio 3 : 1

$\therefore$ Position vector of point R, $\overrightarrow{OR}=\dfrac{3\overrightarrow{OQ}+\overrightarrow{OP}}{4}$

$$=\frac{3\vec{q}+\vec{p}}{4}$$

Also, S is the mid-point of PR

$\therefore$ Position vector of point S, $\overrightarrow{OS}=\dfrac{\overrightarrow{OP}+\overrightarrow{OR}}{2}$

$$=\frac{\vec{p}+\left(\dfrac{3\vec{q}+\vec{p}}{4}\right)}{2}$$

$$=\frac{5\vec{p}+3\vec{q}}{8}$$

Hence, option (D) is correct.

Assertion – Reason Based Questions

Direction : In questions numbers **19** and **20**, two statements are given one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the following options :

(A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

19. **Assertion (A) :** The vectors

$\vec{a}=6\hat{i}+2\hat{j}-8\hat{k}$

$\vec{b}=10\hat{i}-2\hat{j}-6\hat{k}$

$\vec{c}=4\hat{i}-4\hat{j}+2\hat{k}$

represent the sides of a right-angled triangle.

Reason (R) : Three non-zero vectors of which none of two are collinear forms a triangle if their resultant is zero vector or sum of any two vectors is equal to the third.

Answer (B)

Sol. Given,

$$\vec{a} = 6\hat{i} + 2\hat{j} - 8\hat{k}$$

$$\vec{b} = 10\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\vec{c} = 4\hat{i} - 4\hat{j} + 2\hat{k}$$

Let ABC be a triangle such that

$$\overrightarrow{AB} = \vec{a}$$

$$\overrightarrow{BC} = \vec{b}$$

and $\overrightarrow{AC} = \vec{c}$

Hence, $|\overrightarrow{AB}| = |\vec{a}| = \sqrt{6^2 + 2^2 + (-8)^2}$

$$= \sqrt{104}$$

$$|\overrightarrow{BC}| = |\vec{b}| = \sqrt{10^2 + (-2)^2 + (-6)^2}$$

$$= \sqrt{140}$$

$$|\overrightarrow{AC}| = |\vec{c}| = \sqrt{4^2 + (-4)^2 + 2^2}$$

$$= \sqrt{36}$$

$$= 6$$

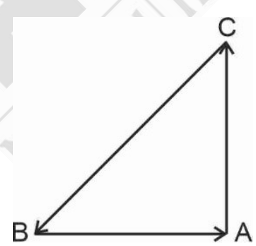
As, we can observe that

$AB^2 + AC^2 = 104 + 36$

$= 140$

$= BC^2$

So, ΔABC is a right-angled triangle



Also, $\vec{a} = 6\hat{i} + 2\hat{j} - 8\hat{k}$

and $\vec{c} = 4\hat{i} - 4\hat{j} + 2\hat{k}$

$\Rightarrow$ $\vec{a} + \vec{c} = 10\hat{i} - 2\hat{j} - 6\hat{k}$

$= \vec{b}$

So, sum of two vectors $\vec{a}$ and $\vec{c}$ is equal to third vector $\vec{b}$

Hence, both Assertion (A) and Reason (R) are true and Reason (R) **is not** the correct explanation of the Assertion (A).

Hence, option (B) is correct.

20. **Assertion (A) :** Domain of y = $\cos^{-1}(x)$ is [–1, 1].

Reason (R) : The range of the principal value branch of y = $\cos^{-1}(x)$ is $[0, \pi]-\left\{\frac{\pi}{2}\right\}$.

Answer (C)

Sol. Given, $y = \cos^{-1} x$

Domain of y is equivalent to the range of value of x for which y exists

Let $y = \theta$

$\Rightarrow$ $x = \cos\theta$

as we know range of $\cos\theta$ is [–1, 1]

therefore range of x is [–1, 1]

hence, domain of y is [–1, 1]

also, when x = 0

then $y = \cos^{-1}(0)$

$$= \frac{\pi}{2}$$

Hence, $\frac{\pi}{2}$ is included in the principal value branch of y

$\therefore$ Assertion (A) is correct but Reason (R) is false.

So, option (C) is correct.

SECTION - B

This section has **5** Very Short Answer questions of **2 marks** each.

21. If $a = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$ and $b = \tan^{-1}\left(\sqrt{3}\right) - \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ then find the value of a + b.

Sol. $a = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$

$$b = \tan^{-1}\left(\sqrt{3}\right) - \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$a = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$$

$$= \sin^{-1}\left(\sin\frac{\pi}{4}\right) + \cos^{-1}\left(\cos\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{4} + \frac{2\pi}{3}$$

$$a = \frac{11\pi}{12}$$

and $b=\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$$=\frac{\pi}{3}-\left(\pi-\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$=\frac{\pi}{3}-\left(\pi-\frac{\pi}{3}\right)$$

$$=\frac{\pi}{3}-\frac{2\pi}{3}$$

$b=-\frac{\pi}{3}$

$\therefore$ a + b = $\frac{11\pi}{12}+\left(\frac{-\pi}{3}\right)=\frac{11\pi}{12}-\frac{4\pi}{12}=\frac{7\pi}{12}$

22. (a) Find : $\int \cos^3 x\, e^{\log \sin x}dx$

OR

(b) Find : $\int \frac{1}{5+4x-x^2}dx$

Sol. (a) Let I = $\int \cos^3 x\, e^{\log \sin x}dx$

$$I=\int \cos^3 x.\sin x dx$$

Putting cos x = t

–sinx dx = dt

or, sinxdx = –dt, we get

$$I=-\int t^3 dt$$

$$=-\frac{t^4}{4}+c$$

$$=\frac{-\cos^4 x}{4}+c \quad [\because\ t=\cos x]$$

(b) Let $I=-\int \frac{1}{x^2-4x-5}dx$

$$=-\int \frac{1}{x^2-4x+4-4-5}dx$$

$$=-\int \frac{1}{(x-2)^2-9}dx$$

We know that, $\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

$$\therefore \quad I = -\int \frac{1}{(x-2)^2-(3)^2}dx$$

$$= -\frac{1}{6}\log\left|\frac{x-2-3}{x-2+3}\right| + c$$

$$= -\frac{1}{6}\log\left|\frac{x-5}{x+1}\right| + c$$

23. Sand is pouring from a pipe at the rate of 15 cm^3/minute. The falling sand forms a cone on the ground such that the height of the cone is always one-third of the radius of the base. How fast is the height of the sand cone increasing at the instant when the height is 4 cm?

Sol. Let r = radius; h = height; v = Volume of sand cone and t = time.

Given, h = 4 cm; $\frac{dV}{dt} = 15\ cm^3/min$ and $h = \frac{1}{3}r$

As r = 3h

$\therefore \quad V = \frac{1}{3}\pi r^2 h$ $\qquad$ ($\because$ volume of cone = $\frac{1}{3}\pi r^2 h$)

$$= \frac{1}{3}\pi(3h)^2 h$$

$$V = 3\pi h^3$$

Differentiating both side w.r.t. t

$$\therefore \quad \frac{dv}{dt} = 3\pi.3h^2\frac{dh}{dt}$$

$$15 = 3\pi.3(4)^2\frac{dh}{dt}$$

$$\therefore \quad \frac{dh}{dt} = \frac{5}{48}cm/min.$$

24. Find the vector equation of the line passing through the point (2, 3, –5) and making equal angles with the co-ordinate axes.

Sol. Let angle formed with *x*-axis, y-axis and z-axis are α, β and γ respectively.

$\because \alpha = \beta = \gamma$

Now, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \quad 3\cos^2\alpha = 1$ $\qquad$ [$\because \alpha = \beta = \gamma$]

$$\cos^2\alpha = \frac{1}{3}$$

$$\cos\alpha = \pm\frac{1}{\sqrt{3}}$$

$\therefore \quad (l, m, n) = (\cos\alpha, \cos\beta, \cos\gamma)$

$$= \pm\frac{1}{\sqrt{3}}(1, 1, 1)$$

$\therefore$ Direction ratio of line $\vec{l} = (1, 1, 1)$

We know that equation of line is $\vec{r} = \vec{a} + \lambda\vec{l};\ \lambda \in R$

$$\vec{r} = 2\hat{i} + 3\hat{j} - 5\hat{k} + \lambda(\hat{i} + \hat{j} + \hat{k})$$

25. (a) Verify whether the function f defined by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$ or not.

OR

(b) Check for differentiability of the function f defined by f(x) = |x – 5|, at the point $x = 5$.

Sol. (a) For a continuous function $\underset{x\to a}{\text{LHL}} = \underset{x\to a}{\text{RHL}} = f(a)$

Let LHL,

$$= \lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right)$$

$$= \lim_{h\to 0}(0-h)\sin\left(\frac{1}{0-h}\right)$$

$$= \lim_{h\to 0} -h\sin\left(\frac{-1}{h}\right)$$

$$= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) \qquad [\because \sin(-\theta) = -\sin\theta]$$

$= 0 \sin(\infty)$

$= 0$

Now, RHL,

$$= \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right)$$

$$= \lim_{h\to 0}(0+h)\sin\left(\frac{1}{0+h}\right)$$

$$= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right)$$

$= 0\cdot\sin(\infty)$

$= 0$

So, here LHL = RHL = f(0)

$\therefore$ Function is continuous.

(b) For f(x) to be differentiable,

LHD = RHD

Now LHD,

$$\lim_{x\to 5^-}\frac{f(x)-f(5)}{x-5}$$

$$=\lim_{h\to 0}\frac{f(5-h)-0}{5-h-5}$$

$$=\lim_{h\to 0}\frac{5-(5-h)}{-h}=\lim\frac{h}{-h}=-1$$

Now RHD,

$$\lim_{x\to 5^+}\frac{f(x)-f(5)}{x-5}$$

$$=\lim_{h\to 0}\frac{f(5+h)-0}{5+h-5}=\lim_{h\to 0}\frac{5+h-5}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

Since, LHD $\neq$ RHD, f(x) = (x – 5) is not differentiable at x = 5

SECTION - C

There are **6** short answer questions in this section. Each is of **3 marks**.

26. (a) Find the particular solution of the differential equation

$$\frac{dy}{dx}-2xy=3x^2e^{x^2};y(0)=5.$$

OR

(b) Solve the following differential equation:

$x^2dy + y(x + y)dx = 0$

Sol. (a) $\frac{dy}{dx}-2xy=3x^2e^{x^2}$

Compare it with $\frac{dy}{dx}+py=Q$

So, I.F. = $e^{\int pdx}$

Here, I.F. = $e^{-\int 2xdx}$

$= e^{-x^2}$

As it is linear differential equation.

So, y. IF. = $\int I.F.Qdx$

$$y\cdot e^{-x^2}=\int e^{-x^2}\cdot e^{x^2}\cdot 3x^2dx$$

$$y\cdot e^{-x^2}=\int 3x^2dx$$

$$\Rightarrow\quad y\cdot e^{-x^2}=x^3+C$$

Given y(0) = 5

So, $5.e^{o} = 0 + C$

$\Rightarrow \quad C = 5$

Since, $ye^{-x^2} = x^3 + 5$, is our required solution.

(b) $x^2dy + y(x + y)dx = 0$

$x^2dy = -y(x + y)dx$

$$\Rightarrow \frac{dy}{dx} = \frac{-y(x+y)}{x^2} \qquad \dots (1)$$

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + \frac{xdv}{dx}$$

Put this value in equation (1),

$$v + \frac{xdv}{dx} = \frac{-vx(x+vx)}{x^2}$$

$$\Rightarrow v + \frac{xdv}{dx} = \frac{-vx^2(1+v)}{x^2}$$

$$\Rightarrow v + \frac{xdv}{dx} = -v(1+v)$$

$$\Rightarrow v + \frac{xdv}{dx} = -v - v^2$$

$$\Rightarrow \frac{xdv}{dx} = -2v - v^2$$

$$\Rightarrow -\frac{dv}{2v+v^2} = \frac{dx}{x}$$

$$\Rightarrow -\int \frac{dv}{v^2+2v} = \int \frac{dx}{x}$$

$$\Rightarrow \log|x| = -\int \frac{dv}{(v+1)^2 - 1}$$

$$\Rightarrow \log|x| = -\frac{1}{2}\log\left|\frac{v+1-1}{v+1+1}\right| + C$$

$$\Rightarrow \log|x| = -\frac{1}{2}\log\left|\frac{v}{v+2}\right| + C$$

Put $v = \frac{y}{x}$

$$\Rightarrow \log|x| = -\frac{1}{2}\log\left|\frac{y/x}{y/x + 2}\right| + C$$

$$\Rightarrow \log|x| = -\frac{1}{2}\log\left|\frac{y}{y+2x}\right| + C$$

27. Find the values of a and b so that the following function is differentiable for all values of x:

$$f(x)=\begin{cases} ax+b, & x>-1 \\ bx^2-3, & x\le -1 \end{cases}$$

Sol. We have,

$$f(x)=\begin{cases} ax+b, & x>-1 \\ bx^2-3, & x\le -1 \end{cases}$$

f(x) is differentiable for all values of x.

So, f(x) must be continuous as well for all values of x.

So, f(x) is continuous at x = –1

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^+} f(x)=f(-1)$$

$$\lim_{x\to -1^-}\left(bx^2-3\right)=\lim_{x\to -1^+}(ax+b)=b-3$$

$\Rightarrow$ b – 3 = – a + b

$\Rightarrow$ a = 3

Now, f(x) is differentiable at x = –1

(LHD at x = –1) = (RHD at x = –1)

$$\lim_{x\to -1^-}\frac{f(x)-f(-1)}{x-(-1)}=\lim_{x\to -1^+}\frac{f(x)-f(-1)}{x-(-1)}$$

$$\Rightarrow \lim_{x\to -1^-}\frac{bx^2-3-(b-3)}{x+1}=\lim_{x\to -1^+}\frac{ax+b-(b-3)}{x+1}$$

$$\Rightarrow \lim_{x\to -1^-}\frac{bx^2-b}{x+1}=\lim_{x\to -1^+}\frac{ax+3}{x+1}$$

$$\lim_{x\to -1^-}\frac{b\left(x^2-1\right)}{x+1}=\lim_{x\to -1^+}\frac{ax+3}{x+1}$$

$$\lim_{x\to -1^-}\frac{b(x-1)(x+1)}{(x+1)}=\lim_{x\to -1^+}\frac{3x+3}{x+1}\quad \text{(as a = 3)}$$

$$\lim_{x\to -1^-} b(x-1)=\lim_{x\to -1^+}\frac{3(x+1)}{(x+1)}$$

$$\lim_{x\to -1^-} b(x-1)=\lim_{x\to -1^+} 3$$

$\Rightarrow$ b(–2) = 3

$\Rightarrow$ –2b = 3

$\Rightarrow$ $b=-\dfrac{3}{2}$

So, for given f(x), a = 3, $b=-\dfrac{3}{2}$

28. (a) Find $\frac{dy}{dx}$, if $(\cos x)^y = (\cos y)^x$.

OR

(b) If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, prove that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Sol. (a) Given that, $(\cos x)^y = (\cos y)^x$

We need to find $\frac{dy}{dx}$.

$(\cos x)^y = (\cos y)^x$

Taking log both sides,

$\log(\cos x)^y = \log(\cos y)^x$

$y\log(\cos x) = x\log(\cos y)$

(As $\log(a^b) = b \log a$)

Now, differentiate both sides with respect to x.

$$\frac{d(y\log(\cos x))}{dx}=\frac{d(x\log((\cos y)))}{dx}$$

Using product rule here,

$$\left(\frac{d(uv)}{dx}=\frac{udv}{dx}+\frac{du}{dx}\cdot v\right)$$

$$\frac{dy}{dx}\cdot\log\cos x+\frac{d(\log(\cos x))}{dx}\cdot y=\frac{dx}{dx}\log(\cos y)+\frac{d}{dx}(\log((\cos y)))\cdot x$$

$$\Rightarrow \frac{dy}{dx}\log\cos x+\frac{1}{\cos x}\frac{d(\cos x)}{dx}\cdot y=\log\cos y+\frac{1}{\cos y}\frac{d(\cos y)}{dx}\cdot x$$

$$\Rightarrow \frac{dy}{dx}\cdot\log\cos x+\frac{1}{\cos}(-\sin x)\cdot y=\log\cos y+\frac{1}{\cos y}(-\sin y)\frac{dy}{dx}\cdot x$$

$$\Rightarrow \frac{dy}{dx}\cdot\log(\cos x)-\tan x.y=\log\cos y-\tan y\cdot x\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(\log(\cos x)+x\tan y)=\log\cos y+y\tan x$$

$$\Rightarrow \frac{dy}{dx}=\frac{\log\cos y+y\tan x}{\log\cos x+x\tan y}$$

OR

(b) We have given,

$$\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$$

We need to prove $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$

Let x = sinA and y = sinB ...(1)

Take f(x) as,

$$f(x)=\sqrt{1-\sin^2 A}+\sqrt{1-\sin^2 B}=a(\sin A-\sin B)$$

$\Rightarrow$ cosA + cosB = a(sinA – sinB)

as 1 – $\sin^2x = \cos^2x$

$$\Rightarrow \quad 2\left[\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right]=a\left[2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right]$$

$$\Rightarrow \quad 2\cos\left(\frac{A-B}{2}\right)=a.2\sin\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}=a$$

$$\Rightarrow \quad \cot\left(\frac{A-B}{2}\right)=a \quad \left[\because \frac{\cos\theta}{\sin\theta}=\cot\theta\right]$$

$$\Rightarrow \quad \frac{A-B}{2}=\cot^{-1}a$$

$\sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$

Differentiate both sides,

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

29. (a) Evaluate : $\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

OR

(b) Find : $\int\frac{2x+1}{(x+1)^2(x-1)}dx$

Sol. (a) $\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

Let $I = \int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$ ...(1)

Now, using property here,

$$\int_0^a f(x)dx=\int_0^a f(a-x)dx$$

$$I = \int_0^{\pi}\frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}dx$$

$$I = \int_0^{\pi}\frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}dx \quad \text{...(2)}$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx + \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} dx$$

$$2I = \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}} dx$$

$$2I = \int_0^{\pi} dx = [x]_0^{\pi}$$

$$2I = \pi$$

$$I = \frac{\pi}{2}$$

(b) $\int \frac{2x+1}{(x+1)^2 (x-1)} dx$

Applying form of partial fraction here,

$$\frac{2x+1}{(x+1)^2 (x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)}$$

$\Rightarrow$ $2x + 1 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$

Put $x = 1$

$\Rightarrow$ $3 = C(1 + 1)^2$

$\Rightarrow$ $3 = 4C$

$\Rightarrow$ $C = \frac{3}{4}$

Now, put x = – 1

$\Rightarrow$ $-1 = B(-1 -1)$

$\Rightarrow$ $-1 = -2B$

$\Rightarrow$ $B = \frac{1}{2}$

Now put $x = 0$

$\Rightarrow$ $1 = -A - B + C$

$\Rightarrow$ $1 = -A - \frac{1}{2} + \frac{3}{4}$

$\Rightarrow$ $A = \frac{3}{4} - \frac{1}{2} - 1$

$\Rightarrow$ $A = -\frac{3}{4}$

$$\int \frac{2x+1}{(x+1)^2 (x-1)} dx = \int \frac{-3}{4(x+1)} dx + \frac{1}{2}\int \frac{dx}{(x+1)^2} + \frac{3}{4}\int \frac{dx}{(x-1)}$$

$$\Rightarrow \int \frac{2x+1}{(x+1)^2 (x-1)} dx = -\frac{3}{4}\log|(x+1)| + \frac{1}{2}\int \frac{dx}{(x+1)^2} + \frac{3}{4}\log|(x-1)|$$

Take I = $\frac{1}{2}\int \frac{dx}{(x+1)^2}$

Let $x + 1 = t$

$dx = dt$

$$I = \frac{1}{2}\int \frac{dt}{t^2} = \frac{1}{2}\left(-\frac{1}{t}\right)$$

So, I = $\frac{1}{2}\left[-\frac{1}{(x+1)}\right]$

$$\Rightarrow \quad \int \frac{2x+1}{(x+1)^2(x-1)}dx = -\frac{3}{4}\log|(x+1)| - \frac{1}{2(x+1)} + \frac{3}{4}\log|(x-1)| + C$$

$$\Rightarrow \quad \int \frac{2x+1}{(x+1)^2(x-1)}dx = \frac{3}{4}\log\left|\left(\frac{x-1}{x+1}\right)\right| - \frac{1}{2(x+1)} + C$$

30. Given $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 3\hat{i} - \hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 2\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}\cdot\vec{d}$ = 3.

Sol. We have,

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{b} = 3\hat{i} - \hat{k}$$

Vector which is perpendicular to both $\vec{a}$ and $\vec{b}$ must be parallel to $\vec{a}\times\vec{b}$.

So, here $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{vmatrix}$

$= \hat{i}(1) - \hat{j}(2-3) + 3\hat{k}$

$$\vec{a}\times\vec{b} = \hat{i} + 5\hat{j} + 3\hat{k}$$

So, $\vec{d}$ is parallel to $\vec{a}\times\vec{b}$

So, let $\vec{d}$ = $\lambda(\vec{a}\times\vec{b}) = \lambda(\hat{i} + 5\hat{j} + 3\hat{k})$

Also, we have given that $\vec{c}\,.\,\vec{d}$ = 3

Here, $\vec{c}$ = $2\hat{i} + \hat{j} - 2\hat{k}$

So, $\vec{c}\cdot\vec{d}$ = $(2\hat{i} + \hat{j} - 2\hat{k}).(\lambda)(\hat{i} + 5\hat{j} + 3\hat{k}) = 3$

$\Rightarrow$ $2\lambda + 5\lambda - 6\lambda = 3$

$\Rightarrow$ $\lambda = 3$

So, $\vec{d} = 3(\hat{i} + 5\hat{j} + 3\hat{k})$

Hence, the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ & $\vec{b}$ and $\vec{c}\cdot\vec{d} = 3$ is given by = $3(\hat{i} + 5\hat{j} + 3\hat{k})$

31. Bag I contains 3 red and 4 black balls, Bag II contains 5 red and 2 black balls. Two balls are transferred at random from Bag I to Bag II and then a ball is drawn at random from Bag II. Find the probability that the drawn ball is red in colour.

Sol. Bag I contains 3 red balls and 4 black balls

Bag II contains 5 red balls and 2 black balls

Two balls are transferred at random from Bag I to Bag II

Here we make cases.

Case I: When both transferred balls are red.

Then Bag II has 7 red balls and 2 black balls.

So required probability = $\dfrac{\text{Number of red balls}}{\text{Total balls}}$

$$= \frac{7}{7+2} = \frac{7}{9}$$

Case II: When 1 ball is red and 1 ball is black

Then Bag II has 6 red balls and 3 black balls.

Required probability $= \dfrac{6}{9}$

Case III: When both balls are black

Then, Bag II has 5 red balls and 4 black balls

Then, required probability $= \dfrac{5}{9}$

Now, probability of choosing 2 red balls from Bag I $= \dfrac{{}^3C_2}{{}^7C_2}$

Probability of choosing 1 red ball and 1 black ball $= \dfrac{{}^3C_1 \times {}^4C_1}{{}^7C_2}$

Probability of choosing 2 black balls $= \dfrac{{}^4C_2}{{}^7C_2}$

So, required probability

$$= \frac{{}^3C_2}{{}^7C_2} \times \frac{7}{9} + \frac{{}^3C_1 \times {}^4C_1}{{}^7C_2} \times \frac{6}{9} + \frac{{}^4C_2}{{}^7C_2} \times \frac{5}{9}$$

$$= \frac{3}{21} \times \frac{7}{9} + \frac{3 \times 4}{21} \times \frac{6}{9} + \frac{6}{21} \times \frac{5}{9}$$

$$= \frac{1}{9} + \frac{8}{21} + \frac{10}{63} = \frac{7+24+10}{63} = \frac{41}{63}$$

SECTION - D

There are **4** long answer questions in this section. Each question is of **5 marks**.

32. (a) Find the co-ordinates of the foot of the perpendicular drawn from the point (2, 3, –8) to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$.

Also, find the perpendicular distance of the given point from the line.

OR

(b) Find the shortest distance between the lines L_1 and L_2 given below:

L_1 : The line passing through (2, –1, 1) and parallel to $\frac{x}{1}=\frac{y}{1}=\frac{z}{3}$

L_2 : $\vec{r}=\hat{i}+(2\mu+1)\hat{j}-(\mu+2)\hat{k}$.

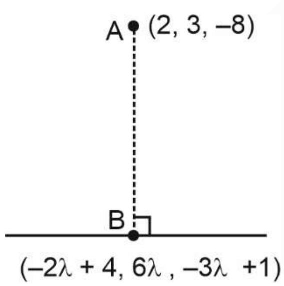
Sol. (a) Given equation of line can be written as,

$\frac{x-4}{-2}=\frac{y}{6}=\frac{z-1}{-3}=\lambda$ (let)

General points of line are

$(-2\lambda + 4, 6\lambda, -3\lambda + 1)$

(i) Foot of perpendicular (B)



Direction ratio of line segment AB is

$-2\lambda + 2, 6\lambda - 3, -3\lambda + 9$

$\therefore$ AB is perpendicular to given line

$\therefore$ $-2(-2\lambda+2)+6(6\lambda-3)-3(-3\lambda+9)=0$

$\Rightarrow$ $\lambda = 1$

$\therefore$ Coordinates of B $\equiv$ (2, 6, –2)

$\therefore$ Perpendicular distance of point (2, 3, –8) from given line

= perpendicular distance of point (2, 3, –8) from point (2, 6, –2)

$=\sqrt{0+9+36}$

$=\sqrt{45}$

$=3\sqrt{5}$

(b) The equation of line passing through (2, –1, 1) and parallel to $\frac{x}{1}=\frac{y}{1}=\frac{z}{3}$ is

$$L_1 \equiv \vec{r} = \left(2\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}+\hat{j}+3\hat{k}\right)$$

and $L_2 \equiv \vec{r} = \left(\hat{i}+\hat{j}-2\hat{k}\right)+\mu\left(2\hat{j}-\hat{k}\right)$

∴ Shortest distance between lines L_1 and L_2 is

$$d = \frac{\left|\left(\vec{a_2}-\vec{a_1}\right)\cdot\left(\vec{b_1}\times\vec{b_2}\right)\right|}{\left|\vec{b_1}\times\vec{b_2}\right|}$$

where,

$$\vec{a_2}-\vec{a_1} = -\hat{i}+2\hat{j}-3\hat{k}$$

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 3 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= \hat{i}(-7)-\hat{j}(-1)+\hat{k}(2)$$

$$= -7\hat{i}+\hat{j}+2\hat{k}$$

$$\therefore \left|\vec{b_1}\times\vec{b_2}\right| = \sqrt{49+1+4} = \sqrt{54}$$

and $\left(\vec{a_2}-\vec{a_1}\right)\cdot\left(\vec{b_1}\times\vec{b_2}\right) = 7+2-6 = 3$

$$\therefore d = \frac{3}{\sqrt{54}} = \frac{1}{\sqrt{6}}$$

33. (a) If $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{bmatrix}$, then find A^{-1} and hence solve the following system of equations :

x + 2y – 3z = 1

2x – 3z = 2

x + 2y = 3

OR

(b) Find the product of the matrices $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$ and hence solve the system of linear equations :

x + 2y – 3z = –4

2x + 3y + 2z = 2

3x – 3y – 4z = 11

Sol. (a) Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{bmatrix}$

$\because \quad A^{-1} = \frac{1}{|A|} \text{adj } A$

$|A| = 1(6) - 2(3) - 3(4)$

$= 6 - 6 - 12 = -12$

and $\text{adj } A = \begin{bmatrix} 6 & -6 & -6 \\ -3 & 3 & -3 \\ 4 & 0 & -4 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{-1}{12}\begin{bmatrix} 6 & -6 & -6 \\ -3 & 3 & -3 \\ 4 & 0 & -4 \end{bmatrix}$

Given system of liner equations can be written as,

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{-1}{12}\begin{bmatrix} 6 & -6 & -6 \\ -3 & 3 & -3 \\ 4 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{-1}{12}\begin{bmatrix} -24 \\ -6 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{2}{3} \end{bmatrix}$$

$\therefore$ Solution of given system is

$x = 2,\ y = \frac{1}{2}$ and $z = \frac{2}{3}$

(b) The product of the matrices

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix} = \begin{bmatrix} 67 & 0 & 0 \\ 0 & 67 & 0 \\ 0 & 0 & 67 \end{bmatrix}$$

$= 67I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}^{-1} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

Given system of linear equations can be written as,

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}^{-1}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$= \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix}$$

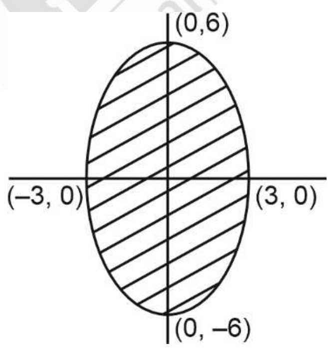
$$= \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ Solution of given system of linear equation is $x = 3$, $y = -2$ and $z = 1$

34. Find the area of the region bounded by the curve $4x^2 + y^2 = 36$ using integration.

Sol. We have to find the area of region bounded by $4x^2 + y^2 = 36$ which can be written as,

$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1$ which represents an ellipse.



Area of region

$$= 4\int_0^3 \sqrt{36 - 4x^2}\, dx$$

$$= 8\int_0^3 \sqrt{3^2 - x^2}\, dx$$

$$= 8\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= 8\left[\frac{9}{2}\sin^{-1}(1)\right]$$

$$= 8 \times \frac{9}{2} \times \frac{\pi}{2}$$

$= 18\pi$

35. Solve the following Linear Programming problem graphically:

Maximise Z = 300x + 600y

Subject to $x + 2y \le 12$

$2x + y \le 12$

$x + \frac{5}{4}y \ge 5$

$x \ge 0, y \ge 0.$

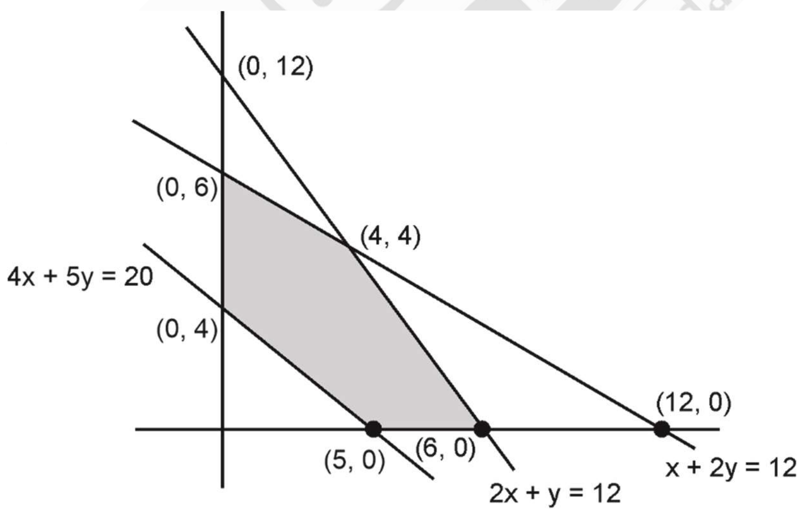
Sol. Maximise Z = 300x + 600y

Subject to $x + 2y \le 12$

$2x + y \le 12$

$4x + 5y \ge 0$

$x \ge 0, y \ge 0$



∴ Corner points are (0, 4), (0, 6)

(4, 4), (5, 0), (6, 0)

We have to check the value of Z at these points

Corner points	Z = 300x + 600y
(0, 4)	2400
(0, 6)	3600
(4, 4)	3600
(5, 0)	1500
(6, 0)	1800

Maximum value of Z = 3600

SECTION - E

In this section, there are **3** case study questions of **4 marks** each.

36. A departmental store sends bills to charge its customers once a month. Past experience shows that 70% of its customers pay their first month bill in time. The store also found that the customer who pays the bill in time has the probability of 0.8 of paying in time next month and the customer who doesn't pay in time has the probability of 0.4 of paying in time the next month.

Based on the above information, answer the following questions:

(i) Let E_1 and E_2 respectively denote the event of customer paying or not paying the first month bill in time. Find $P(E_1)$, $P(E_2)$

(ii) Let A denotes the event of customer paying second month's bill in time, then find $P(A|E_1)$ and $P(A|E_2)$.

(iii) Find the probability of customer paying second month's bill in time.

OR

(iii) Find the probability of customer paying first month's bill in time if it is found that customer has paid the second month's bill in time.

Sol. E_1 = customer paying the first month bill on time.

E_2 = customer not paying the first month bill on time.

(i) $P(E_1) = \frac{70}{100} = 0.7$ Ans.

$P(E_2) = \frac{30}{100} = 0.3$ Ans.

(ii) A = customer paying second month bill on time

$P(A|E_1)$ = P (customer pay second month bill on time given that first month bill on time)

= 0.8 Ans.

$P(A|E_2)$ = P (customer paying 2nd month bill on time given that 1st month bill not on time)

= 0.4 Ans.

(iii) $P(A) = P(E_1)\cdot P(A|E_1) + P(E_2)\cdot P(A|E_2)$

$= 0.7 \times 0.8 + 0.3 \times 0.4$

$= 0.56 + 0.12$

$= 0.68$ Ans.

OR

$$P(E_1/A) = \frac{P(E_1)\cdot P(A|E_1)}{P(E_1)P(A|E_1)+P(E_2)\cdot P(A|E_2)}$$

$$= \frac{0.7\times 0.8}{0.7\times 0.8+0.3\times 0.4}$$

$$= \frac{56}{68}$$

$$= \frac{14}{17} \text{ Ans.}$$

37. (a) Students of a school are taken to a railway museum to learn about railways heritage and its history.

An exhibit in the museum depicted many rail lines on the track near the railway station. Let L be the set of all rail lines on the railway track and R be the relation on L defined by

$R = \{l_1, l_2\} : l_1$ is parallel to $l_2\}$

On the basis of the above information, answer the following questions :

(i) Find whether the relation R is symmetric or not.

(ii) Find whether the relation R is transitive or not.

(iii) If one of the rail lines on the railway track is represented by the equation y = 3x + 2, then find the set of rail lines in R related to it.

OR

(b) Let S be the relation defined by $S = \{(l_1, l_2) : l_1$ is perpendicular to $l_2\}$ check whether the relation S is symmetric and transitive.

Sol. (a) $R = \{(l_1, l_2) : l_1$ is parallel to $l_2\}$

(i) l_1 is parallel to l_2, then l_2 is parallel to l_1.

$\therefore$ If $(l_1, l_2) \in R$, then $(l_2, l_1) \in R$.

$\therefore$ R is symmetric.

(ii) If l_1 is parallel to l_2 and l_2 is parallel to l_3, then l_1 is parallel to l_3.

So, if $(l_1, l_2) \in R$, $(l_2, l_3) \in R$, then $(l_1, l_3) \in R$.

$\therefore$ R is transitive.

(iii) $R = \{(l_1, l_2) : l_1$ is parallel to $l_2\}$

Set of all lines related to y = 3x + 2, is set of all lines that are parallel to y = 3x + 2

Let equation of line parallel to y = 3x + 2 be y = mx + c, where m is slope of line.

$\because$ y = 3x + 2 and y = mx + c are parallel

$\therefore$ Slope of both the lines will be equal.

$\therefore$ m = 3

$\because$ Required line is y = 3x + c, where $c \in R$.

OR

(b) S = {(l_1, l_2) : l_1 is perpendicular to l_2}

If l_1 is perpendicular to l_2, then l_2 is perpendicular to l_1.

So, (l_1, l_2) ∈ S, then (l_2, l_1) ∈ S.

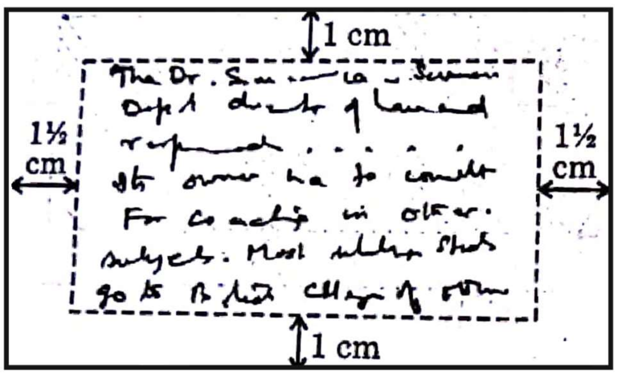
∴ S is symmetric.

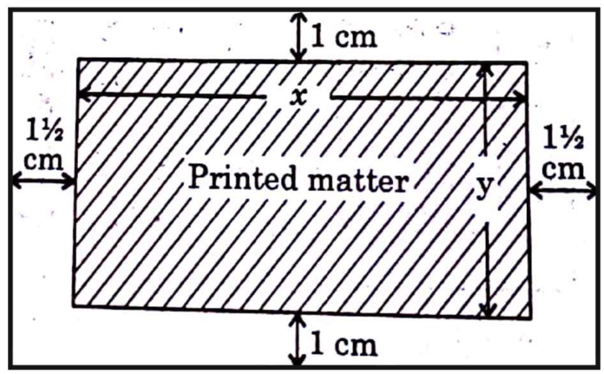
Checking for transitive :

If l_1 is perpendicular to l_2 and l_2 is perpendicular to l_3, then l_1 is not perpendicular to l_3. If is parallel to l_3. So, if (l_1, l_2) ∈ S, (l_2, l_3) ∈ S, then (l_1, l_3) ∉ S.

∴ S is not transitive.

38. A rectangular visiting card is to contain 24 sq. cm. of printed matter. The margins at the top and bottom of the card are to be 1 cm and the margins on the left and right are to be 1½ cm as shown below :



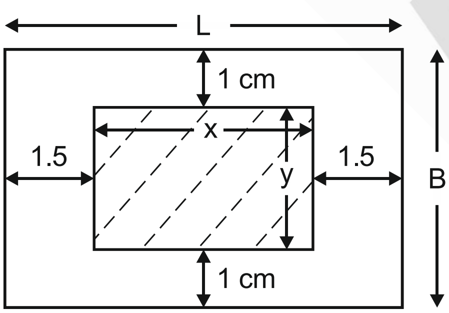


On the basis of the above information, answer the following questions :

(i) Write the expression for the area of the visiting card in terms of x.

(ii) Obtain the dimensions of the card of minimum area.

Sol.



Area of printed matter = 24 cm^2

∴ xy = 24

$$y = \frac{24}{x}$$

(i) Area of visiting card = L × B

$= (x + 3)(y + 2)$

$$= (x+3)\left(\frac{24}{x}+2\right)$$

$$= 24 + 2x + \frac{72}{x} + 6$$

$$= 2x + \frac{72}{x} + 30$$

(ii) $$\frac{dA}{dx} = \frac{d}{dx}\left(2x + \frac{72}{x} + 30\right)$$

$$= 2 - \frac{72}{x^2}$$

For maximum/minimum area,

$$\frac{dA}{dx} = 0$$

$$\therefore \quad 2 - \frac{72}{x^2} = 0$$

$\Rightarrow$ $2x^2 - 72 = 0$

$\Rightarrow$ $x^2 = 36$ $\Rightarrow$ $x = 6$ (As dimension cannot be negative)

Now, $$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(2 - \frac{72}{x^2}\right)$$

$$= \frac{144}{x^3}$$

$\therefore$ Area is minimum, when $x = 6$

$$\therefore \quad y = \frac{24}{6} = 4$$

So, dimensions are $x = 6$ cm, $y = 4$ cm

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