# Chapter 2

# Complex Numbers and Quadratic Equations

## COMPLEX NUMBERS

### IMAGINARY NUMBERS

Square root of a negative number is called imaginary number. While solving the equations $x^2 + 1 = 0$, a quantity $\sqrt{-1}$ is obtained and is denoted by $i$ (iota) which is imaginary.

Further $\sqrt{-2}$ is an imaginary number and can be written as

$$\sqrt{-2} = \sqrt{2} \times \sqrt{-1} \quad = \sqrt{2}\, i$$

If $a < 0$, then $\sqrt{a} = \sqrt{|a|}\, i$

### Integral Powers of *i*

We have $i = \sqrt{-1}$ so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

For any $n \in N$, we have

$$i^{4n+1} = i\,,\ i^{4n+2} = -1$$

$$i^{4n+3} = -i,\ i^{4n} = 1$$

Thus any integral power of $i$ can be expressed as $\pm 1$ or $\pm i$.

In other words $i^n = \begin{cases} (-1)^{n/2}, & \text{if } n \text{ is even integer} \\ (-1)^{\frac{n-1}{2}} i, & \text{if } n \text{ is odd integer} \end{cases}$.

Also $i^{-n} = \dfrac{1}{i^n}$

***Note :*** *$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true iff at least one of a and b are non-negative. If a < 0 and b < 0, then*

$$\sqrt{a} \times \sqrt{b} = \sqrt{-|a|} \times \sqrt{-|b|}$$

$$= i\sqrt{|a|} \times i\sqrt{|b|}$$

$$= -\sqrt{ab}$$

*Thus $\sqrt{ab} = \sqrt{a}.\sqrt{b}$ only when at most one of a & b is negative*

## COMPLEX NUMBERS

The formal addition, '$a + ib$', where $a, b \in R$ and the collection of all such expressions is called the set of complex numbers. For the complex number, $z = a + ib$, '$a$' is called as real part of $z$ and is denoted by $Re(z)$ while '$b$' is called as imaginary part of $z$ and is denoted by Im($z$).

Set of complex numbers is denoted by $C$, which includes the set of real numbers $R$. *i.e.*, $R \subset C$

A complex number $z$ is said to be purely real if $I_m(z) = 0$ and is said to be purely imaginary if $Re(z) = 0$. The complex number $0 + 0i = 0$ is both purely real and purely imaginary.

All purely imaginary numbers except zero are imaginary numbers but an imaginary number may or may not be purely imaginary.

For *e.g.,* $4 + 3i$ is imaginary but not purely imaginary.

### Equality of Complex numbers

Two complex numbers $a + ib$ and $c + id$ are said to be equal, if and only if, $a = c$ and $b = d$. *i.e.*, the corresponding real and imaginary parts are equal.

If $a + ib = C_1$ and $c + id = C_2$

then either $C_1 = C_2$

or $C_1 \neq C_2$

**Note :** *For imaginary numbers, the property of order is not defined because i is neither positive, zero nor negative.*

*i.e.,* $C_1 > C_2$ *or* $C_1 < C_2$ *is meaningless if b and d are not equal to zero.*

## ALGEBRA OF COMPLEX NUMBERS

(i) Addition : $(a + ib) + (c + id) = (a + c) + i\,(b + d)$

(ii) Subtraction : $(a + ib) - (c + id) = (a - c) + i\,(b - d)$

(iii) Multiplication : $(a + ib) \cdot (c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc)$

(iv) Division : $\dfrac{a+ib}{c+id} = \dfrac{ac+bd}{c^2+d^2} + i\dfrac{(bc-ad)}{c^2+d^2}$ (When atleast one of $c$ and $d$ is non-zero)

### Conjugate Complex Number

For $z = a + ib$, its conjugate is defined as $\bar{z} = a - ib$. Here the complex conjugate is obtained just by changing sign of $i$.

#### Properties of Conjugate Complex Number :

(i) $\overline{(\bar{z})} = z$

(ii) $z = \bar{z}$ iff $z$ is purely real

(iii) $z + \bar{z} = 2Re(z) \Rightarrow Re(z) = Re(\bar{z}) = \dfrac{z+\bar{z}}{2}$

(iv) $z - \bar{z} = 2i\,Im(z) \Rightarrow Im(z) = \dfrac{z-\bar{z}}{2i}$

(v) $z = -\bar{z}$ iff $z$ is purely Imaginary

(vi) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(vii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

(ix) $\overline{\left(z^n\right)} = \left(\overline{z}\right)^n$

(x) $z_1\overline{z_2} + \overline{z_1}z_2 = 2\text{Re}(\overline{z_1}z_2) = 2\text{Re}(z_1\overline{z_2})$

(xi) If $z = f(z_1)$, then $\overline{z} = f(\overline{z}_1)$

## Modulus of a Complex Number

The modulus of a complex number $z = x + iy$ is defined as $|z| = \sqrt{x^2 + y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$ . In other way distance of a complex number $z$ from origin while represented on argand plane is called as modulus of a complex number denoted by mod($z$) or $|z|$, or $r$.

Here $OP = r = \sqrt{x^2 + y^2}$

$|z|$ is also called absolute value of $z$.

> ***Note :*** *$|z_1 - z_2|$ is the distance between $z_1$ and $z_2$.*

### Properties of Modulus :

(i) $|z| \geq 0$; $|z| = 0$ iff real and imaginary parts are zero.

(ii) $|z_1z_2| = |z_1||z_2|$. In general $|z_1z_2......z_n| = |z_1||z_2|....|z_n|$

(iii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, (z_2 \neq 0)$

(iv) $|z| = |\overline{z}| = |-z| = |-\overline{z}|$

(v) $z\overline{z} = |z|^2$

(vi) $-|z| \leq \text{Re}(z) \leq |z|, \quad -|z| \leq \text{Im}(z) \leq |z|$

(vii) $|z^n| = |z|^n$

(viii) $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2)$

$= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2$

$= |z_1|^2 + |z_2|^2 + 2\text{Re}\,(z_1\overline{z}_2)$

(ix) $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z}_1 - \overline{z}_2)$

$= z_1\overline{z}_1 - z_1\overline{z}_2 - z_2\overline{z}_1 + z_2\overline{z}_2$

$= |z_1|^2 + |z_2|^2 - 2\text{Re}\left(z_1\overline{z}_2\right)$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

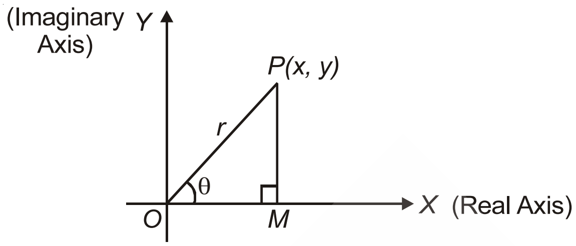
(xi) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ where $a, b \in R$

## Representation of a Complex Number

### Geometrical Representation

The complex number, $z = x + iy$ can be associated with the ordered pair $P(x, y)$.

We consider two perpendicular lines $OX$ and $OY$ (analogous to the Cartesian system) called as the real axis and imaginary axis respectively and where $O$ denotes the origin of reference. The resulting plane is called as Argand plane or Gaussian plane or complex plane and $z$ is represented by the point $P$ corresponding to the ordered pair $(x, y)$. The point $P$ is called as affix of $z$.




### Argument or Amplitude of $z$

The argument of $z$, denoted by arg $z$ or amp $z$ is the angle which $OP$ makes with the positive direction of real axis, the angle being measured in anticlockwise sense.

If $z = x + iy$ then the angle $\theta$ given by $\tan\theta = \frac{y}{x}$ is said to be the argument of $z$.

or $\quad \arg(z) = \text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$

> ***Note :*** *Arg (z) is not unique; if $\theta_1$ is one value, then $\theta_1 + 2k\pi \;:\; k \in I$ is the set of all possible values of arg z. Any two arguments of a complex number differ by $2k\pi$.*

Principal argument of a complex number $z$ is that value of argument ($z$) which lies in the interval $(-\pi, \pi]$.

For $z = x + iy$, $\theta = \tan^{-1}\left|\frac{y}{x}\right|$, then principal argument depends on the quadrant in which point $(x, y)$ lies.

(i) $x > 0, y = 0 \;\Rightarrow$ Principal argument = 0 (Positive Real Axis)

(ii) $x > 0, y > 0 \;\Rightarrow$ Prinicpal argument = $\theta$ (Ist quadrant)

(iii) $x = 0, y > 0 \;\Rightarrow$ Prinicpal argument = $\frac{\pi}{2}$ (Positive Imaginary Axis)

(iv) $x < 0, y > 0 \;\Rightarrow$ Principal argument = $\pi - \theta$ (IInd quadrant)

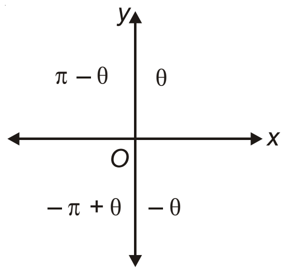
(v) $x < 0, y = 0 \;\Rightarrow$ Prinicpal argument = $\pi$ (Negative Real Axis)

(vi) $x < 0, y < 0 \;\Rightarrow$ Principal argument = $-\pi + \theta$ (IIIrd quadrant)

(vii) $x = 0, y < 0 \;\Rightarrow$ Principal argument = $-\frac{\pi}{2}$ (Negative Imaginary Axis)

(viii) $x > 0, y < 0 \;\Rightarrow$ Principal argument = $-\theta$ (IVth quadrant)

### Properties of Argument




(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi \;:\; k \in I$

(ii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi \;:\; k \in I$

(iii) $\arg(z^2) = 2\arg(z) + 2k\pi \;:\; k \in I$

(iv) If arg (z) = 0 or $\pi$ then z is real

(v) $\arg\left(\dfrac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi \;:\; k \in I$

(vi) $\arg(z^n) = n\arg(z) + 2k\pi \;:\; k \in I$

(vii) If $\arg\left(\dfrac{z_2}{z_1}\right) = \theta$, then $\arg\left(\dfrac{z_1}{z_2}\right) = 2k\pi - \theta; k \in I$

(viii) $\arg(\bar{z}) = -\arg(z)$

(ix) $\arg(z - \bar{z}) = \pm\dfrac{\pi}{2}$

(x) $\arg(z) - \arg(-z) = \pm\pi$

## Polar form (Trigonometric form) of a Complex Number

Let $OP = r$, then $x = r\cos\theta$, and $y = r\sin\theta$

$\Rightarrow$ $z = x + iy = r\cos\theta + ir\sin\theta, = r(\cos\theta + i\sin\theta)$. This is known as Trigonometric (or Polar) form of a complex Number. Here we should take the principal value of $\theta$.

For general values of the argument

$z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ (where $n$ is an integer)

> ***Note :*** *Sometimes $\cos\theta + i\sin\theta$, in short it written as cis($\theta$).*

## Exponential form of a Complex Number (Euler's Form)

According to Euler's Theorem, $e^{i\theta} = \cos\theta + i\sin\theta$ and therefore $z = r(\cos\theta + i\sin\theta)$ can be written as $z = re^{i\theta}$ which is called as exponential form of a complex number.

Replacing $\theta$ by $-\theta$ in $e^{i\theta}$, we obtain

$e^{-i\theta} = \cos\theta - i\sin\theta$

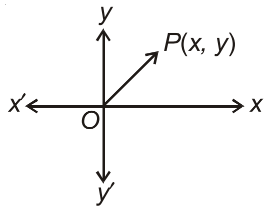
Hence $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{x}{\sqrt{x^2 + y^2}}$

and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{y}{\sqrt{x^2 + y^2}}$

The formulae $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$; $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ are known as Euler's formulae.

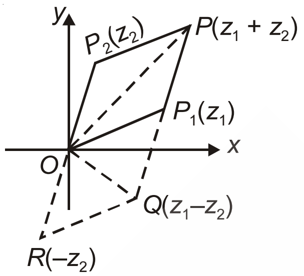
## Vector representation of a Complex Number

A complex number $z = x + iy$ can be represented by the position vector $OP$ of point $P(x, y)$ in a two dimensional plane becuase a complex number depends on two things viz (i) its modulus and (ii) its argument which are also the requirements of a vector on a plane.




## Geometrical meaning of Algebraic Operations

(i) Let $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ be represented by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively.




Then, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ is represented by the point $P$ (where $OP_1PP_2$ is a parallelogram). Similarly $z_1 - z_2 = z_1 + (-z_2)$ is represented by the point $Q$ (as shown).

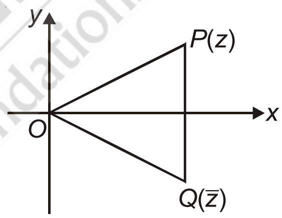
In vector notation $z_1 + z_2 = \overrightarrow{OP_1} + \overrightarrow{OP_2} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = \overrightarrow{OP}$

and $z_1 - z_2 = \overrightarrow{OP_1} - \overrightarrow{OP_2} = \overrightarrow{OP_1} + \overrightarrow{OR} = \overrightarrow{OQ}$

(ii) If $z = x + iy$, then, the conjugate of $z$,

denoted by $\bar{z}$ is the number $\bar{z} = x - i\,y$,

which is the mirror image of $z$ along real axis.




***Note :***

1. *If z lies in 2nd quadrant, then $\bar{z}$ lies in 3rd quadrant, and if z lies in 1st quadrant, then $\bar{z}$ lies in 4th quadrant.*
2. *If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be represented by point $Q_1$ $(x_1, y_1)$ and $Q_2$ $(x_2, y_2)$ respectively then $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$ is represented by the point P (as shown)*
3. *In polar form, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then the multiplication $z_1z_2 = r_1r_2e^{i}\left(\theta_1\theta_2\right)$ increases or decreases the modulus of $z_2$ by $r_1$ times and rotates $z_2$ anticlockwise by an angle equal to argument of $z_1$ about an axis passing through its tail and perpendicular to the argand plane in which the two complex vectors are lying.*
4. *Multiplication of a complex number $z(= re^{i\theta})$ by i rotates z anticlockwise about origin by $\frac{\pi}{2}$ and mulitplication of z by –i rotates z clockwise about origin by $\frac{\pi}{2}$.*

## TRIANGLE INEQUALITY

In any triangle, sum of any two sides is greater than the third side and difference of any two sides is less than the third side, we have

(i) $|z_1| + |z_2| \geq |z_1 + z_2|$; here equality holds when $\arg\left(\frac{z_1}{z_2}\right) = 0$ *i.e.*, position vectors representing two complex numbers $z_1$ and $z_2$ are parallel.

(ii) $\left\| z_1 \right| - \left| z_2 \right\| \le \left| z_1 - z_2 \right|$; here equality holds when $\arg\left(\dfrac{z_1}{z_2}\right) = 0$ *i.e.*, position vectors representing two complex numbers $z_1$ and $z_2$ are parallel.

### Parallelogram law :

In the parallelogram $OP_1PP_2$, the sum of the squares of its sides is half of the sum of the squares of its diagonals

$$\Rightarrow \quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

### Square Root of a Complex Number:

Let $z = x + iy$ be a complex number such that $\sqrt{x + iy} = a + ib$ where $x, y \in R$

$\sqrt{x + iy} = a + ib \quad \Rightarrow \quad (a + ib)^2 = x + iy$

$\Rightarrow \quad a^2 - b^2 + 2abi = x + iy$

On equating real and imaginary parts,

$x = a^2 - b^2$ ......(i)

and $y = 2ab$ ......(ii)

Solving (i) and (ii), we get $a = \pm\sqrt{\dfrac{1}{2}\left(x + \sqrt{x^2 + y^2}\right)}$

and $b = \pm\sqrt{\dfrac{1}{2}\left(\sqrt{x^2 + y^2} - x\right)}$

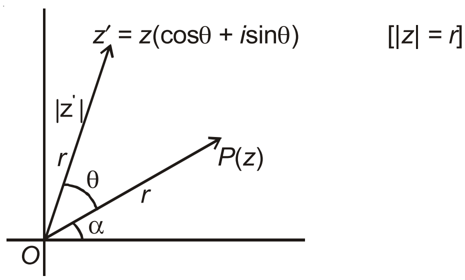
$$\text{Hence } \sqrt{x + iy} = \begin{cases} \pm\left[\sqrt{\dfrac{|z| + x}{2}} + i\sqrt{\dfrac{|z| - x}{2}}\right] & \text{for } y > 0 \\ \pm\left[\sqrt{\dfrac{|z| + x}{2}} - i\sqrt{\dfrac{|z| - x}{2}}\right] & \text{for } y < 0 \end{cases}$$

***Note :** This result need not to be memorised.*

**Remember :**

$$\sqrt{i} = \pm\left(\frac{1+i}{\sqrt{2}}\right), \sqrt{-i} = \pm\left(\frac{1-i}{\sqrt{2}}\right)$$

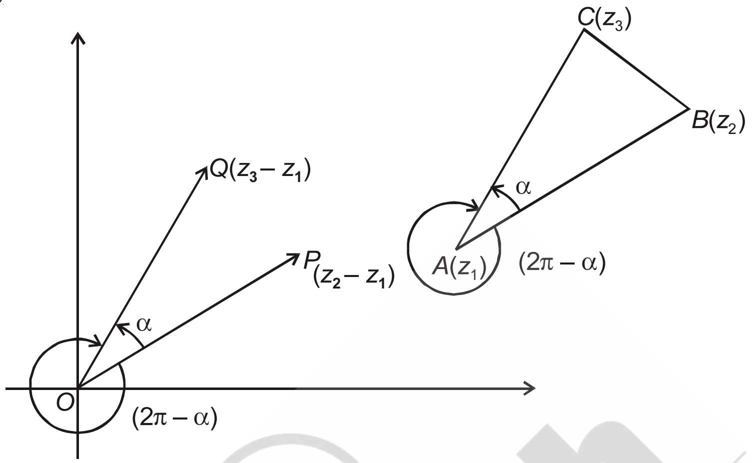
**Concept of Rotation :** Multiplication of a complex number $z$ represented by point $P$, with $e^{i\theta} = (\cos\theta + i\sin\theta)$ rotates the line $OP$ by an angle $\theta$ in anti-clockwise direction about $O$.



Let $z_1$, $z_2$, $z_3$ be the affixes of the vertices of a $\Delta ABC$ described in the counter-clockwise sense. Draw $OP$ and $OQ$ parallel and equal to $AB$ and $AC$ respectively. Then the point $P$ is $z_2 - z_1$ and $Q$ is $z_3 - z_1$ and

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP}(\cos\alpha + i\sin\alpha) = \frac{CA}{BA}e^{i\alpha} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right|e^{i\alpha}$$

or $\quad \text{amp}\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \alpha$




> ***Note*** **:** We can also write as $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}e^{-i(2\pi - \alpha)}$

In this case, we are rotating $\overrightarrow{OP}$ in clockwise direction by an angle $(2\pi - \alpha)$. Since the rotation is in clockwise direction, we are taking negative sign with angle $(2\pi - \alpha)$.

## de Moivre's Theorem

**Case I :**

Statement : If $n$ is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$

**Case II :**

Statement : If $p, q \in Z$ and $q \neq 0$, then

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(2k\pi + \frac{p\theta}{q}\right) + i\sin\left(2k\pi + \frac{p\theta}{q}\right)$$

where $k = 0, 1, 2, 3, \ldots, q - 1$

**Cube roots of unity**

(i) The cube roots of unity are $1, \dfrac{-1 + i\sqrt{3}}{2}, \dfrac{-1 - i\sqrt{3}}{2}$.

(ii) If $\omega$ is one of the imaginary cube roots of unity, then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3. Also $\omega^3 = 1$

(iii) In polar form the cube roots of unity are

$$\cos 0 + i\sin 0; \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}; \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

(iv) The three cube roots of unity when plotted on the argand plane constitutes the vertices of an equilateral triangle.

(v) The following factorization should be remembered :

($a$, $b$, $c \in R$ and $\omega$ is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$; $a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

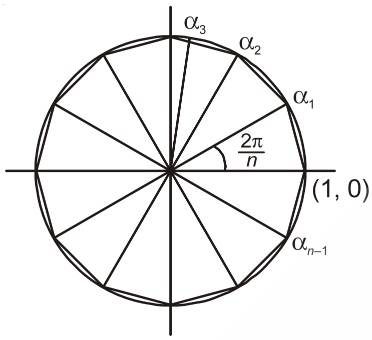
## $n^{th}$ roots of unity

If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the $n^{th}$ root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$.

(ii) $1^p + (\alpha_1)^p + (\alpha_2)^p + \ldots + (\alpha_{n-1})^p = \begin{cases} 0, \text{ if } p \text{ is not an integral multiple of } n. \\ n, \text{ if } p \text{ is an integral multiple of } n. \end{cases}$

(iii) $1.\alpha_1.\alpha_2 \ldots \alpha_{n-1} = (-1)^{n-1}$




## Application of Complex Numbers to Geometrical Application

**1. Distance Formula**

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex number represented by points $P$ and $Q$ on the argand plane.

Then the distance $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Also $z_2 - z_1 = (x_2 + iy_2) - (x_1 + iy_1) = (x_2 - x_1) + i(y_2 - y_1)$

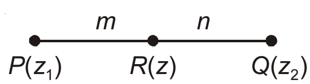
$\therefore \quad |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = PQ$

Hence distance between two points $z_1$ and $z_2$ is given by $|z_2 - z_1|$

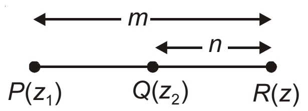
**2. Section formula**

If the point R($z$) divides the line segment $PQ$ (where $P$ and $Q$ represent the complex numbers $z_1$ and $z_2$ respectively)

(i) Internally in the ratio $m : n$ then, $z = \dfrac{mz_2 + nz_1}{m + n}$




(ii) Externally in the ratio $m : n$ then, $z = \dfrac{mz_2 - nz_1}{m - n}$




(iii) Mid point of line segment $PQ = \dfrac{z_1 + z_2}{2}$

**3. Area of Triangle**

If the vertices of a triangle $ABC$ represent the complex numbers $z_1$, $z_2$ and $z_3$ respectively then area of the triangle is the modulus of $\dfrac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}$.

**4. Slope of line segment joining two points**

If $P$ and $Q$ represent complex numbers $z_1$ and $z_2$ respectively in the Argand plane, then the complex slope of $PQ$ is defined to be $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

**5. Condition for collinearity :** Three points $A(z_1)$ $B(z_2)$ and $C(z_3)$ will be collinear.

(i) If there exists a relation $az_1 + bz_2 + cz_3 = 0$, such that $a + b + c = 0$ (where $a$, $b$, $c$ are not all zero real numbers)

(ii) If the area of $\Delta ABC$ formed by $z_1$, $z_2$ and $z_3$ is zero

*i.e.,* $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

(iii) If slope of $AB$ = slope of $BC$ = slope of $AC$

*i.e.,* $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \dfrac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \dfrac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$

**Useful Result**

If $\left|z + \dfrac{1}{z}\right| = a$, the greatest and least values of $|z|$ are respectively $\dfrac{a + \sqrt{a^2 + 4}}{2}$ and $\dfrac{-a + \sqrt{a^2 + 4}}{2}$

# QUADRATIC EQUATIONS

## Quadratic Equations with Real Coefficients

The expression, $ax^2 + bx + c$, where $a$, $b$, $c \in R$ and $a \neq 0$ is called as a quadratic expression in $x$. The quadratic expression when equated to zero, $ax^2 + bx + c = 0$, is called as a quadratic equation in $x$, where the numbers $a$, $b$, $c$ are called the coefficients of the equation.

1. The values of $x$ which satisfy the quadratic equation is called roots (also called solutions or zeros) of the quadratic equation.
2. This equation has two roots which are given by $x = \dfrac{-b \pm \sqrt{D}}{2a}$,

   where $D$ (or $\Delta$) = $b^2 - 4ac$ is called as discriminant of the quadratic equation.
3. If $\alpha$ and $\beta$ denote the roots of $ax^2 + bx + c = 0$, then,

   (i) $\alpha + \beta = -\dfrac{b}{a}$ (ii) $\alpha\beta = \dfrac{c}{a}$ (iii) $|\alpha - \beta| = \dfrac{\sqrt{D}}{|a|}$

## Solution of Quadratic Equations

The solution of $ax^2 + bx + c = 0$ is the value of $x$ for which the equation is satisfied. By the following illustrations we can understand very well the solution of equations.

### Identity

If a quadratic equation in $x$ is satisfied by more than two roots *i.e.*, the coefficient of $x^2 = 0$, the coefficient of $x = 0$ and the constant term = 0, then it is an identity in $x$ and it will be satisfied by all values of $x$.

### Identical Equations

Two equations having same roots are called identical equations.

## NATURE OF ROOTS

**A.** (i) $D > 0 \Leftrightarrow$ roots are real and unequal.

(ii) $D = 0 \Leftrightarrow$ roots are real and equal.

(iii) $D < 0 \Leftrightarrow$ roots are complex.

(iv) If $D$ is a perfect square of a rational numbers ($a$, $b$ and $c$ are rationals), then the roots are rational and unequal.

(v) If $D > 0$ and is not a perfect square of a rational numbers ($a$, $b$ and $c$ are rationals), then the roots are irrational and unequal.

**B. (i)** **Imaginary roots** always occur in pair if $a, b, c$ are real numbers. *i.e.*, if $\alpha + i\beta$ is one root of quadratic equation, then $\alpha - i\beta$ will be its other root.

**(ii)** **Irrational roots** of a quadratic equation with rational coefficients always occur in pair. *i.e.*, if $m+\sqrt{n}$ is one root of equation, then its other root will be $m-\sqrt{n}$.

## RELATION BETWEEN ROOTS AND CO-EFFICIENTS

If $\alpha$ and $\beta$ are the roots of $ax^2 + bx + c = 0$ then,

$ax^2 + bx + c = a(x - \alpha)(x - \beta) = a(x^2 - (\alpha + \beta)x + \alpha\beta)$

If $\Delta = b^2 - 4ac > 0$.

Now consider the following cases.

| | | Nature of roots |
|---|---|---|
| **Case-I** | $a > 0,\ b > 0,\ c > 0 \Rightarrow \alpha + \beta < 0,\ \alpha\beta > 0$ | Both roots are negative. |
| **Case-II** | $a > 0,\ b > 0,\ c < 0 \Rightarrow \alpha + \beta < 0,\ \alpha\beta < 0$ | Both roots are opposite in sign; Magnitude of negative root is more than the magnitude of positive root. |
| **Case-III** | $a > 0,\ b < 0,\ c > 0 \Rightarrow \alpha + \beta > 0,\ \alpha\beta > 0$ | Both roots are positive. |
| **Case-IV** | $a > 0,\ b < 0,\ c < 0 \Rightarrow \alpha + \beta > 0,\ \alpha\beta < 0$ | Roots are opposite in sign. Magnitude of positive root is more than magnitude of negative root. |

**Remember :**

(i) Roots are rational $\Leftrightarrow$ D is a perfect square provided $a$, $b$, $c$ are rationals.

(ii) Roots are irrational $\Leftrightarrow$ D is positive but not a perfect square.

(iii) (a) If $a + b + c = 0$, then 1 is a root of the equation $ax^2 + bx + c = 0$ and other root is $\frac{c}{a}$

(b) If $a - b + c = 0$, then $-1$ is a root of the equation $ax^2 + bx + c = 0$ and other root is $-\frac{c}{a}$.

(iv) If $a$ and $c$ are of opposite sign, the roots must be of opposite sign.

(v) If the roots are equal in magnitude but opposite in sign, then $b = 0$, $ac < 0$.

(vi) If the roots are reciprocal of each other, then $c = a \neq 0$.

(vii) The quadratic equation whose roots are reciprocals of the roots of $ax^2 + bx + c = 0$ is $cx^2 + bx + a = 0$ (*i.e.*, the coefficients are written in reverse order) provided $c \neq 0$.

(viii) If $a = 1$, $b$, $c \in Z$ and the roots are rational numbers, then these roots must be integers, where $Z$ is the set of integers.

(ix) If $ax^2 + bx + c = 0$ is satisfied by more than two values of $x$, then it is an identity in $x$ and $a = b = c = 0$ and vice-versa.

(x) If $P(x) = a_1x^2 + b_1x + c_1$ and $Q(x) = a_2x^2 + b_2x + c_2$, then $P(x).\ Q(x) = 0$ have at least two real roots if $D_1 + D_2 > 0$.

**Condition for a root of a quadratic equation $ax^2 + bx + c = 0$ to be reciprocal of the root of $a'x^2 + b'x + c' = 0$ is $(cc' - aa')^2 = (ba' - b'c)(ab' - bc')$.**

## COMMON ROOTS

### One Root Common :

If $\alpha \neq 0$ is a common root of the equation

$$a_1x^2 + b_1x + c_1 = 0 \quad \text{...(i)}$$

and $a_2x^2 + b_2x + c_2 = 0$ ...(ii)

then we have

$$a_1\alpha^2 + b_1\alpha + c_1 = 0 \text{ and } a_2\alpha^2 + b_2\alpha + c_2 = 0$$

These give $\dfrac{\alpha^2}{b_1c_2 - b_2c_1} = \dfrac{\alpha}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1} (a_1b_2 - a_2b_1 \neq 0)$.

Thus, the required condition for one common root is $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$ and the value of the common root is

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \text{ or } \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}.$$

### Both Roots Common :

If the equations (i) and (ii) have both roots common, then these equations will be identical. Thus the required condition for both roots common is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \quad \text{(If no root is equal to zero)}$$

**Remember :**

(i) To find the common root of two equations, make the coefficient of second degree terms in two equations equal and subtract. The value of $x$ so obtained is the required common root.

(ii) If two quadratic equations with real and rational coefficients have a common imaginary or irrational root, then both roots will be common and the two equations will be identical. The required condition is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

(iii) If $\alpha$ is a repeated root of the quadratic equation

$$f(x) = ax^2 + bx + c = 0,$$

then $\alpha$ is also a root of the equation $f'(x) = 0$.

## FORMATION OF EQUATION WITH GIVEN ROOTS

If $\alpha$ and $\beta$ are the roots of $ax^2 + bx + c = 0$, $a \neq 0$

then $ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} \equiv x^2 - (\alpha + \beta)x + \alpha\beta$$

Hence, $x^2$ – (sum of roots) $x$ + product of roots = 0 is the required quadratic equation.

## SYMMETRIC FUNCTION OF THE ROOTS

A function of $\alpha$ and $\beta$ is said to be a symmetric function if it remains unchanged when $\alpha$ and $\beta$ are interchanged.

For example, $\alpha^2 + \beta^2 + 2\alpha\beta$ is a symmetric function of $\alpha$ and $\beta$ whereas $\alpha^2 - \beta^2 + 3\alpha\beta$ is not a symmetric function of $\alpha$ and $\beta$.

In order to find the value of a symmetric function of $\alpha$ and $\beta$, express the given function in terms of $\alpha + \beta$ and $\alpha\beta$. The following results may be useful.

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

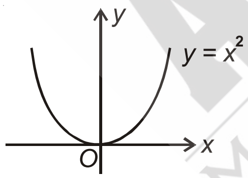
## QUADRATIC EXPRESSION

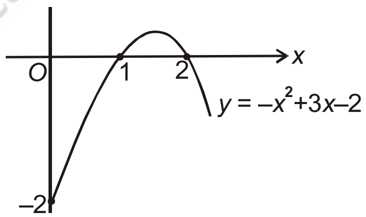
Let $y = ax^2 + bx + c$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right) \quad \ldots(1)$$

(1) represents a parabola with vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ and axis of the parabola is $x = \frac{-b}{2a}$

If $a > 0$, the parabola opens upward while if $a < 0$, the parabola opens downward. The parabola cuts the $x$-axis at points corresponding to roots of $ax^2 + bx + c = 0$. If this equation has




(i) $D > 0$, the parabola cuts $x$-axis at two real and distinct points.

(ii) $D = 0$, the parabola touches $x$-axis at $x = \frac{-b}{2a}$.

(iii) $D < 0$, then;

if $a > 0$, parabola lies above $x$-axis.

if $a < 0$, parabola lies below $x$-axis.

*e.g.* 1. $y = x^2$

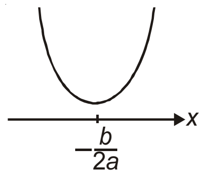
2. $y = -x^2 + 3x - 2$

## Graphs of Quadratic Expressions

Let $f(x) = ax^2 + bx + c$ and $\alpha, \beta\ :\ \alpha < \beta$, be its roots

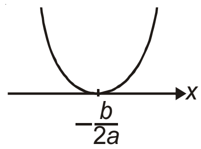
(i) $a > 0$ and $D < 0 \Leftrightarrow f(x) > 0 \quad \forall \quad x \in R$

$\alpha$ and $\beta$ are complex conjugates




(ii) $a > 0$ and $D = 0 \Leftrightarrow f(x) \geq 0 \quad \forall \quad x \in R$

$f(x) = 0$ at $x = \dfrac{-b}{2a}$

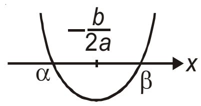



(iii) $a > 0$ and $D > 0$; then,

$f(x) > 0\ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$

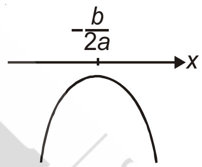
$f(x) < 0\ \forall\ x \in (\alpha, \beta)$

$f(x)_{\min} = -\dfrac{D}{4a}$ at $x = -\dfrac{b}{2a}$




(iv) $a < 0$ and $D < 0 \Leftrightarrow f(x) < 0 \quad \forall \quad x \in R$

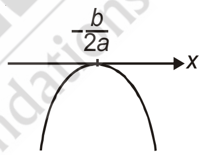
$\alpha$ and $\beta$ are complex conjugates




(v) $a < 0$ and $D = 0 \Leftrightarrow f(x) \leq 0 \quad \forall \quad x \in R$

$f(x) \leq 0$ for all $x \in R$

$f(x) = 0$ at $x = \dfrac{-b}{2a}$

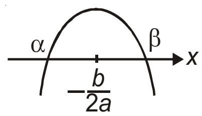



(vi) $a < 0$ and $D > 0$; then (for $\alpha < \beta$)

$f(x) < 0 \quad \forall \quad x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) > 0 \quad \forall \quad x \in (\alpha,\ \beta)$

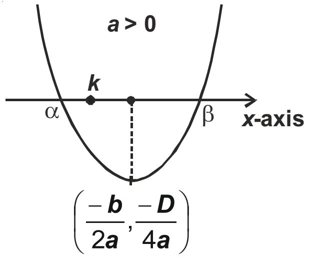
$f(x)_{\max} = -\dfrac{D}{4a}$ at $x = -\dfrac{b}{2a}$

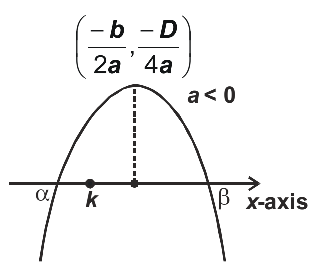



## Location of Roots

Let $f(x) = ax^2 + bx + c, a, b, c \in R, a \neq 0$ and $\alpha, \beta$ be roots of $f(x) = 0$

(1) A real number $k$ lies between the roots of $f(x) = 0$

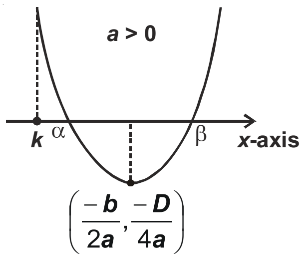


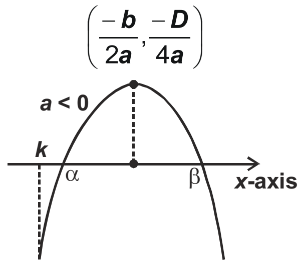




(i) $D > 0$

(ii) $af(k) < 0$, where $\alpha < \beta$

(2) If both roots of quadratic equation $f(x) = 0$ are greater than $k$, then



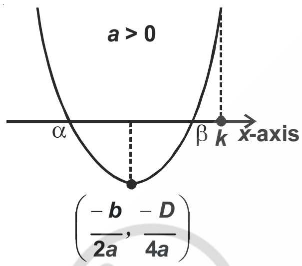



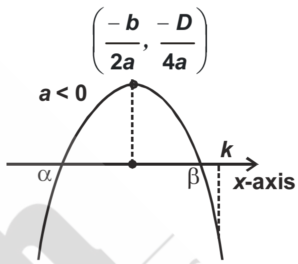

(i) $D \geq 0$

(ii) $af(k) > 0$

(iii) $k < \frac{-b}{2a}$, where $\alpha \leq \beta$

(3) If both roots are less than real number $k$, then



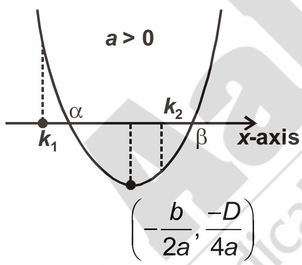



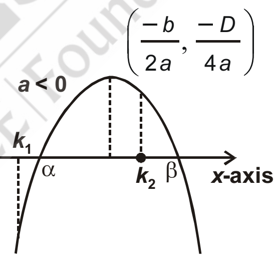

(i) $D \geq 0$

(ii) $af(k) > 0$

(iii) $k > \frac{-b}{2a}$, where $\alpha \leq \beta$

(4) Exactly one root lies between real numbers $k_1$ and $k_2$, where $k_1 < k_2$.

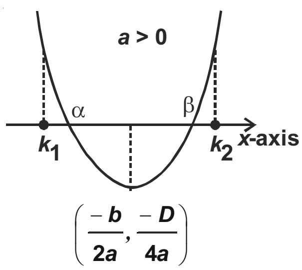


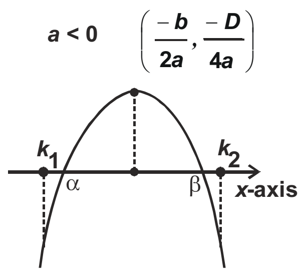




(i) $D > 0$

(ii) $f(k_1).f(k_2) < 0$, where $\alpha < \beta$

(5) If both roots of $f(x) = 0$ are confined between real numbers $k_1$ and $k_2$, where $k_1 < k_2$.



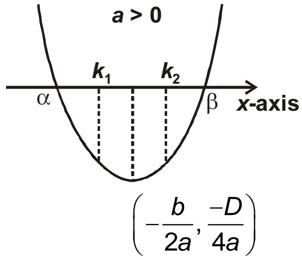

(i) $D \geq 0$

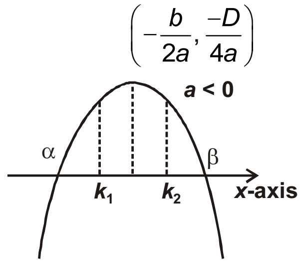
(ii) $af(k_1) > 0$

(iii) $af(k_2) > 0$

(iv) $k_1 < \frac{-b}{2a} < k_2$, where $\alpha \leq \beta$

(6) If real numbers $k_1$ and $k_2$ lie between roots of $f(x) = 0$, where $k_1 < k_2$.



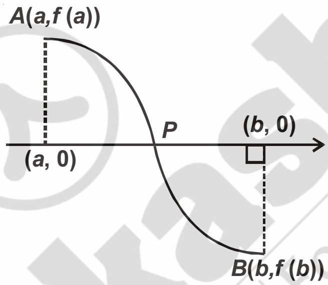

(i) $D > 0$

(ii) $af(k_1) < 0$

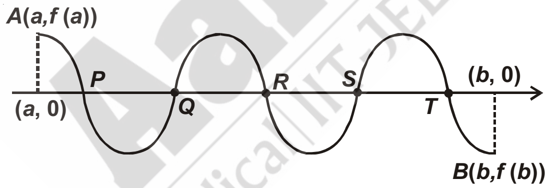
(iii) $af(k_2) < 0$, where $\alpha < \beta$

## NUMBER OF ROOTS OF A POLYNOMIAL EQUATION

1. If $f(x)$ is an increasing function in $[a, b]$, then $f(x) = 0$ will have atmost one root in $[a, b]$.
2. Let $f(x) = 0$ be a polynomial equation in $x$ and $a, b$ are two real numbers. Then $f(x) = 0$ will have at least one real root or odd number of real roots in $(a, b)$ if $f(a)$ and $f(b)$ $(a < b)$ are of opposite signs.

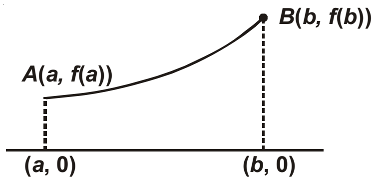



**One real root**

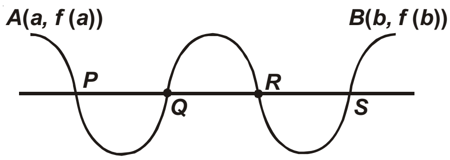



**Odd number of real roots**

But if $f(a)$ and $f(b)$ are of same signs, then either $f(x) = 0$ has no real root or if $f(x) = 0$ possess a root then even number of real roots exists in $(a, b)$.




**No real root**




**Even number of real roots**

3. If the equation $f(x) = 0$ has two real roots $a$ and $b$ then $f'(x) = 0$ will have atleast one real root lying between $a$ and $b$ (using Rolle's Theorem).

## EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

1. **Reciprocal equation** of the standard form can be reduced to an equation of half its dimension.
2. **Equations of the form**
   (a) $(x-a)(x-b)(x-c)(x-d) = A$ where $a < b < c < d$ can be solved by change of variable.
   $$y = x - \left(\frac{a+b+c+d}{4}\right)$$
   (b) $(x-a)(x-b)(x-c)(x-d) = Ax^2$ where $ab = cd$ can be solved by assumption $y = x + \frac{ab}{x}$.
3. **For the equation of the type $(x-a)^4 + (x-b)^4 = A$**

   Substitute $y = \frac{(x-a)+(x-b)}{2}$

## Important Theorems and Results

1. Rolle's Theorem : Let $f(x)$ be a real valued function defined on [a, b] such that
   (i) $f(x)$ is continuous on $[a, b]$
   (ii) $f(x)$ is differentiable on $(a, b)$ and
   (iii) $f(a) = f(b)$
   Then there exists at least one real number $c \in (a, b)$ such that $f'(c) = 0$
2. **Lagrange's theorem :** Let $f(x)$ be a function defined on $[a, b]$ such that
   (i) $f(x)$ is continuous on [a, b] and
   (ii) $f(x)$ is derivable on $(a, b)$. Then there exists at least one real number $c \in (a, b)$ such that
   $$f'(c) = \frac{f(b)-f(a)}{b-a}$$
3. **Factor theorem :** If $\alpha$ is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x - \alpha)$ and conversely, if $f(x)$ is exactly divisible by $(x - \alpha)$ then $\alpha$ is a root of the equation $f(x) = 0$.
4. Every equation of an odd degree has atleast one real root.
5. Every equation of an even degree whose last term is negative has at least two real roots, one positive and one negative, provided that the coefficient of the first term is positive.
6. If an equation has no odd powers of $x$, then all roots of the equation are complex provided all the coefficients of the equation are having positive sign.
7. If $x = \alpha$ is a root repeated $m$ times in $f(x) = 0$,
   ($f(x) = 0$ is an $n$th degree equation in $x$) then
   $$f(x) = (x-\alpha)^m g(x)$$
   where $g(x)$ is a polynomial of degree $(n - m)$ and the root $x = \alpha$ is repeated $(m - 1)$ times in $f'(x) = 0$, $(m - 2)$ times in $f''(x) = 0, \ldots (m - (m - 1))$ times in $f^{m-1}(x) = 0$.
8. The condition that a quadratic function
   $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factor if
   $$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \text{ or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \text{ is 0.}$$

**9. Law of proportions**

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=...,$ then each of these ratios is also equal to :

(i) $\frac{a+c+e+...}{b+d+f+...}$

(ii) $\left(\frac{pa^{n}+qc^{n}+re^{n}+...}{pb^{n}+qd^{n}+rf^{n}+...}\right)^{1/n}$ (when $p$, $q$, $r$, $n \in R$)

(iii) $\frac{\sqrt{ac}}{\sqrt{bd}}=\frac{\sqrt[n]{(ace...)}}{\sqrt[n]{(bdf...)}}$

❑ ❑ ❑