Chapter 13

Co-ordinate Geometry
(Straight Lines, Circles and Conic Sections)

CO-ORDINATE GEOMETRY

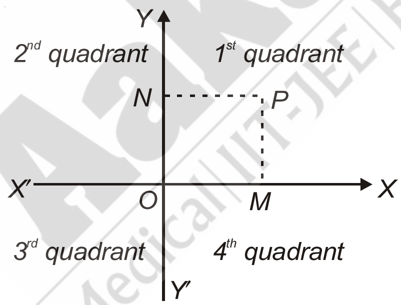
Coordinate geometry is the study of geometry with the help of algebra. The plane curves which would be considered are : straight line, circle, parabola, ellipse and hyperbola.

The system of coordinates used here is cartesian system of coordinates, introduced by philosopher Descartes. It is by far the most important system.

Cartesian System of Rectangular Co-ordinates in a plane

To identify a point P in the two dimensional plane, we associate with it, an ordered pair of numbers, in the following way:

Let OX and OY be two fixed perpendicular straight lines in the plane of the paper.



The line OX is called, the axis of x ; the line OY is called, the axis of y ; the two together are called the axes of coordinates. The point O is called the origin (of reference) and OXY is referred to as the Cartesian system. We shall restrict ourselves to rectangular Cartesian system (where OX and OY are perpendicular).

From any point P in the plane, draw lines PM and PN parallel to Y and X axes respectively. The distance $OM = x$ is called as abscissa of P ; the distance $ON = y$ is called as ordinate of P ; the two together, i.e. (x, y), are referred to as the coordinates of P.

Distances measured along OX are positive and those measured opposite to it are negative. Distances measured in the direction of OY are positive and those measured opposite to it are negative.

OXY = 1st quadrant, Abscissa and ordinate both +ve

$OX'Y$ = 2nd quadrant, Abscissa –ve and ordinate +ve

$OX'Y'$ = 3rd quadrant, Abscissa –ve and ordinate –ve

OXY' = 4th quadrant, Abscissa +ve and ordinate –ve

Distance formula

Distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ *i.e.* length of line segment PQ is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

> ***Note :*** *This formula is true for all points i.e. the points lying in any quadrant.*

Distance of point $P(x_1, y_1)$ from origin is given by $OP = \sqrt{x_1^2 + y_1^2}$

Area of triangle

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ form a triangle then area of ΔABC is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ or } \Delta = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right| = \frac{1}{2}\left|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right| \text{ 'Stair-method'}$$

Area of ΔOAB, where O is origin of co-ordinate system is given by $\Delta = \frac{1}{2}(x_1y_2 - x_2y_1)$

> ***Note :*** *When point A, B, C are taken in anticlockwise order, the area comes out to be positive otherwise negative. As all areas are necessarily positive quantities so care should be taken. Whenever area of a triangle is given take $\pm$ signs.*

Cor. : If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the sides of a triangle, then area of triangle

$$\Delta = \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where C_1, C_2, C_3 are cofactors of c_1, c_2, c_3 in the determinant above.

Condition for Collinearity of 3-points

If A, B & C are collinear, the area of triangle ABC has to be zero *i.e.*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ or } x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

Area of a plane polygon

Let A_1, A_2 ....A_n are the vertices of a n sided plane polygon, its area is given by Stair method

$$\text{Area of Polygon} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

i.e. $$= \frac{1}{2}\left|[x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + ... + (x_ny_1 - x_1y_n)]\right|$$

Important results

To prove that a given four sided figure is :

1. **Square** : Prove that four sides are equal & diagonals are equal
2. **Rhombus** : Four sides equal, diagonals unequal.
3. **Rectangles** : Opposite sides equal & diagonals equal.
4. **Parallelogram** : Opposite sides equal & diagonals unequal.

Note : *Diagonals bisect each other in all the above.*

Section formula

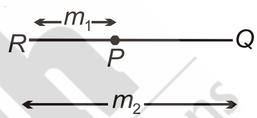
Coordinates of the point $R(\bar{x}, \bar{y})$ which divides the join of the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(i) internally, in the ratio $m_1 : m_2$, are

$$\bar{x} = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \; ; \; \bar{y} = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

(ii) externally, in the ratio $m_1 : m_2$, are

$$\bar{x} = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \; ; \; \bar{y} = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$



where $m_1 \neq m_2$

Cor 1 : Mid point ($m_1 = m_2$) of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Cor 2 : If $R(\bar{x}, \bar{y})$ divides $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $\lambda : 1$ ($\lambda > 0$) then

$$\bar{x} = \frac{\lambda x_2 \pm x_1}{\lambda \pm 1} \; ; \; \bar{y} = \frac{\lambda y_2 \pm y_1}{\lambda \pm 1}$$

Here '+' sign is taken for internal division and '–' sign for external division.

Cor 3 : For finding the ratio of division, use $\lambda : 1$. If λ comes out to be positive, it indicates internal division otherwise external if λ is negative.

Cor 4 : Line $Ax + By + C = 0$ divides join of $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $\left[-\frac{(Ax_1 + By_1 + C)}{(Ax_2 + By_2 + C)}\right]$

Locus and its Equation

When a point moves, so as always to satisfy a given condition or conditions, the path it traces out is called as its locus under the given conditions.

Equation to the locus (or curve) is the relation between coordinates of an arbitrarily chosen point on the curve and which relation holds for no other points except those lying on the curve.

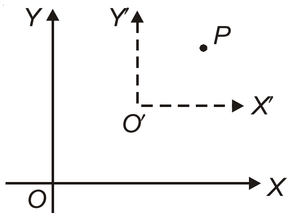
Standard method for writing equation of a locus

1. Assume the point whose locus (path) is to be found as (h, k).
2. Make the equation involving the coordinates of the moving point as per the conditions given.
3. Eleminate variables (if present) to relate h and k.
4. Substitute h with x and k with y in the simplified form of equation & we get the equation of locus.

Translation of Axes

(i) Shifting of Origin

Change of axes, by changing origin, the direction of axes remaining the same.

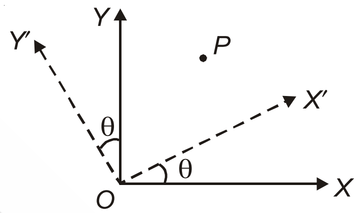


Let OXY and $O'X'Y'$ be two rectangular Cartesian system of axes.

Let P be any point in the plane of the axes and let P and O' have coordinates (x, y) and (h, k) respectively with respect to OXY system. Then the coordinates (x', y') of P with respect to the system $OX'Y'$ are given by $x = x' + h$; $y = y' + k$ and equivalently $x' = x - h$; $y' = y - k$.

(ii) Rotation of axes

Change of axes (without changing the origin), by changing the direction of axes, both systems of coordinates being rectangular.



If a point P in the plane of OXY has coordinates (x, y) and (x', y') with respect to the system OXY and $OX'Y'$ respectively, then

$x = x'\cos\theta - y'\sin\theta$ or equivalently $x' = x\cos\theta + y\sin\theta$

$y = x'\sin\theta + y'\cos\theta$ $y' = y\cos\theta - x\sin\theta$

Slope of a line

One of the simplest loci we come across in geometry is a straight line ; the geometrical property which is true in case of this locus is that slope between any two points on a line, is a constant m. This constant is known as slope of the line.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two points then slope between A and B is defined as :

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta \ : \ 0 \le \theta < \pi; \ \ \theta \ne \frac{\pi}{2}$$

where θ is the angle of inclination of the line joining A and B with the positive direction of x-axis. If $x_1 = x_2$ then slope is not defined.

Points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are **collinear** (i.e. lie on a straight line) if slope between A and B is equal to slope between B and C.

Slope of line $ax + by + c = 0$ is $\left(-\frac{a}{b}\right)$

Parallel and perpendicular lines

Two lines with slopes m_1 and m_2 are

(i) Parallel if $m_1 = m_2$

(ii) Perpendicular if $m_1m_2 = -1$ (Here m_1, m_2 are finite)

Intercepts of a Line on the coordinate axes

Let a line intersects OX - axes at A and OY - axes at B, then OA and OB are called x-intercept and y-intercept of line respectively.

The value of the intercept of a line may be positive, negative and zero.

STRAIGHT LINES

Various forms of equations of straight line

(i) General equation

An equation of the form $ax + by + c = 0$ where a & b both are not zero simultaneously, represents a straight line.

$\boxed{\text{Equation of straight line is of first degree in } x \text{ and } y.}$

Slope of a line is denoted by $m = \tan\theta$; where θ is the inclination of line to the positive direction of x-axis. (θ measured anticlockwise from +ve x-axis is taken as positive).

$\boxed{\text{Slope} = \text{gradient} = m = \tan\theta}$

Slope of x-axis is zero

Slope of y-axis is Infinite (undefined)

Slope of line equally inclined with both axes is either 1 or –1 ($\theta = 45°$ or $135°$).

Slope of a line joining two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

$\boxed{\text{Slope of the straight line is constant not variable.}}$

Slope of line $ax + by + c = 0$ is

$\boxed{m = -a/b}$

The intercepts of a line $ax + by + c = 0$ on x and y-axes can be found by substituting $y = 0$ and $x = 0$ respectively in the equation i.e.,

x intercept $\Rightarrow$ $ax + c = 0$ i.e., $x = -c/a$

y intercept $\Rightarrow$ $by + c = 0$ i.e., $y = -c/b$

Hence length of x intercept $= |-c/a|$ and length of y intercept is $|-c/b|$

Equation of x-axis is $y = 0$ and the equation of y-axis is $x = 0$

Equation of line (a is a non-zero constant)

(i) parallel to x-axis is $y = a$

and (ii) parallel to y-axis is $x = a$

(ii) Point slope form: Line with slope m and passes through the given point (x_1, y_1)

is $y - y_1 = m(x - x_1)$

(iii) Two point form: Line with two given points (x_1, y_1) and (x_2, y_2) on it is given by

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

(iv) Slope intercept form: Line with given slope m and intercept c on y-axis is given by

$y = mx + c$

(v) Intercept form: Line with given intercepts a and b on x and y axes respectively is

$$\frac{x}{a}+\frac{y}{b}=1$$

(vi) Normal or perpendicular form: Line at perpendicular distance p from the origin and where, the perpendicular makes angle α with OX is given by

$$x\cos\alpha+y\sin\alpha=p$$

(vii) Symmetric or parametric form: Line making an angle θ with OX and passing through (x_1, y_1) and r as the directed distance of any point $P(x, y)$ on the line from (x_1, y_1) is given by $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r$

i.e. $x=x_1+r\cos\theta$; $y=y_1+r\sin\theta$

(viii) Reduction of General Equation to Different Standard Forms

The general equation of straight line $ax + by + c = 0$ can be put in the form.

$\Rightarrow$ $y=\dfrac{-ax}{b}-\dfrac{c}{b}$ Slope - intercept form ($y = mx + c$)

where $m=\dfrac{-a}{b}$ & y intercept = $\dfrac{-c}{b}$

Hence, slope = $-\dfrac{\text{coefficient of } x}{\text{coefficient of } y}$

The general equation of the straight line $ax + by + c = 0$ can be reduced to

$\Rightarrow$ $ax + by = -c$

$\Rightarrow$ $\dfrac{x}{(-c/a)}+\dfrac{y}{(-c/b)}=1$ intercept form $\left(\dfrac{x}{a}+\dfrac{y}{b}=1\right)$

Here x intercept = $-c/a$ and intercept on y-axis is $-c/b$.

The equation $ax + by + c = 0$ can be rewritten as

$ax + by = -c$

$\Rightarrow$ $\dfrac{a}{\sqrt{a^2+b^2}}x+\dfrac{b}{\sqrt{a^2+b^2}}y=-\dfrac{c}{\sqrt{a^2+b^2}}$ which is in normal or perpendicular form ($x\cos\alpha + y\sin\alpha = p$)

(a) If $c < 0$ then $-c > 0$

Hence, $p=\dfrac{-c}{\sqrt{a^2+b^2}}$ and $\cos\alpha=\dfrac{a}{\sqrt{a^2+b^2}}, \sin\alpha=\dfrac{b}{\sqrt{a^2+b^2}}$

(b) If $c > 0$ then $\dfrac{-a}{\sqrt{a^2+b^2}}x-\dfrac{by}{\sqrt{a^2+b^2}}=\dfrac{c}{\sqrt{a^2+b^2}}$

$p=\dfrac{c}{\sqrt{a^2+b^2}}, \cos\alpha\dfrac{-a}{\sqrt{a^2+b^2}}$ and $\sin\alpha=\dfrac{-b}{\sqrt{a^2+b^2}}$

Length of perpendicular from origin to the straight line $ax+by+c=0$ is $\dfrac{\lvert c\rvert}{\sqrt{a^2+b^2}}$

Point of intersection of two lines

By solving the two equations of straight lines simultaneously, the point of intersection can be found

for lines $l_1 = a_1x + b_1y + c_1 = 0$

and $l_2 = a_2x + b_2y + c_2 = 0$

Point of intersection is

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right); \; a_1b_2 \neq a_2b_1$$

distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by $d = \frac{\lvert c_1 - c_2 \rvert}{\sqrt{a^2 + b^2}}$

Angle between Two Lines

$\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$ where ϕ is acute angle and m_1 and m_2 are slopes of two lines

Otherwise $\tan\phi = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$; one sign gives acute angle and the other gives obtuse angle

Here m_1 or m_2 or both should not be infinite.

If both m_1 and m_2 are undefined, angle = 0

If m_1 is undefined and $m_2 = \tan\theta$, then angle between lines is $\left(\frac{\pi}{2} - \theta\right)$

For parallel lines $m_1 = m_2$ and for perpendicular lines $m_1m_2 = -1$

(i) The two lines which make angle ϕ with the given line $y = mx + c$ and passing through the point (x_1, y_1) are given by $y - y_1 = m_1(x - x_1)$ and $y - y_1 = m_2(x - x_1)$

where $m_1 = \frac{m - \tan\phi}{1 + m\tan\phi}$; $m_2 = \frac{m + \tan\phi}{1 - m\tan\phi}$

i.e., $m_1 = \tan(\theta - \phi)$ and $m_2 = \tan(\theta + \phi)$ if $m = \tan\theta$.

(ii) The length r of the line segment drawn through a given point (x_1, y_1) and making an angle θ with x-axis, to meet the line $ax + by + c = 0$ is given by $r = -\frac{ax_1 + by_1 + c}{a\cos\theta + b\sin\theta}$

(iii) Condition for two lines to be intersecting, parallel or perpendicular. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(a) Coincident then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(b) Parallel but not coincident then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(c) Perpendicular then $a_1a_2 + b_1b_2 = 0$

(d) Intersecting if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(iv) The line parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is variable to be find out.

(v) The line perpendicular to $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is variable to be find out.

Condition of concurrence of three lines

If the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ are concurrent then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Points in Relation to a Line *ax + by + c = 0*

1. The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same side or on the opposite side of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have same sign or opposite sign.
2. The ratio in which the line $ax + by + c = 0$ divides the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $-\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$.
3. The length of the perpendicular from $P(x_1, y_1)$ to $ax + by + c = 0$ is $p = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$
4. Coordinates of the foot of the perpendicular drawn from $P(x_1, y_1)$ to the line $ax + by + c = 0$ are given by $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = -\dfrac{ax_1 + by_1 + c}{a^2 + b^2}$
5. Coordinates of the image of $P(x_1, y_1)$ in the line $ax + by + c = 0$ are given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

Angle Bisectors between Two Lines

The equations of the bisectors of the angles between two intersecting lines

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

Let ϕ be the angle between one of the bisectors and one of the lines $a_1x + b_1y + c_1 = 0$. If $|\tan\phi| < 1$ i.e. $\phi < 45°$, then that bisector is the acute angle bisector of the two given lines. The other equation represents the obtuse angle bisector.

Rule for writing a particular bisector

Write the equations of the lines so that constant terms are positive

If $a_1a_2 + b_1b_2 > 0$ then $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$ …(1)

gives the obtuse angle bisector. If $a_1a_2 + b_1b_2 < 0$ then (1) gives the acute angle bisector. '+' signs gives the bisector of angle containing origin.

> **Note :** *If the equations are written so that the constant terms have same sign, then (1) gives bisector of the angle in which the origin lies (and not necessarily the acute angle bisector)*

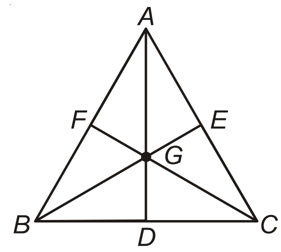
Coordinate of centroid, orthocentre, circumcentre, incentre, Excentres

Consider triangle ABC in which $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$ and $AB = c$, $BC = a$, $AC = b$.

(i) **Centroid:** Centroid of a triangle is the point of intersection of medians.

Here AD, BE, CF are the medians and their point of intersection is G which is called centroid.

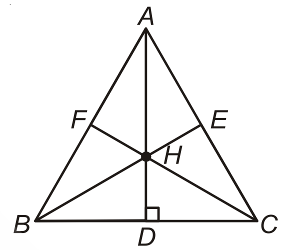
$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$



(ii) **Orthocentre:** Orthocentre is the point of intersection of perpendiculars drawn from vertices to the opposite sides.

Here AD, BE, CF are perpendicular to the sides BC, AC, AB respectively 'H' represents orthocentre.

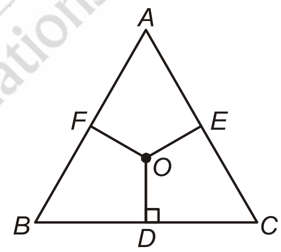
$$H \equiv \left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$



(iii) **Circumcentre:** This is the point of intersection of perpendicular bisector of sides.

Here OD, OE, OF are perpendicular bisectors of sides. O is the circumcentre.

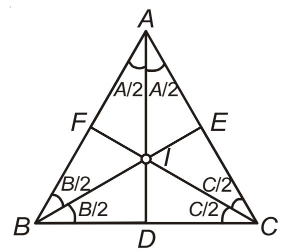
$$O \equiv \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$



(iv) **Incentre:** This is the point of intersection of internal angle bisectors.

Here AD, BE, CF are internal angle bisectors. I is the incentre.

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

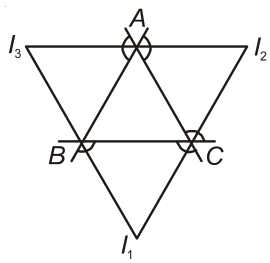


(v) **Excentres:** These are the points of intersection of external angle bisectors. I_1, I_2, I_3 are excentres with respect to vertices A, B, C.

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$$

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right)$$



(vi) **Important points about these centres:**

(a) If the vertices have rational coordinates then centroid, circumcentre, orthocentre have rational coordinates but incentre and excentres may have irrational coordinates.

(b) Except equilateral triangle, orthocentre, centroid, circumcentre are collinear and centroid divides the distance between orthocentre and circumcentre in the ratio 2 : 1.

(c) In case of right angled triangle the middle point of hypotenuse is the circumcentre and the vertex of right angle is called orthocentre.

(d) Circumcentre is a point equidistant from all vertices and incentre is the point equidistant from all sides.

(e) If the coordinates of vertices of triangle are rational then it cannot be equilateral.

Equation of family of lines passing through the point of intersection of two lines

(i) The family of lines passing through the points of intersection of the line $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv a_2x + b_2y + c_2 = 0$ is $L_1 + \lambda L_2 = 0$, λ is a variable.

or $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$, where λ is to be find out by any given condition.

Here the point of intersection of lines is called the fixed point.

(ii) If a, b, c are variable then to find the fixed point for the family of lines $ax + by + c = 0$, we may use comparison method.

Some standard results

(i) Area of Parallelogram or Rhombus

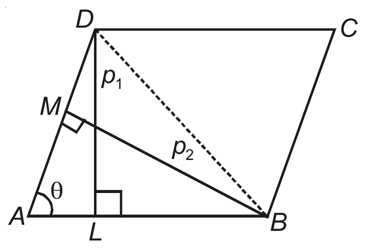
Area of a parallelogram or a rhombus, equations of whose sides are given, can be obtained by using the following formula

$$Area = \frac{p_1p_2}{\sin\theta}$$

Where $p_1 = DL$ = distance between lines AB and CD,

$p_2 = BM$ = distance between lines AD and BC,

θ = angle between adjacent sides AB and AD.



In the case of a rhombus, $p_1 = p_2$. Thus, Area of rhombus $= \dfrac{p_1^2}{\sin\theta}$

Also, area of rhombus $= \dfrac{1}{2}d_1d_2$

where d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus.

(ii) If (x_1, y_1) and (x_2, y_2) are the two vertices of an equilateral triangle, then the third vertex is given by,

$$\left[\frac{(x_1+x_2)\pm\sqrt{3}(y_1-y_2)}{2}, \frac{y_1+y_2\mp\sqrt{3}(x_1-x_2)}{2}\right]$$

(iii) If (x_1, y_1) and (x_2, y_2) are the vertices of the hypotenuse of a right angle triangle, then the third vertex is given by

$$\left[\frac{(x_1+x_2)\pm(y_1-y_2)}{2}, \frac{(y_1+y_2)\pm(x_1-x_2)}{2}\right]$$

(iv) Area of the triangle, whose sides are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ is given by

$$A = \frac{1}{2}\left|\sum \frac{(c_1 - c_2)^2}{m_1 - m_2}\right|$$

(v) Area of rhombus, formed by $ax \pm by \pm c = 0$, is given by,

$$A = \left|\frac{2c^2}{ab}\right|$$

(vi) Area of the parallelogram formed by the lines

$a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + d_2 = 0$ is given by

$$A = \left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|$$

Among all curves, circle is the simplest curve. It is common in almost every sphere of life. A wheel, a circular object, has revolutionised the transportation like all motor vehicles, railways. pullies, gears, various rings are all circular.

CIRCLE

Standard Equation of Circle

A circle is the locus of a point which moves such that its distance from a fixed point, called as centre, is equal to a given distance, called as radius. Thus, equation to a circle with centre (α, β) and radius a is :

$$(x - \alpha)^2 + (y - \beta)^2 = a^2$$

If the centre is origin then its equation reduces to $x^2 + y^2 = a^2$

General Equation of Circle

The general equation of a circle is :

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

It has three arbitrary constants. Its centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$. The circle is real, point circle or imaginary according as $g^2 + f^2 - c > 0$, $= 0$, or < 0 (We are concerned to real circles only. Actually the concept of imaginary circles is beyond our scope.) $c = 0$ if circle passes through origin.

Other forms of the circle are as following:

(i) **General Equation of Second Degree**

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ shall represent a circle if

Coefficient of x^2 & y^2 are equal i.e. $a = b \neq 0$ & Coefficient of xy is zero i.e. $h = 0$ and $g^2 + f^2 - ac \geq 0$

(ii) **Parametric Form**

Circle with centre (α, β) and radius a is $(x - \alpha)^2 + (y - \beta)^2 = a^2$

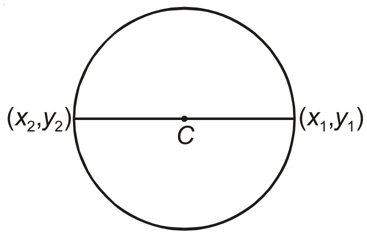
It can be represented in parametric form as

$$\left.\begin{array}{l} x = \alpha + a\cos\theta \\ y = \beta + a\sin\theta \end{array}\right] \begin{array}{l} \theta \text{ is parameter} \\ 0 \leq \theta < 2\pi \end{array}$$

A circle with centre (0, 0) and radius a is

$$\left.\begin{array}{l} x = a\cos\theta \\ y = a\sin\theta \end{array}\right] \quad \theta \text{ is parameter}$$

(iii) **Equation of Circle in Diametric Form**

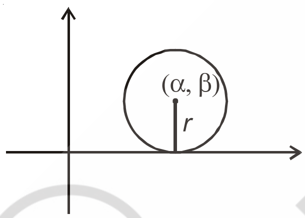


If two diametrically opposite points on a circle are (x_1, y_1) and (x_2, y_2) then centre = $\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)$

Radius = $\dfrac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

and equation of circle is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

(iv) **Equation of circle with centre (α, β) and touching the *x*–axis**

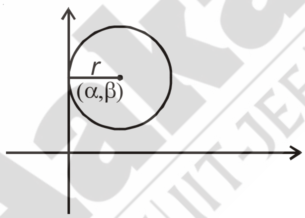


As it is clear from figure that $r = \beta$

$\therefore\ (x-\alpha)^2 + (y-\beta)^2 = \beta^2$

or $x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 = 0$

(v) **Equation of circle with centre (α, β) and touching the *y*–axis**

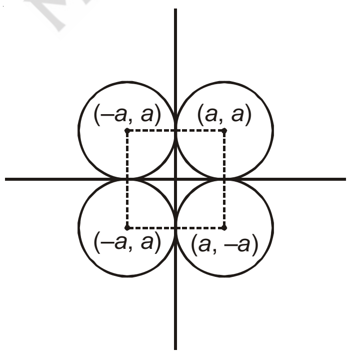


Here $r = \alpha$

$\therefore\quad (x-\alpha)^2 + (y-\beta)^2 = \alpha^2$

$\Rightarrow\quad x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$

(vi) **Equation of circle with radius a and touching both the axes**



There will be four such circles

There centres are $(\pm a, \pm a)$ and radius *a*.

$\therefore\quad (x \pm a)^2 + (y \pm a)^2 = a^2$

$\Rightarrow\quad x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$

(vii) **Circle through three non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3)**

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Then all three points must satisfy it

$$\Rightarrow \left.\begin{array}{l} x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \\ x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \\ x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \end{array}\right] \text{ simultaneous solution of these equations}$$

will give g, f & c and hence we can write the equation of circle.

Alternate : The equation of required circle passing through the non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Intersection of a line and circle

(i) Intercept made on x-axis by $S \equiv 0$ is : $2\sqrt{g^2 - c}$

(ii) Intercept made on y-axis by $S \equiv 0$ is : $2\sqrt{f^2 - c}$

(iii) If $g^2 = c$ (resp., $f^2 = c$), the circle touches the x-axis (resp. y-axis).

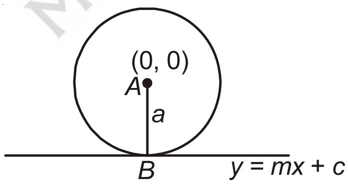
if $c = g^2 = f^2$ then circle touches both axes.

(iv) Length of the chord intercepted by $x^2 + y^2 = a^2$ on the line $y = mx + c$ is $2\sqrt{\dfrac{a^2(1+m^2) - c^2}{1+m^2}}$

(v) Intercepts are always positive.

Condition for a line to be a tangent to a circle

If a line is tangent to a circle then length of perpendicular from centre of circle to the line is equal to the radius of the circle. Let us discuss the condition for $x^2 + y^2 = a^2$. If the line $y = mx + c$ is tangent to $x^2 + y^2 = a^2$ then $c = \pm a\sqrt{1+m^2}$.



The centre of the circle is (0, 0). AB is the length of perpendicular from (0, 0) to the line.

$$AB = \left|\frac{0 - 0 - c}{\sqrt{1+m^2}}\right| = a$$

$$\Rightarrow \quad c = \pm a\sqrt{1+m^2}$$

Hence the line $y = mx \pm a\sqrt{1+m^2}$ always touches the circle $x^2 + y^2 = a^2$.

Equation of tangent to a circle

Corresponding to any point $P(x_1, y_1)$, we define the expressions

$$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

(i) Equation of **tangent at (x_1, y_1)** to $S \equiv 0$ is : $T = 0$

(ii) The equation of tangent of $x^2 + y^2 = a^2$ at (x_1, y_1) is $xx_1 + yy_1 - a^2 = 0$

(iii) The line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ at

real and distinct points if $c^2 < a^2(1 + m^2)$

real and coincident points if $c^2 = a^2(1 + m^2)$

imaginary points (means the line neither touches nor cuts the circle) if $c^2 > a^2(1 + m^2)$

$\therefore$ $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$

(iv) The line $y = mx + a\sqrt{1+m^2}$ is a tangent to $x^2 + y^2 = a^2$ at $\left(\dfrac{-am}{\sqrt{1+m^2}}, \dfrac{a}{\sqrt{1+m^2}}\right)$

The line $y = mx - a\sqrt{1+m^2}$ is tangent to $x^2 + y^2 = a^2$ at $\left(\dfrac{am}{\sqrt{1+m^2}}, \dfrac{-a}{\sqrt{1+m^2}}\right)$.

Some other standard result about the circle

(i) A point (x_1, y_1) lies inside, on or outside the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0, = 0$, or > 0

(ii) The length of tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ or $\sqrt{S_1}$.

(iii) The equation of tangent of $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$ or $T = 0$, where $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

(iv) Angle of intersection of two circles $S_1 = 0$ and $S_2 = 0$

Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1$

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2$

Let S_1 and S_2 have centres C_1 and C_2 respectively and radii r_1 and r_2 respectively. The angle of intersection θ of the circles is the angle between their tangents at points of intersection and is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - (C_1C_2)^2}{2r_1r_2} = \frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}.\sqrt{g_2^2 + f_2^2 - c_2}}$$

(v) Orthogonal circles :

Circles $S_1 \equiv 0$ and $S_2 \equiv 0$ are said to intersect orthogonally if $\theta = 90°$;

i.e., if $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

(vi) **Equation of a Circle through Intersection Points of Circle and Line**

Let the circle be $S = 0$ and line be $L = 0$

$S + \lambda L = 0$ is a circle which passes through the points of intersection of both.

(a) Equation of any circle that passes through two given points (x_1, y_1) and (x_2, y_2) is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

Let $u \equiv ax+by+k=0$

(b) If $u \equiv 0$ is a tangent to $S \equiv 0$ at the point P then $S+\lambda u \equiv 0$ is the equation of circles touching $S \equiv 0$ at P.

(c) Equation of the circles which touch the line $u \equiv 0$ at (x_1, y_1) is $(x-x_1)^2+(y-y_1)^2+\lambda u \equiv 0$

(vii) **Equation of a Circle through Intersection of Two Circles**

(a) If $S_1 \equiv 0$ and $S_2 \equiv 0$ be two circle, intersecting in real points, then $S_1+\lambda S_2=0$ $(\lambda \neq -1)$ is the equation of the family of circles passing through the common points of $S_1 \equiv 0$ and $S_2 \equiv 0$.

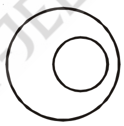
(b) If $S_1 \equiv 0$ and $S_2 \equiv 0$ intersect, then $S_1 - S_2 \equiv 0$ is the equation of their common chord.

(c) If $S_1 \equiv 0$ and $S_2 \equiv 0$ touch each other then $S_1 - S_2 \equiv 0$ is the equation of their common tangent at the point of contact.

(viii) **Common Tangents to Two Circles**

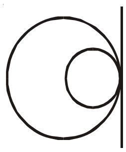
Let circles $S_1 \equiv 0$ and $S_2 \equiv 0$ have centers O_1 and O_2 respectively and radii r_1 and r_2 respectively. The number of common tangents of $S_1 \equiv 0$ and $S_2 \equiv 0$ is

(a) **zero,** if one circle lies totally inside the other



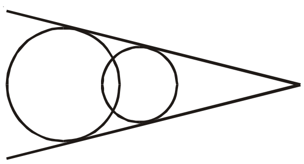
$O_1O_2 < |r_1 - r_2|$

(b) **one,** if circles touch each other internally



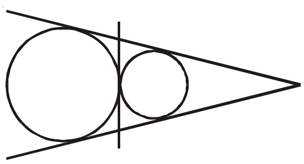
$O_1O_2 = |r_1 - r_2|$

(c) **two,** if circles intersect at two distinct points



$|r_1 - r_2| < O_1O_2 < r_1 + r_2$

(d) **three,** if circles touch each other externally

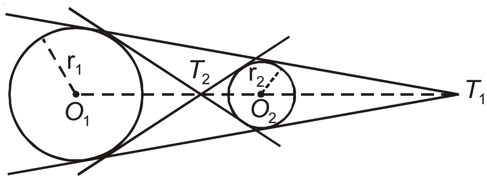


$O_1O_2 = r_1 + r_2$

(e) **four,** if circles lie outside each other

Two of the tangents are **direct common tangents.**

Two of the tangents are **transverse common tangents.**



$O_1O_2 > r_1 + r_2$

If direct common tangents meet the line O_1O_2 in T_1 then T_1 divides the segment O_1O_2 externally in the ratio $r_1 : r_2$.

If the transverse common tangents meet the line O_1O_2 in T_2 then T_2 divides the segment O_1O_2 internally in the ratio $r_1 : r_2$.

The coordinates of T_1 and T_2 having been found, the corresponding tangents are straight lines through it such that perpendiculars on them from O_1 are each equal to r_1.

Note : *If $S_1 = 0$ and $S_2 = 0$ touch each other, then their point of contact can be obtained by solving $S_1 = 0$ and $S_1 - S_2 = 0$ simultaneously.*

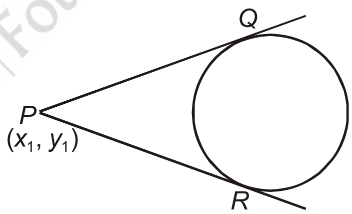
(ix) **Radical Axis**

The radical axis of $S_1 \equiv 0$ and $S_2 \equiv 0$ is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

The equation to the radical axis is : $S_1 - S_2 = 0$.

(x) **Equation of the pair of tangents :**

Combined equation of the tangents PQ and PR drawn from $P(x_1, y_1)$ to the circle $S = 0$ is given by



$\boxed{SS_1 = T^2}$

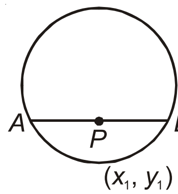
where $S = x^2 + y^2 + 2gx + 2fy + c$

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

(xi) **Equation of the chord whose middle point is given :**

If $P(x_1, y_1)$ be the middle point of a chord AB of the circle $S = 0$, then equation of the chord is given by



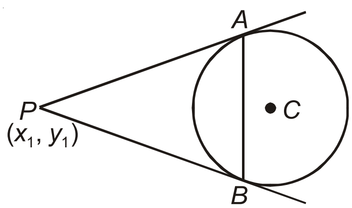
$\boxed{T = S_1}$

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$

(xii) **Chord of contact**

It is the line joining the points of contact of the tangents drawn from a point outside the circle.

If from a point $P(x_1, y_1)$, two tangents PA and PB are drawn. The line AB is the chord of contact of the point P with respect to the given circle.



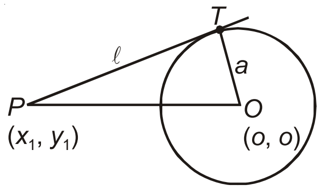
Equation of the chord of contact with respect to the

(i) Circles $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

(ii) Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(xiii) **Length of the tangent from a point to the circle**

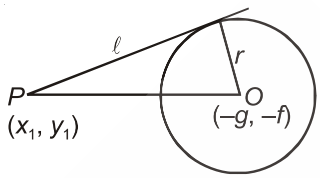
Let $P(x_1, y_1)$ be a point and PT is the tangent drawn from P to the circle $S \equiv x^2 + y^2 - a^2 = 0$, then



$$PT = \sqrt{OP^2 - OT^2}$$

$$= \sqrt{x_1^2 + y_1^2 - a^2}$$

If the circle is $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then



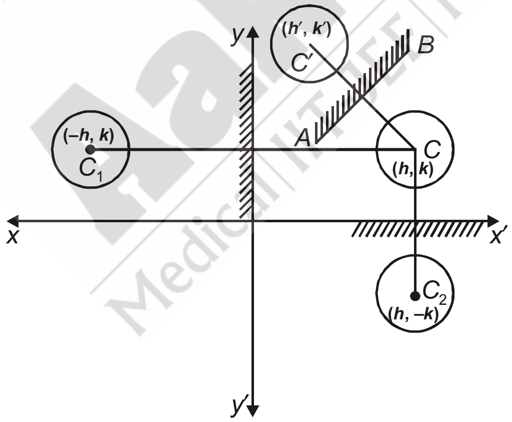
$$PT = \sqrt{OP^2 - OT^2}$$

$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

(xiv) **Image of the circle through a line**

Let $S \equiv (x - h)^2 + (y - k)^2 - a^2 = 0$

Radius of the image circle will be same as the radius of S, but centre will be changed



(a) Centre of the given circle (h, k)

(b) Centre of the image circle through y-axis is $c_1(-h, k)$

(c) Centre of the image circle through x-axis is $c_2(h, -k)$

(d) For the image through the line AB.

Let equation of AB is $\ell x + my + n = 0$ and let centre of the image circle be (h', k'). Then slope of CC' × slope of $AB = -1 \Rightarrow \left(\frac{k-k'}{h-h'}\right)\times\left(-\frac{\ell}{m}\right) = -1$ and middle point of $CC' = \left(\frac{h+h'}{2}, \frac{k+k'}{2}\right)$ lies on $\ell x + my + n = 0$ solving for h' and k'.

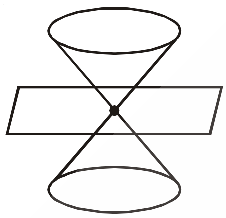
CONIC SECTIONS

Conic sections, namely a point, pair of straight lines, circle, ellipse, parabola and hyperbola are called so because they can be obtained when a cone (or double cone) is cut by a plane.

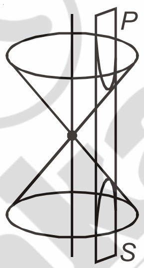
The mathematicians associated with the study of conics were Euclid, Aristarchus and Apollonius. Most of the objects around us and in space have shape of conic-sections. Hence study of these becomes a very important tool for present knowledge and further exploration.

Conic-sections as Sections of Right Circular Cone(s)

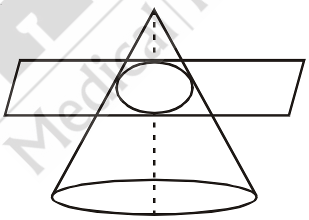
(i) When a double right circular cone is cut by a plane parallel to base at the common vertex, the cutting profile is a point.



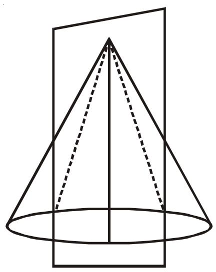
(ii) When a double right circular cone is cut by plane, parallel to its common axis, the cut profile is hyperbola



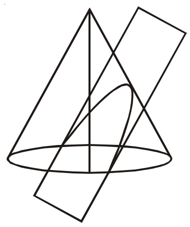
(iii) When a right circular cone is cut by a plane parallel to its base the cutting profile is a circle.



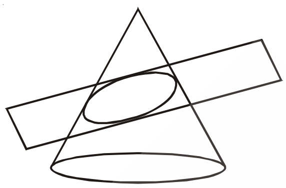
(iv) When a right circular cone is cut by any plane through its vertex, the cutting profile is a pair of straight lines through its vertex



(v) When a right circular cone is cut by a plane parallel to a generator of cone, the cutting profile is a parabola.



(vi) When a right circular cone is cut by a plane which is neither parallel to any generator / axis nor parallel to base, the cutting profile is an ellipse.

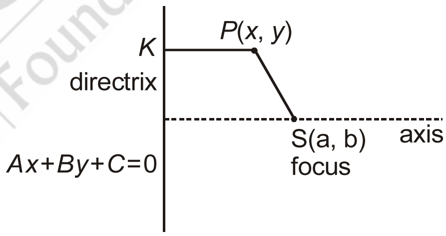


Hence a point, a pair of intersecting straight lines, circle, parabola, ellipse and hyperbola, all are conic-sections. All the conic sections are plane or two dimensional curves.

Conic sections (Definition of Parabola, Ellipse, Hyperbola)

The conic section is the locus of a point which moves such that the ratio of its distance from a fixed point (focus) to the distance from a fixed straight line (directrix) is constant (e), e is called eccentricity of conic *i.e.*,

$$\frac{PS}{PK} = e$$



For e = 1 the conic section is called parabola

For e < 1 the conic section is called ellipse

For e > 1 the conic section is called hyperbola

A line through focus and perpendicular to directrix is called - axis.
The vertex of conic is that point where the curve intersects the axis.

$$\frac{PS}{PK} = e \Rightarrow PS^2 = e^2 PK^2$$

or $(x-\alpha)^2 + (y-\beta)^2 = e^2\left(\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right)^2$

Simplification shall lead to the equation of the form

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ ... (*)

So equation of a conic is the general equation of second degree. If

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

if $\Delta = 0 \Rightarrow$ the equation (*) represents a pair of straight lines

if $\Delta \neq 0$ and $h = 0$ and $a = b \neq 0 \Rightarrow$ the equation (*) represents a circle provided $g^2 + f^2 - ac \geq 0$

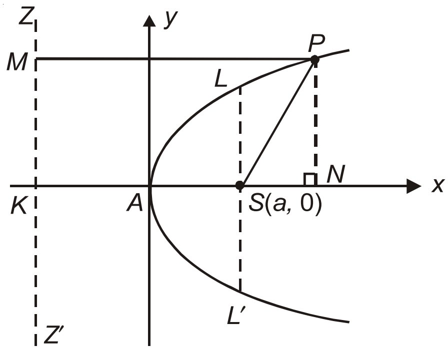
if $\Delta \neq 0$ and $h^2 = ab \Rightarrow$ the equation (*) represents a parabola

if $\Delta \neq 0$ and $h^2 < ab \Rightarrow$ the equation (*) represents an ellipse

if $\Delta \neq 0$ and $h^2 > ab \Rightarrow$ the equation (*) represents a hyperbola

PARABOLA

Standard equation of parabola :



To find the equation of parabola, let the focus is $S(a, 0)$ and the directrix is $x = -a$. P is general point on the parabola.

Here $PS = PM \Rightarrow PS^2 = PM^2$

If the coordinate of P is (x, y) then

$PS^2 = (x - a)^2 + (y - 0)^2$, $PM^2 = (x + a)^2$

But $PS^2 = PM^2 \Rightarrow (x - a)^2 + y^2 = (x + a)^2 + y^2$

$\Rightarrow y^2 = 4ax$

For this parabola

(i) Vertex A is $(0, 0)$

(ii) Focus S is $(a, 0)$

(iii) Equation of directrix ZZ' is $x = -a$.

(iv) Equation of axis AX is $y = 0$

(v) The parametric equation of $y^2 = 4ax$ are $x = at^2$, $y = 2at$.

Focal chord : A chord of a parabola which passes through the focus is called as focal chord. If $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are ends of a focal chord, then $t_1t_2 = -1$.

Focal distance : The distance, of a point on the parabola, from the focus is called as focal distance of the point. The semilatus rectum is the harmonic mean between the segments of focal chord.

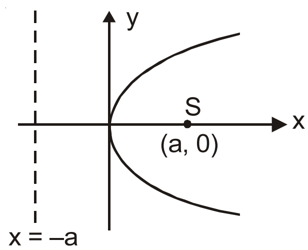
Latus rectum : The double ordinate LSL' through the focus is the latus rectum. Its equation is $x = a$ and its length is $4a$ units. Ends of latus rectum are $(a, 2a)$ and $(a, -2a)$.

Note : *(i) For parabola $PN^2 = 4AS.AN$.*

(ii) Two parabolas are said to be equal when their latus recta are equal.

Equations of Parabola in Standard Forms

(i) The equation $y^2 = 4ax$ represents a parabola whose



Focus : $(a, 0)$

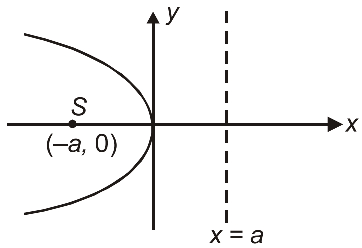
Directrix : $x = -a$

Vertex : $(0, 0)$

Axis : $y = 0$

Parametric coordinate : $x = at^2$, $y = 2at$

(ii) $y^2 = -4ax$



Focus : $(-a, 0)$

Directrix : $x = a$

Vertex : $(0, 0)$

Axis : $y = 0$

Parametric coordinate : $x = -at^2$, $y = 2at$

(iii) $x^2 = 4ay$

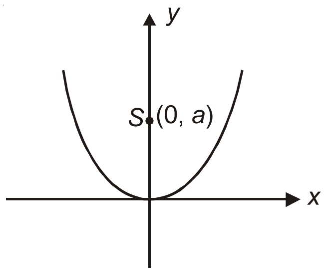
Focus : $(0, a)$

Directrix : $y = -a$

Vertex : $(0, 0)$

Axis : $x = 0$

Parametric coordinate : $x = 2at$, $y = at^2$



(iv) $x^2 = -4ay$

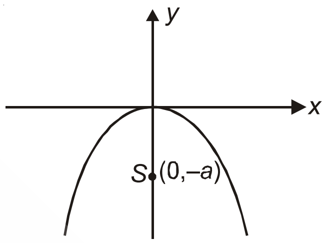
Focus : $(0, -a)$

Directrix : $y = a$

Vertex : $(0, 0)$

Axis : $x = 0$

Parametric coordinate : $x = 2at$, $y = -at^2$



Condition of tangency

If the line $y = mx + c$ touches the parabola $y^2 = 4ax$ then the number of solutions of $y = mx + c$ and $y^2 = 4ax$ will be one

$y = mx + c$ ... (i)

$y^2 = 4ax$ ... (ii)

By (i), (ii)

$(mx + c)^2 = 4ax$

$\Rightarrow \quad x^2(m^2) + x(2mc - 4a) + c^2 = 0$

Putting discriminant = 0 we get $c = \dfrac{a}{m}$

Hence if $y = mx + c$ touches the parabola $y^2 = 4ax$ then $c = \dfrac{a}{m}$.

(i) Equation of tangent at $P(x_1, y_1)$ to $y^2 = 4ax$ is $T \equiv 0$

i.e. $yy_1 = 2a(x + x_1)$

(ii) Equation of tangent in slope form is $y = mx + \dfrac{a}{m}$

Its point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

(iii) The equation of tangent at $(at^2, 2at)$ is $yt = x + at^2$

(iv) The line $y = mx + c$ is a tangent to $y^2 = 4ax$ if $c = \dfrac{a}{m}$

Some other standard results and equations

(i) Equation of pair of tangents

Through any given external point $P(x_1, y_1)$ there pass, in general, two tangents to $y^2 = 4ax$. When P is external to the parabola, the joint equation to the pair of tangents from P is $SS_1 = T^2$

where $S : y^2 - 4ax$, $S_1 : y_1^2 - 4ax_1$, $T : yy_1 - 2a(x + x_1)$

(ii) Points of intersection of two tangents

The tangents to the parabola $S \equiv y^2 - 4ax = 0$ drawn at the points $(at_1^2,\ 2at_1)$ and $(at_2^2,\ 2at_2)$ intersect at the point $(at_1t_2,\ a(t_1 + t_2))$.

(iii) Chord of contact

Equation of chord of contact of the point $P(x_1, y_1)$ (i.e. of tangents from P) with respect to the parabola $y^2 = 4ax$ is $T \equiv 0$ i.e., $yy_1 = 2a(x + x_1)$

(iv) Chord with middle point $P(x_1, y_1)$ has equation : $T \equiv S_1$

(v) Normals to the Parabola

(a) Equation of normal to the parabola $y^2 = 4ax$ at the point $P(x_1, y_1)$ is $y - y_1 = -\dfrac{y_1}{2a}(x - x_1)$

(b) Equation of normal to the parabola $y^2 = 4ax$ in slope form is $y = mx - 2am - am^3$, which is normal to the parabola at $(am^2, -2am)$

(c) Equation of normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y = -tx + 2at + at^3$

(d) The normals at the points $(at_1^2,\ 2at_1)$ and $(at_2^2,\ 2at_2)$ intersect at the point

$$(2a + a(t_1^2 + t_1t_2 + t_2^2),\ -at_1t_2(t_1 + t_2))$$

(e) Through a given point, maximum of three normals (at least one) can be drawn.

(f) The sum of the slopes of the normals drawn from a given point to a parabola, is zero.

(g) The sum of the ordinates of the feet of the normals drawn from a given point to a parabola is zero.

(h) If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \dfrac{2}{t_1}$

(vi) Point (x_1, y_1) lies inside, on, or outside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 < 0, = 0, > 0$.

(vii) The tangents at the ends of focal chord are perpendicular and meet at directrix.

(viii) The length of semilatus rectum of a parabola is the harmonic mean of focal radii. If the length of focal radii are b and c then the length of semilatus-rectum = $\dfrac{2bc}{b+c}$

ELLIPSE

Standard equation of ellipse

An ellipse is the locus of a point which moves such that its distance from a fixed point is e (< 1) times its distance from a fixed straight line. An ellipse has

(i) two foci

(ii) two directrices – one corresponding to each focus.

(iii) a center – the point such that all chords through it are bisected at it.

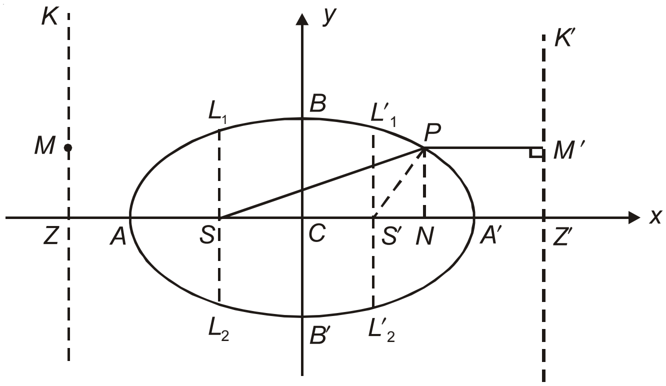
(iv) two axes – major and minor axes

(v) two vertices

(vi) two latus recta

Thus, the ellipse shown, has the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Whose

(i) Center is $C(0, 0)$

(ii) Vertices are $A(-a, 0)$ and $A'(a, 0)$, $B(0, b)$ and $B'(0, -b)$

(iii) Foci are $S(ae, 0)$ and $S'(-ae, 0)$

(iv) Directrix $K'M'Z'$ corresponds to focus S' and has equation $x=\frac{a}{e}$.

Directrix KMZ corresponds to focus S and has equation $x=-\frac{a}{e}$

(v) L_1L_2 and $L_1' L_2'$ are double ordinates through the foci and are called latus rectum. Each has length $\frac{2b^2}{a}$.

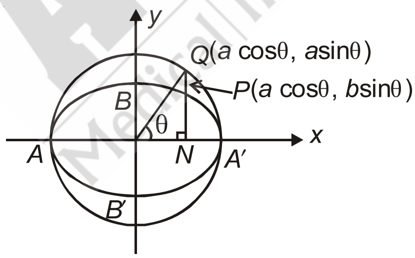
(vi) AA' is the major axis and has length $2a$. BB' is the minor axis and has length $2b$ where $a > b$ and $b^2 = a^2(1-e^2)$

Note : 1. *An ellipse is the locus of a point which moves such that the sum of its distances from two fixed points remains constant. In fact, $PS + PS' = 2a$ for all points $P(x, y)$ on the ellipse.*

2. *The constant in the above definition should be greater than the distance between the fixed points. This definition is called the physical definition of the ellipse.*

(vii) **Auxillary Circle**

The circle which is described on the major axis AA' of an ellipse as diameter is called auxillary circle of the ellipse.



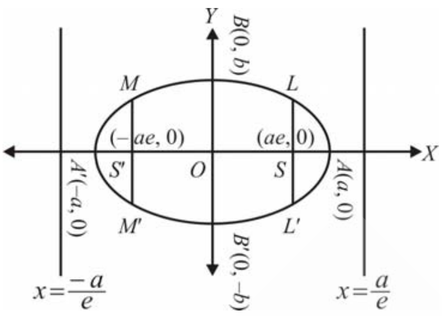
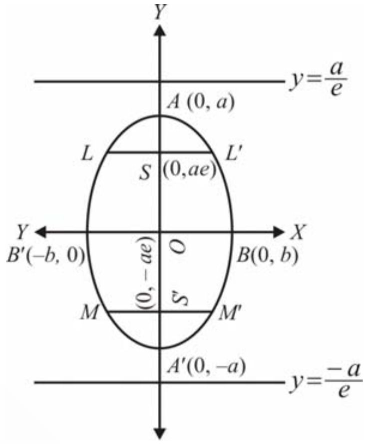
(viii) **Equation of ellipse in parametric form**

$x = a\cos\theta$, $y = b\sin\theta$; $0 \le \theta < 2\pi$

where θ is the parameter (as shown in the figure above) and is called as eccentric angle corresponding to the point P on the ellipse. The point Q has co-ordinates $(a\cos\theta, a\sin\theta)$. Further, $\frac{PN}{QN}=\frac{b}{a}$.

(ix) **Position of a point with respect to the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The point $R(x', y')$ lies within, upon or outside the ellipse according as $\frac{(x')^2}{a^2}+\frac{(y')^2}{b^2}-1$ is negative, zero or positive.

Two Standard Forms of the Ellipse		
Standard Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$ (Horizontal Form of an Ellipse)	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1\,(a>b)$ (Vertical Form of an Ellipse)
Shape of the Ellipse		
Centre	(0, 0)	(0, 0)
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Length of major axis	$2a$	$2a$
Length of minor axis	$2b$	$2b$
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{a}{e}$
Eccentricity	$e=\sqrt{\frac{a^2-b^2}{a^2}}$	$e=\sqrt{\frac{a^2-b^2}{a^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Ends of latra-recta	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{b^2}{a}, \pm ae\right)$
Parametric coordinates	$(a\cos\theta, b\sin\theta)$	$(b\cos\theta, a\sin\theta)$
Focal radii	$SP = a - ex_1$ and $S'P = a + ex_1$	$SP = a - ey_1$ and $S'P = a + ey_1$
Sum of focal radii $SP + S'P =$	$2a$	$2a$
Distance between foci	$2ae$	$2ae$
Distance between directrices	$\frac{2a}{e}$	$\frac{2a}{e}$
Tangents at the vertices	$x = \pm a$	$y = \pm a$

Condition of tangency for $y = mx + c$

$y = mx + c$ ... (i)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (ii)

By (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$\Rightarrow$ $b^2x^2 + a^2(mx + c)^2 - a^2b^2 = 0$

$\Rightarrow$ $x^2(b^2 + a^2m^2) + x(2mca^2) + a^2(c^2 - b^2) = 0$ ... (iii)

If line $y = mx + c$ is tangent to the ellipse then the equation (iii) has equal roots, hence discriminant = 0

$\Rightarrow$ $(2mca^2)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$

$\Rightarrow$ $c = \pm\sqrt{a^2m^2 + b^2}$

Hence if $y = mx + c$ is tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $c = \pm\sqrt{a^2m^2 + b^2}$

Equation of tangent and point of tangency

Corresponding to any point $P(x_1, y_1)$ we define the following expressions for the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

viz., $$T \equiv axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c$$

$$S_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

(i) The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at

two distinct points if $c^2 < a^2m^2 + b^2$

two coincident points if $c^2 = a^2m^2 + b^2$

imaginary points if $c^2 > a^2m^2 + b^2$

(ii) Length of the chord intercepted by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the line $y = mx + c$ is

$$\frac{2ab\sqrt{1+m^2}\sqrt{a^2m^2 + b^2 - c^2}}{a^2m^2 + b^2}$$

(iii) Equation of tangent at the point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$T \equiv 0$; *i.e.*, $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(iv) Equation of tangent at ($a\cos\theta$, $b\sin\theta$) is $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$

(v) Equation of tangent in slope form is $y=mx\pm\sqrt{a^2m^2+b^2}$

The point of contact of $y=mx+\sqrt{a^2m^2+b^2}$ is $\left(\dfrac{-a^2m}{\sqrt{a^2m^2+b^2}},\ \dfrac{b^2}{\sqrt{a^2m^2+b^2}}\right)$

and point of contact of $y=mx-\sqrt{a^2m^2+b^2}$ is $\left(\dfrac{a^2m}{\sqrt{a^2m^2+b^2}},\ \dfrac{-b^2}{\sqrt{a^2m^2+b^2}}\right)$

(vi) **Point of intersection of two tangents**

Tangents at ($a\cos\theta_1$, $b\sin\theta_1$) and ($a\cos\theta_2$, $b\sin\theta_2$) intersect at $\dfrac{a\cos\left(\dfrac{\theta_1+\theta_2}{2}\right)}{\cos\left(\dfrac{\theta_1-\theta_2}{2}\right)},\ \dfrac{b\sin\left(\dfrac{\theta_1+\theta_2}{2}\right)}{\cos\left(\dfrac{\theta_1-\theta_2}{2}\right)}$

(vii) **Equation of pair of tangents**

Through any given external point $P(x_1, y_1)$ there pass, in general, two tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

When P is external to the ellipse, the joint equation of the pair of tangents from P is $SS_1=T^2$

Standard results:

(i) Chord of contact

The equation of chord of contact of tangents drawn from $P(x_1, y_1)$ is $T\equiv 0$ i.e. $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}-1=0$

(ii) Director Circle

It is the locus of point of intersection of tangents which are perpendicular to each other. Its equation is $x^2+y^2=a^2+b^2$

(iii) Equation of Chord in Mid Point Form

Equation of the chord with middle point $P(x_1, y_1)$ is $T\equiv S_1$ i.e. $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}$

(iv) Normals to the Ellipse

In general, four normals can be drawn from any point to an ellipse and the sum of the eccentric angles of their feet is equal to an odd multiple of two right angles

(a) Equation of normal at $P(x_1, y_1)$ is $\dfrac{x-x_1}{\dfrac{x_1}{a^2}}=\dfrac{y-y_1}{\dfrac{y_1}{b^2}}$

(b) Equation of normal at ($a\cos\theta$, $b\sin\theta$) is $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$

(c) Equation of normal in slope form is $y=mx\pm\dfrac{m(a^2-b^2)}{\sqrt{b^2m^2+a^2}}$

(v) Equation of chord joining two points

The equation of chord joining the points (a cos α, b sin α) and (a cos β, b sin β) is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

HYPERBOLA

Standard equation of hyperbola

The hyperbola is the locus of a point which moves such that its distance from a fixed point is e (> 1) times its distance from a fixed straight line. As in case of an ellipse, the hyperbola has

(i) two foci

(ii) two directrices – one corresponding to each focus.

(iii) a center

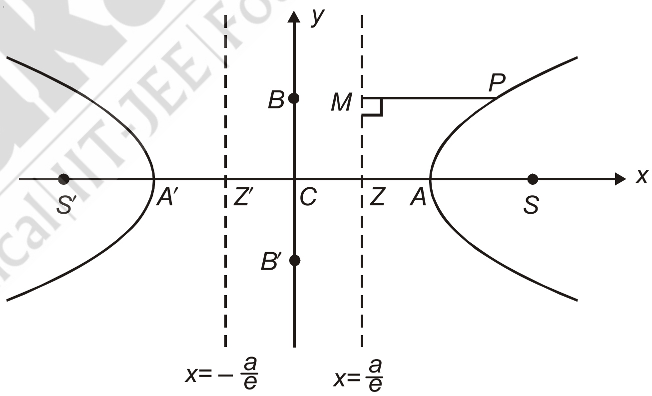
(iv) two axes – the transverse axis and conjugate axis.

(v) two vertices

(vi) two latus recta

Thus, the hyperbola shown in the adjacent figure has the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$



(i) Centre is $C(0, 0)$

(ii) Vertices are $A'(-a, 0)$ and $A(a, 0)$

(iii) Foci are $S(ae, 0)$; $S'(-ae, 0)$

(iv) Directrices are $x=\pm\frac{a}{e}$

(v) AA' is the transverse axis and has length $2a$; BB' is the conjugate axes where B and B' are two points on the axis of y equidistant from C such that $CB = B'C = b$ and where $b^2=a^2(e^2-1)$.

(vi) Length of latus rectum is $\frac{2b^2}{a}$.

Note : *The difference of the focal distances of any point on the hyperbola is equal to length of transverse axis ; thus,* $|PS-PS'|=2a$.

(vii) **Auxillary Circle**

The circle described on the transverse axis AA' of a hyperbola as diameter is called as auxillary circle of the hyperbola and its equation is $x^2 + y^2 = a^2$.

(viii) **Equation of Hyperbola in Parametric Form**

$x = a \sec\theta$; $y = b \tan\theta$, where $0 \le \theta < 2\pi$ (θ is the parameter)

(ix) **Position of a Point with respect to Hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The point $R(x', y')$ lies within, upon or outside the hyperbola according as $\frac{(x')^2}{a^2} - \frac{(y')^2}{b^2} - 1$ is positive, zero or negative.

(x) **Equilateral or Rectangular Hyperbola**

The hyperbola $x^2 - y^2 = a^2$, whose asymptotes are at right angles ($y = \pm x$) is called as an equilateral or rectangular hyperbola. If this hyperbola is rotated such that the coordinate axes coincides with the asymptotes, then, its equation reduces to $xy = c^2$ where $c^2 = \frac{a^2}{2}$.

(a) Parametric form of $xy = c^2$ is $x = ct$; $y = \frac{c}{t}$

(b) Centre is (0, 0)

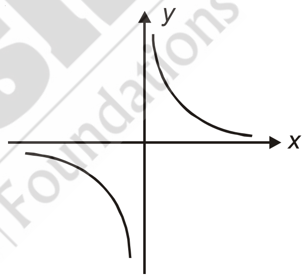
(c) Transverse axis has equation $y = x$

Conjugate axis has equation $y = -x$

(d) Its eccentricity is $e = \sqrt{2}$.

Tangent at $\left(ct, \frac{c}{t}\right)$ is $t^2 y + x = 2ct$

Normal at $\left(ct, \frac{c}{t}\right)$ is $xt^3 - ty - ct^4 + c = 0$



$xy = c^2$

Condition of tangency for $y = mx + c$

$y = mx + c$ ... (i)

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ... (ii)

If $y = mx + c$ is tangent to hyperbola then the roots of the equation $\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$ will be qual.

For that the condition is $c = \pm\sqrt{a^2m^2 - b^2}$. Hence if the line $y = mx + c$ is tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then $c = \pm\sqrt{a^2m^2 - b^2}$.

Equation of tangent and point of tangency

Corresponding to any point $P(x_1, y_1)$, we define the following expressions for the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

viz., $T \equiv axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$

$$S_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

(i) The line $y = mx + c$ intersects the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at

real and distinct point if $c^2 > a^2m^2 - b^2$

real and coincident points if $c^2 = a^2m^2 - b^2$

imaginary points if $c^2 < a^2m^2 - b^2$

(ii) Equation of tangent at the point (x_1, y_1) lying on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T \equiv 0$;

i.e., $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(iii) Equation of tangent at $(a \sec\theta, b \tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

(iv) Equation of tangent in slope form is $y = mx \pm \sqrt{a^2m^2 - b^2}$

The point of contact of $y = mx + \sqrt{a^2m^2 - b^2}$ is $\left(\frac{-a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{-b^2}{\sqrt{a^2m^2 - b^2}}\right)$

The point of contact of $y = mx - \sqrt{a^2m^2 - b^2}$ is $\left(\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$

(v) **Point of Intersection of Two Tangents**

The tangents at $(a \sec\theta_1, b \tan\theta_1)$ and $(a \sec\theta_2, b \tan\theta_2)$ intersect at

$$\left(\frac{a\sin(\theta_2 - \theta_1)}{\sin\theta_2 - \sin\theta_1}, \frac{b(\cos\theta_1 - \cos\theta_2)}{\sin\theta_2 - \sin\theta_1}\right)$$

(vi) **Equation to Pair of Tangents**

Through any given external point $P(x_1, y_1)$ there pass in general, two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. When $P(x_1, y_1)$ is external to the hyperbola, the joint equation to the pair of tangents from P is $SS_1 = T^2$

i.e., $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$

Standard results

(i) **Chord of Contact**

Equation of chord of contact of the point (x_1, y_1) with respect to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is $T \equiv 0$

(ii) **Director Circle**

It is the locus of point of intersection of tangents which are perpendicular to each other. Its equation is $x^2+y^2=a^2-b^2$

(iii) **Equation of Chord in Mid Point Form**

Equation of the chord with middle point (x_1, y_1) is $T \equiv S_1$

(iv) **Equation of Normal**

(a) Equation of normal to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point (x_1, y_1) is $y-y_1=-\dfrac{a^2 y_1}{b^2 x_1}(x-x_1)$

(b) Equation of normal at $(a \sec\theta, b\tan\theta)$ is $ax\cos\theta+by\cot\theta=a^2+b^2$

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