Date: 28/01/2023

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Answers & Solutions

Time : 3 hrs. *for* Max. Marks : 100

Indian National Astronomy Olympiad
(INAO) 2023

(For Class XI & XII Students)

INSTRUCTIONS TO CANDIDATES

(1) There are total **6** questions. All questions are compulsory.

(2) Maximum marks are indicated in front of each sub-question.

(3) For all questions, the process involved in arriving at the solution is more important than the final answer. Valid assumptions / approximations are perfectly acceptable. Please write your method clearly, explicitly stating all reasonings / assumptions / approximations.

(4) Use of non-programmable scientific calculator is allowed.

1. For the spacecrafts travelling from one planet to another, the travel path can be designed in multiple ways. The most energy efficient of these methods is called 'Hohmann Transfer Orbit'. In this method, the spacecraft travels in an elliptical orbit with the Sun at the principal focus. Thus, its motion is governed by Kepler's Laws and hardly any fuel is spent for this journey. Step-wise process of Hohmann orbit is as follows:

- Step 1: The spacecraft is in the close proximity of Earth.
- Step 2: The spacecraft engine is fired to allow it to escape the sphere of influence of the Earth and enter the region where the Sun's gravity is the dominating force. The point in Earth's orbit at which engine is fired will also be a point in the new orbit of the spacecraft around the Sun.
- Step 3: The spacecraft travels along an elliptical orbit around the Sun and reaches the orbit of Mars.
- Step 4: The timing of this journey is such that Mars is exactly at the common point of Martian orbit and the elliptical orbit of the spacecraft at the instance when the spacecraft reaches this point.
- Step 5: The spacecraft engine is fired again so that the spacecraft is captured by the Mars' gravity and it remains in close proximity of Mars.

Points to note:

A. The point where the engine is first fired (Step 2) will be spacecraft's shortest distance from the Sun.

B. The point where the spacecraft is captured by Mars (Step 5) will be the spacecraft's farthest distance from the Sun.

C. If Step 5 is not executed, the spacecraft will complete its orbit around the Sun and reach back the point in space where Step 2 was executed.

D. Let us assume that both Earth and Mars are in circular, co-planer orbits around the Sun and the Hohmann transfer orbit is also in the same plane.

E. Assume that all the three bodies (Earth, Mars and the spacecraft) are point masses and moving in anti-clockwise direction (as viewed from the top) in their respective orbits.

F. The time durations for which engine is fired (Step 2 as well as Step 5) is very small. Hence, the thrust is almost instantaneous.

Based on this, answer the following questions:

(a) **(1.5 marks)** Find the orbital period of Mars in years.

(b) **(1.5 marks)** Find the length of semi-major axis of this Hohmann transfer orbit (in au).

(c) **(1.5 marks)** Find time spent by the spacecraft between Step 2 and Step 5.

(d) **(5 marks)** Draw a diagram showing the Hohmann transfer orbit between Earth and mars. Show positions of Earth and Mars at the instance of Step 2 (as E_1 and M_1) and at the instance of Step 5 (E_2 and M_2). Keep position E_1 directly below the Sun. Calculate angles subtended with respect to Sun – E_1 line for the other three points (M_1, E_2 and M_2).

(e) **(2.5 marks)** Calculate orbital velocity of Earth and Mars around Sun.

(f) **(8 marks)** Determine if the following statements are True or False. Explain your reason in one line each.

i. The semi-minor axis of the Hohmann orbit is 1 au.

ii. The semi-major axis of the Hohmann orbit is 1.52 au.

iii. The thrust was applied in Step 2 to increase the speed of the spacecraft.

iv. The thrust was applied in Step 5 to increase the speed of the spacecraft.

v. If Step 5 is missed, we can re-capture the spacecraft in an orbit around the Earth when it again reaches its departure point in space. (see point C above).

vi. The Hohmann transfer orbit only touches (and does not intersect) the Earth's orbit.

vii. The kinetic energy of the spacecraft in Hohmann orbit is lowest just before Step 5.

viii. The spacecraft would be approaching Mars from behind in its orbit.

Sol. (a) If P is the time period in earth years and a is distance of a planet in astronomical units then

$P^2 = a^3$

$\Rightarrow$ $P = (1.524)^{3/2}$ earth years

$= 1.8814$ earth years

(b) Major axis of Hohmann transfer orbit

= earth orbit radius + mars orbit radius

= 1 au + 1.524 au

= 2.524 au

$\Rightarrow$ semi-major axis of Hohmann transfer orbit

= 1.262 au = a' say

(c) Time period of Hohmann transfer orbit

$P' = (a')^{3/2}$

$P' = (1.262)^{1.5}$

= 1.418 earth years

$\Rightarrow$ time spent between step 2 and step 5 is equal to half of the Hohmann orbital period

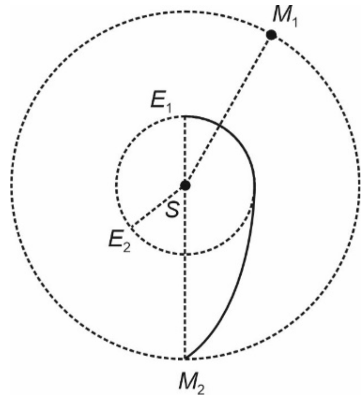
$= \frac{1.418}{2}$ earth years

= 0.709 earth years

(d) Time taken to reach Mars = 0.709 earth years

$= t_1$ (say)

Angle rotated by Earth in time t_1 in its orbit = 0.709 × 360° = 255.24°



Angle rotated by Mars in time t_1 in its orbit = 135.66°

θ_1 = angle between Sun-E_1 and Sun-M_1 line

= 180° – 135.66°

= 44.34° (clockwise)

θ_2 = angle between Sun-E_1 and Sun-E_2 line

= 104.76° (anti-clockwise)

θ_3 = angle between Sun-E_1 and Sun-M_2 line

= 180°

(e) Orbital velocity of Earth

$$= \sqrt{\frac{GM_S}{a_{ES}}} = \sqrt{\frac{6.67\times10^{-11}\times1.989\times10^{30}}{1.496\times10^{11}}}$$

= 2.98×10^4 m/s

= 29.8 km/s

Orbital velocity of Mars

$$= \sqrt{\frac{GM_S}{a_{MS}}} = 24.12 \text{ km/s}$$

(f) Statement (i) $\rightarrow$ False

1.262 (1 – e) = 1 au

1.262 (1 + e) = 1.524 au

$\Rightarrow$ e = 0.21

$$b = 1.262\sqrt{1-(0.21)^2}$$

= 1.2338 au

$\neq$ 1 au

Statement (ii) → False

Semi-major axis of Hohmann orbit is 1.262 au and not 1.52 au

Statement (iii) → True

As the total energy of Hohmann orbit is more than the earth's orbit so at same position keeping the potential energy same kinetic energy needs to be increased.

Statement (iv) → True

As the mars-near orbit should have more energy so the speed here needs to be increased to increase the total energy.

Statement (v) → False

By the time the satellite reaches back at the position of departure point the earth would have moved around the sun up to an angle of 2 × 255.24° i.e., around 510.48°. Or in other word earth would not be at the same position.

Statement (vi) → True

The Hohmann orbit perigee is the point common with earth's orbit i.e., at any other point on the Hohmann orbit satellite would be at a distance greater than 1 au from sun and so this orbit will never cross the earth's orbit.

Statement (vii) → True

At Step-5 the satellite will be at its point of apogee and so it will be at its lowest speed.

Statement (viii) → True

At the point of apogee speed of satellite

$$= \sqrt{\frac{GM_s}{a_{\text{satellite}}}\left(\frac{1-e}{1+e}\right)} = 21.419 \text{ km/s}$$

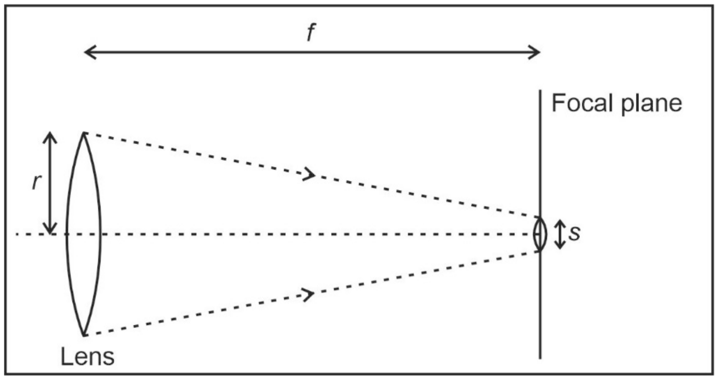
Speed of Mars = 24.12 km/s

⇒ At the point of capture the orbitor would be falling behind the mars and so it will have to speed up be using Step-5 to approach Mars from behind.

2. **(10 marks)** A telescope with objective lens of focal length f = 200 cm is used to image a binary star system using a CCD kept at its focal plane and the image forming symmetrically at the centre (on the optic axis). In the figure, s is the image size of a point object. The image has a finite size due to diffraction. Two nearby objects are considered to be resolved when their image disks are just touching each other externally.

The radius of the objective lens is r = 5 cm. The limiting resolution of the lens is given to be 1″ and the two stars are separated by an angle of 3″. Determine the range dx allowed for the CCD to be displaced from the focal plane towards the lens such that individual stars can still be separately resolved.

(**Note:** This is related to the notion of "depth of field" of a camera lens.)



Sol. We know that by diffraction, spread in the image is given by width of central maxima.

$\Rightarrow \quad s = \dfrac{2\lambda D}{d}$

where $D = f$

& $d = 2r$

$\Rightarrow \quad s = \dfrac{\lambda f}{r}$ ...(1)

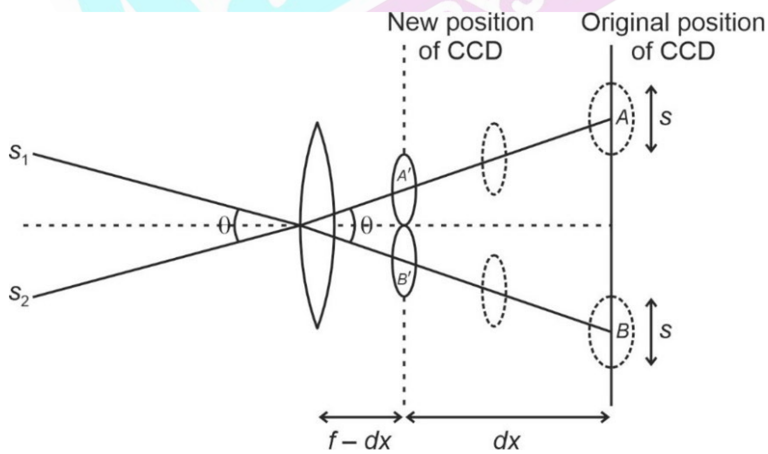
Also, by resolving limit,

$1'' = \dfrac{1.22\lambda}{d}$

$\Rightarrow \quad \dfrac{\lambda}{r} = \dfrac{1''}{0.61} \quad [\because d = 2r]$ ...(2)

From (1) & (2), $s = \dfrac{(1'')(f)}{0.61}$ ...(3)

Now, from the ray diagram:



As we can see from geometry:

$A'B' = s = (f - dx)\theta$

& $AB = f \cdot \theta$

$\Rightarrow \quad dx = f - \dfrac{s}{\theta}$ ...(4)

From (3) & (4)

$$dx = f - \frac{(1'')f}{0.61} \times \frac{1}{3''}$$

$$= f - \frac{f}{1.83}$$

$\Rightarrow \quad dx \simeq 90.7$ cm

Hence, $0 \le dx \le 90.7$ cm

3. A cubical box, with edges of length s, is in a circular orbit of radius d around a star. The time period of revolution of the box is P. The star can be approximated as a perfect blackbody with radius R_0 (s, $R_0 << d$) and temperature T_0.

In all parts of this question, at time $t = 0$, normal vector to one face of the box is exactly pointing towards the star and this face has an albedo of A (where $A \in [0, 1]$). The top face of the box is parallel to the orbital plane of the box.

The box is in thermal equilibrium at each instance $t \ge 0$, *i.e*, its emissivity is exactly equal to its absorptivity. The temperature T in all parts of the box is the same at each instance.

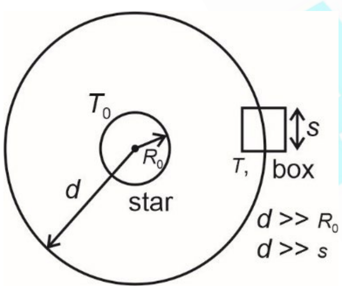
Note: Albedo is the fraction of incident light reflected off the surface of an object.

(a) **(7 marks)** If the same face of the box is locked in towards the star at all times, find its equilibrium temperature T.

(b) **(6 marks)** Let all 6 faces of the box have albedo A. If the box is revolving around the star, but not rotating about itself, derive the expression for the equilibrium temperature T as a function of time t.

(c) **(3 marks)** Now consider the case, where the face of the cube which is towards the star at $t = 0$ and the face opposite to it have albedo A and all other faces of the cube have albedo of $1 - A$. Observe the expressions obtained in part b and write the final expression for temperature in this case.

Sol. (a)



The cubical box has Albedo A, so its absorptivity equal to $1 - A$,

Power received by facing surface of box,

$$P_1 = \text{Power radiated by star} \times \frac{s^2}{4\pi d^2} \times (1 - A)$$

$$= \sigma(4\pi R_0^2)T_0^4 \times \frac{s^2}{4\pi d^2} \times (1 - A)$$

$$= \frac{\sigma R_0^2 s^2 T_0^4 (1 - A)}{d^2} \qquad \ldots(1)$$

And, Power radiated by the box, $P_2 = 6s^2\sigma(1 - A)T^4 \qquad \ldots(2)$

where, $(1 - A)$ is the emissivity of the box from Kirchhoff's law

From (1) and (2):

$P_1 = P_2$ (Thermal equilibrium)

$$\Rightarrow \frac{\sigma R_0^2 s^2 T_0^4 (1-A)}{d^2} = \sigma 6 s^2 (1-A) T^4$$

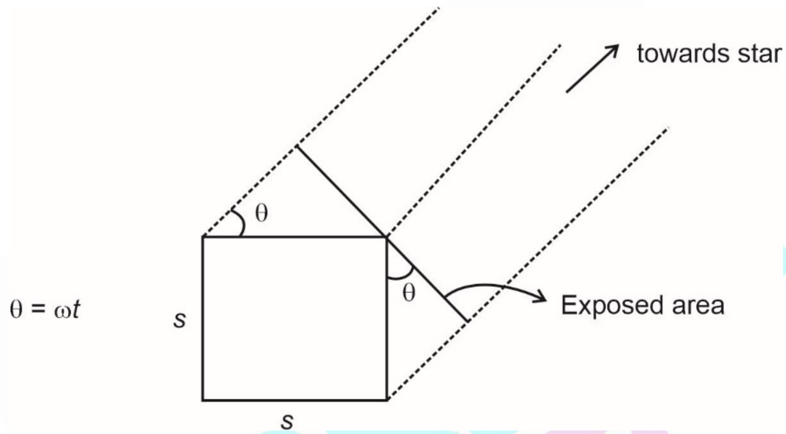
$$\Rightarrow \boxed{T = T_0\left(\frac{R_0^2}{6d^2}\right)^{\frac{1}{4}}}$$

(b) Time period of revolution of box around the star is given as P,

$\therefore$ Angular speed of box, $\omega = \frac{2\pi}{P}$

Now as seen from the box, the star can be considered as to be moving with angular speed ω.

$\therefore$ Exposed area of the box at any time t can be calculated as follows:



$\therefore$ Exposed area = $|s^2[\sin\theta + \cos\theta]|$

$= |s^2[\sin\omega t + \cos\omega t]|$

$$= s^2\sqrt{2}\left|\sin\left(\omega t + \frac{\pi}{4}\right)\right|$$

$\therefore$ Power received by exposed surface of box,

$$P_3 = \sigma(4\pi R_0^2)T_0^4 \times \frac{s^2\sqrt{2}\left|\sin\left(\omega t + \frac{\pi}{4}\right)\right|}{4\pi d^2} \times (1-A) \qquad \ldots(3)$$

And, Power radiated by box, $P_4 = 6s^2\sigma(1-A)T^4$ ...(4)

$\therefore$ From thermal equilibrium, $P_3 = P_4$

$$\Rightarrow \sigma(4\pi R_0^2)T_0^4 \times \frac{s^2\sqrt{2}}{4\pi d^2}\left|\sin\left(\omega t + \frac{\pi}{4}\right)(1-A)\right| = 6s^2\sigma(1-A)T^4$$

$$\Rightarrow \boxed{T = T_0\left[\frac{R_0^2\sqrt{2}\left|\sin\left(\omega t + \frac{\pi}{4}\right)\right|}{6d^2}\right]^{\frac{1}{4}}}$$

(c) Angular speed of box, $\omega = \frac{2\pi}{P}$

Albedo of front and back face of the box is A and remaining faces have Albedo of $(1 - A)$

$\therefore$ Exposed surface area of the box is given by

$a_1 = |s^2[A\sin(\omega t) + (1 - A)\cos\omega t]|$

$\therefore$ Power received by exposed surface of box,

$$P_5 = \frac{\sigma(4\pi R_0^2)T_0^4 \times s^2 |A\sin\omega t + (1-A)\cos\omega t|}{4\pi d^2} \times (1-A)$$

And, Power radiated by the box is

$P_6 = 2s^2\sigma(1 - A)T^4 + 4s^2\sigma A T^4$

$= 2s^2\sigma T^4 (1 + A)$

$\therefore$ From thermal equilibrium, $P_5 = P_6$

$$\Rightarrow \quad \sigma(4\pi R_0^2)T_0^4 \times \frac{s^2}{4\pi d^2}|A\sin\omega t + (1-A)\cos\omega t| \times (1-A) = 2s^2\sigma T^4(1 + A)$$

$$\Rightarrow \quad \boxed{T = T_0\left[\frac{R_0^2}{d^2}\frac{(1-A)}{2(1+A)}|A\sin\omega t + (1-A)\cos\omega t|\right]^{\frac{1}{4}}}$$

4. Indian Space Research Organisation (ISRO) decided to send a spaceship to the planet Jupiter. The spaceship needs to pass through the asteroid belt between Mars and Jupiter which lies between 2.0 au and 3.0 au. The asteroid belt has a thickness (total height perpendicular to the plane of the solar system) of about 0.10 au and there are about 10^{14} asteroids of 1.0 m radius (assumed to be spherical) in it. The spaceship is spherical and is 3.0 m in radius.

(a) **(2 marks)** Calculate the cross-section area within which the centre of an asteroid needs to come from the centre of the spacecraft to collide with it.

(b) **(8 marks)** Assume that the asteroid belt is completely static and the spaceship is launched on a random day of the year to travel radially away from Sun, but within the plane of solar system. Calculate the probability (P_0) that the spaceship will escape the asteroid belt unharmed.

(c) Qualitatively compare the probability in the each of the following cases with P_0:

i. **(2 marks)** The probability (P_A) when the spacecraft's speed through the asteroid belt is doubled.

ii. **(1 mark)** The probability (P_B) when the spacecraft is not moving radially but is in an elliptical orbit with aphelion point at Jupiter (5 au). Here we consider probability only for a single crossing.

iii. **(2 marks)** The probability (P_C) when the spacecraft is not moving radially but is in an elliptical orbit with aphelion point at the centre of the asteroid belt (2.5 au).

Sol. (a) Radius of spaceship, $R = 3$ m

Radius of asteroid, $r = 1$ m

$\therefore$ Required radius of cross-section for collision to be occurred, $V_R = R + r$

$\Rightarrow$ Required cross-section area $= \pi(R + r)^2$

$= 16\,\pi$ m^2

(b) Number density of asteroids in belt is

$$\rho = \frac{N}{\pi\left(r_2^2 - r_1^2\right)h}$$

where, N = total no. of asteroids, r_2 and r_1 are outer and inner radius of belt respectively and h is belt thickness.

And Volume swept by the spaceship through the belt without collision with asteroids, $V = 16\pi \times (r_2 - r_1)$

Now, the probability of spaceship to collide the asteroids in the belt is given by $P_C = \rho V$.

$$\Rightarrow \quad P_C = \frac{N}{\pi\left(r_2^2 - r_1^2\right)h} \times 16\pi(r_2 - r_1)$$

$$\Rightarrow \quad P_C = \frac{16N}{(r_2 - r_1)h}$$

$$= \frac{16 \times 10^{14}}{(3\text{ au} + 2\text{ au})0.1\text{ au}}$$

$$= \frac{16 \times 10^{14}}{5 \times 1.496 \times 10^{11} \times 0.1 \times 1.496 \times 10^{11}}$$

$$= 1.43 \times 10^{-7}$$

$\therefore$ Required probability of spaceship to pass through the belt without collision with asteroids is given by

$P_0 = 1 - P_c$

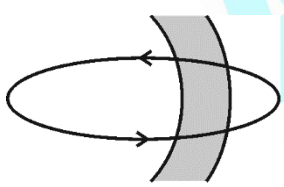
$\Rightarrow$ $P_0 = 1 - 1.43 \times 10^{-7}$

(c) (i) The probability of spaceship to pass through the belt depends on asteroids number density in the belt, so probability is independent of speed of spaceship.

Therefore, P_A, remains the same as P_0.

$P_A = P_0$.

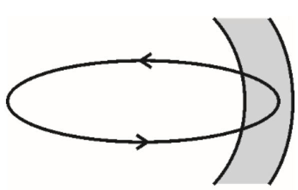
(ii)



In this case, the path length of spaceship through the belt is larger than the case of radial movement, therefore, the probability to pass through the belt would be lesser than previous case.

$\Rightarrow$ $P_B < P_0$.

(iii)



In this case, the path length of spaceship through the belt is further increased, therefore, the probability of spaceship to pass through the belt is further decreased.

$\therefore$ $P_C < P_0$.

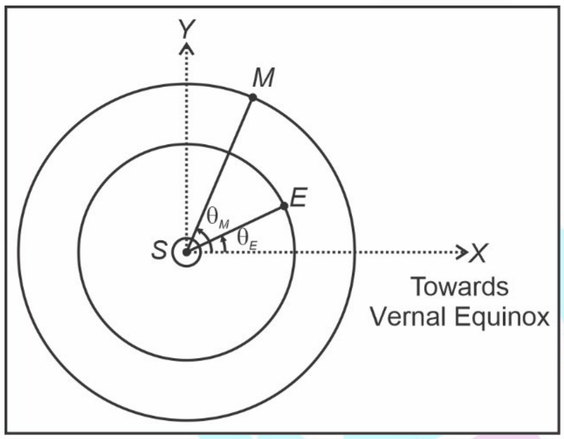
5. **(22 marks)** Assume that the Earth and Mars revolve around the Sun in coplanar circular orbits of radius r_E and r_M respectively. In the figure below, θ_E and θ_M are the angles made by the radius vectors of Earth and Mars, measured at the Sun with respect to the X-axis (which points in the direction of Vernal Equinox - a reference point in space).

On 21 September, θ_E was 0° and θ_m was 42.3°.

In the answer sheet you are given a table to note down your observations. Calculate the values of θ_E and θ_m for the dates mentioned in the table and write them in the respective columns.

The answer sheet also has a particular section of the orbits of Mars and Earth. The dotted lines are markings for angles in the orbits at an interval of 1°. The central black line is when θ_E or θ_M will be 90°. You can also see the distant stellar background near one edge of the page.

For the dates given in the table mark the projected positions of Mars on the stellar background. Also write the corresponding observation numbers near the marked projected positions. Now describe the motion of Mars as seen by an Earth based observer.



Sol. Using Kepler's 3rd law, we can calculate angular speed of Mars (ω_M)

$$\Rightarrow \frac{T_E^2}{T_M^2} = \frac{R_E^3}{R_M^3}$$

where,

T_E : Time period of Earth

T_M : Time period of Mars

R_E : Radius of Earth

R_M : Radius of Mars

$$\Rightarrow T_M = \left[\frac{R_M}{R_E}\right]^{\frac{3}{2}} T_E$$

$$\Rightarrow \omega_M = \left[\frac{R_E}{R_M}\right]^{\frac{3}{2}} \omega_E$$

$$= \left[\frac{1au}{1.524\,au}\right]^{\frac{3}{2}} \cdot \frac{2\pi}{365 \text{ days}}$$

$$\Rightarrow \omega_M \simeq 9.14 \times 10^{-3} \text{ rad/day}$$

Also, $\omega_E \simeq 17.20 \times 10^{-3}$ rad/day

Let d : number of days elapsed starting from 21st September.

$\Rightarrow \quad \theta_E = 0° + \omega_E \cdot d$

$\simeq 0.9863d$ [in degrees] ...(1)

Also, $\theta_M = 42.3° + \omega_M \cdot d$

$= 42.3° + 0.524d$ ...(2)

From equations (1) and (2), we can get the angular position of Earth and Mars with respect to the direction of Vernal Equinox.

Also, using these angular positions, we can mark the projected positions of Mars on the stellar background.

6. Saee is an observer at North Pole during the month of December. On the grid given in your answer sheet:

(a) **(1 mark)** Mark the directions.

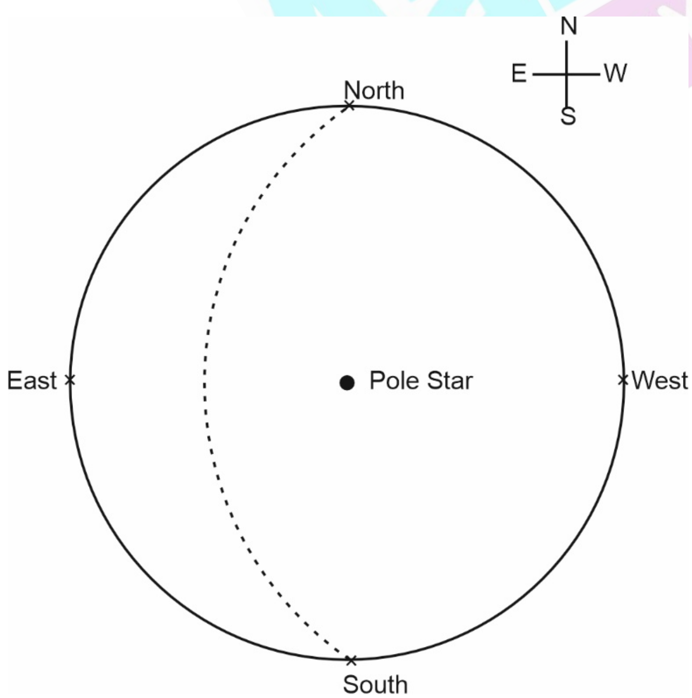
(b) **(1 mark)** Mark the pole star (Polaris).

(c) **(6 marks)** Approximately, for how many days will the Moon be visible in one lunar month assuming clear skies? If you were doing an exact calculation, which secondary effects you will have to consider? List any three effects that may affect your answer by an hour or more.

(d) **(6 marks)** In this month the full Moon was seen on winter solstice day in the Gemini constellation. Saee noted the Moon's position every day at 7 pm during an entire lunar month. Mark the approximate positions of the Moon as seen by her. For each observation label the day number starting with day of first observation.

(e) **(3 marks)** Aadarsh, who is also camping at the North pole with Saee, noted the Moon's position for 24 hours period of the 6th day after full moon. Describe what he will observe.

Sol. Solution for (a) and (b)



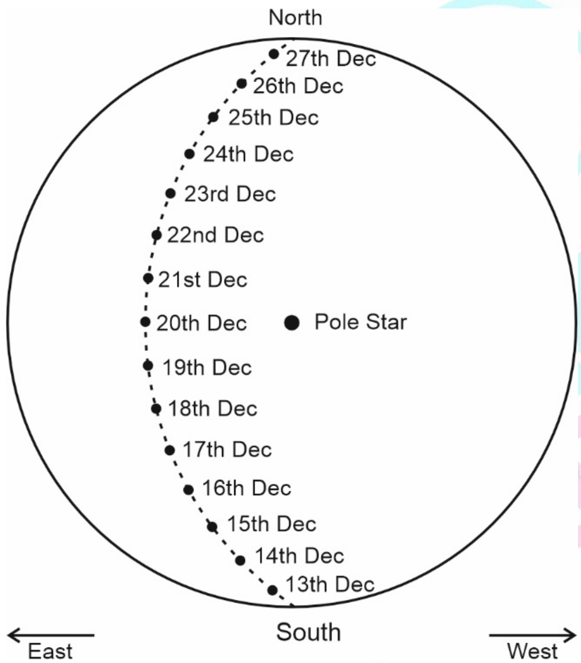
(c) At N-Pole the vernal equinox point will lie at horizon in south and point of autumn equinox will lie at north most point.

That is the ecliptic will pass through south and north point. In December as sun will be below horizon (celestial equator) in west side i.e., the ecliptic will pass through South and North most point through left half of the skymap as shown by dotted line.

The motion of moon will be seen along this dotted line in the sky.

As sun will be below horizon so whenever moon will be above horizon ideally it should be visible so moon will be visible in between two half-moon positions i.e., for half of its cycle or $\frac{29.5}{2}$ or 14.75 days. While doing the exact calculation sun's shift in position across constellation has to be taken into account. Other than this the difference in duration of one solar month and a lunar month will also effect the beginning and ending dates of the said period i.e. the dates of two half-moons.

(d)



(e) After 6th day of full moon, the moon will be nearing its third quarter phase. As the time will pass in the 24 hrs duration the moon will move northwards along the ecliptic approximately waning in size from full moon towards half-moon as it will be passing through Leo constellation.

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