Solutions
for
INMO 2026

Gold Medalist

Aarav Gupta
66th International Mathematical Olympiad (IMO) 2025

Yug Gandhi
Singapore Math Olympiad 2025

Arjun Tyagi
International Olympiad in Artificial Intelligence (IOAI) 2025

Dhruv Advani
34th International Biology Olympiad (IBO) 2023

Rohit Panda
34th International Biology Olympiad (IBO) 2023

Chirag Falor
International Olympiad on Astronomy & Astrophysics 2019

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Divyant J — AIR 80

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Akash Deep — AIR 99

HOME OF PROBLEM SOLVERS

Instruction:

- **Answer all questions. All questions carry equal marks. Maximum marks : 102.**
- **Each answer should start on a new page, clearly indicating the question number**
- **No marks will be awarded for stating an answer without justification.**
- **Calculators (in any form), protractors and electronic devices are not allowed.**
- **Rulers and compasses are allowed. Draw neat and labeled diagrams.**

Q1. Let x_1, x_2, x_3, … be a sequence of positive integers defined as follows: $x_1 = 1$ and for each $n \geq 1$ we have

$$x_{n+1} = x_n + \left[\sqrt{x_n}\right]$$

Determine all positive integers m for which $x_n = m^2$ for some $n \geq 1$. (her $[x]$ denotes the greatest integer less or equal to x for every real number x)

S1. If $k^2 \leq x_n < (k+1)^2$ for some natural number k.

$\Rightarrow \quad \left[\sqrt{x_n}\right] = k$

$\Rightarrow \quad x_{n+1} = x_n + k$

$\Rightarrow \quad k^2 + k \leq x_n + k < (k+1)^2 + k$

Lets assume the next square which appear is $(k + r)^2$, where $r \in \{1, …, k - 1\}$

$\Rightarrow \quad$ till then sequence will keep increasing by k

$\Rightarrow \quad (k + r)^2 - k^2 \equiv 0 \pmod{k}$

$\Rightarrow \quad (r^2 + 2kr) \equiv 0 \pmod{k}$

$\Rightarrow \quad r^2 \equiv 0 \pmod{k}$

$\quad r \in \{1, …, k - 1\}$

Not possible

$\Rightarrow \quad$ Next perfect square is $(k + k)^2 = (2k)^2$

$\Rightarrow \quad$ Since 1st perfect square is 2^0

Next will be $(2 \times 2^0) = 2^1$

$\Rightarrow \quad$ Answer $= 2^k$, $k \in \{0, 1, …\}$

where k is whole number.

Q2. Let $f : N \rightarrow N$ be a function satisfying the following conditions : for each $k > 2026$, the number $f(k)$ equals the maximum number of times a number appears in the list $f(1)$, $f(2)$, …, $f(k - 1)$. Prove that $f(n) = f(n + f(n))$ for infinitely many $n \in N$. (Here N denotes the set $\{1, 2, 3, …\}$ of positive integers)

S2. For $k > 2026$, $f(k)$ depends on multi set $\{f(1), f(2), \ldots f(k-1)\}$

For $k > 2026$, $f(k)$ is determined by previous values. Once number's frequency increasing, maximum frequency cannot decrease.

$f(k) = \max$ {frequency of number in $f(1), f(2) \ldots f(k-1)$}

In another words, we want frequency of mode in set of data.

$\Rightarrow \quad \forall k \geq 2027$, $f(k)$ is non decreasing

Let $M = \max \{f(i) \mid i \in \{1, 2, \ldots, 2027\}\}$

$\Rightarrow \quad f(i) \leq M \ \forall i \in \{1, 2, \ldots, 2027\}$

$\Rightarrow \quad (M+1) \notin \{f(i) \mid i \in \{1, 2, \ldots, 2027\}\}$

Lemma : $f(k)$ is unbounded

Lets assume for sake of contradiction, $f(k)$ is bounded as there exist a $n_0 > 2027$

Such that $f(n_0) = f(n_0 + 1) = f(n_0 + 2) = \ldots = m$

Since, $f(n_0) = m$

$\Rightarrow$ at least m times a number occurs out of $f(1), f(2) \ldots f(n_0 - 1)$, but each of $f(n_0), f(n_0 + 1) \ldots$ add another copy of m.

$\Rightarrow$ Number m appear more than m

$\Rightarrow \quad f(n_0 + m + 1) > m$

$\Rightarrow$ Contradiction $\Rightarrow f$ can't be constant after a while.

Let $n > 2027$ be smallest n such that $f(n) = M + 1$

$\Rightarrow \quad (M+1)$ occurs zero times in $f(1), f(2), \ldots f(n-1)$

Since $f(n) = M + 1$ and f is non decreasing

$f(n) = f(n+1) = \ldots = f(n+M) = M + 1$

$\Rightarrow \quad (M+1)$ appears $(M+1)$ times in $f(1), f(2), \ldots f(n+M)$

$\Rightarrow \quad f(n+M+1) = M + 1$

Since $f(n) = M + 1$

We can replace it in equation

$\Rightarrow \quad f(n + f(n)) = f(n) \ \forall n > 2027$

$\Rightarrow$ There will be in finite number n where n win be such index where new frequency will be introduced.

Q3. Let *ABC* be an acute-angled scalene triangle with circumcircle Γ. Let *M* be the midpoint of *BC* and *N* be the midpoint of the minor are *BC* of Γ. Points *P* and *Q* lie on segments *AB* and *AC* respectively such that *BP = BN* and *CQ = CN*. Point *K ≠ N* lies on line *AN* with *MK = MN*. Prove that ∠*PKQ* = 90°.

S3.
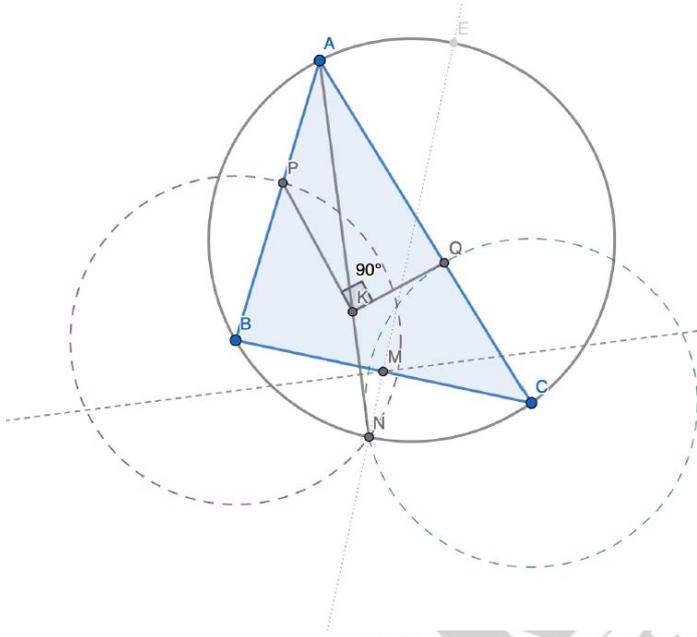
Since *AN* is angle bisector of ∠*BAC*.

Let *I* be incentre of △*ABC*. Let *w* be circum circle of △*BIC*.

Using excentre Incentre lemma, *N* will centre of *w*.

$$Pow_w(A) = AN^2 - r_w^2 = AN^2 - BN^2$$

$$= AC \cdot AC'$$

Where *C′* is inter section of w with *AC*.

∠*NBC* = θ (if ∠*BAC* = 2θ) due on angle on same chord *NC*.

Using sine rule in △*ABN*

$$\frac{AN}{\sin(\theta + B)} = \frac{BN}{\sin(\theta)}$$

$$\Rightarrow \quad \frac{AN}{BN} = \frac{\sin\left(90° + \dfrac{B - C}{2}\right)}{\sin(\theta)}$$

$\Rightarrow$ $AN\sin\theta = BN\cos\left(\dfrac{B-C}{2}\right)$

$NK = 2AN\sin^2\theta$

$\Rightarrow$ $AK = AN(1 - 2\sin^2\theta) = AN\cos(2\theta)$

Now $PQ^2 = AP^2 + AQ^2 - 2AP.AQ\cos2\theta$

$PK^2 = AP^2 + AK^2 - 2AP.AK\cos\theta$

$\Rightarrow$ using cosine law

$QK^2 = AQ^2 + AK^2 - 2AQ.AK\cos\theta$

Now,

$PK^2 + QK^2 = AP^2 + AQ^2 + 2AK^2 - 2AK\cos\theta(AP + AQ)$

Let $AP = x$, $AQ = y$

$\Rightarrow$ $PK^2 + QK^2 = x^2 + y^2 + 2AN^2\cos^2(2\theta) - 2(\cos\theta)(AP + AQ)AN\cos2\theta$

And $PQ^2 = x^2 + y^2 - 2xy\cos2\theta$

If $PQ^2 = PK^2 + QK^2$

$-2xy\cos^2\theta = 2AN^2\cos^2 2q - 2AN\cos\theta\cos2\theta(x + 2)$

$\Rightarrow$ $-xy = AN^2\cos2\theta - AN\cos\theta(x + y)$

Using above relation of AN and $\cos\theta$ we get that $PQ^2 = PK^2 + QK^2$

$\Rightarrow$ $\angle PKQ = 90°$

Q4. Two integers a and b are called companions if every prime number p either divides both or none of a, b. Determine all functions $f : N_0 \to N_0$ such that $f(0) = 0$ and the numbers $f(m) + n$ and $f(n) + m$ are companions for all m, $n \in N_0$. (Here N_0 denotes the set of all non-negative integers)

S4. $\sim$ denote that : $p \mid a \Leftrightarrow p \mid b$

$\Rightarrow$ $a \sim b$

Let prove using induction that $f(m) = m$

Base case $f(0) = 0$, $f(1) = 1$ are verified

Lets assume $f(k) = k \; \forall k < m$, $m \geq 2$

If $f(m) = m$ we are done, so for sake of contradiction $f(m) \neq m$

WLOG, $f(m) > m$, let $d = f(m) - m$ and $f(k) = k \; \forall \; k < m$

(i) $n = 1 \Rightarrow f(m) + 1 \sim 1 + m$

(ii) $n = m - 1 \Rightarrow (f(m) + m - 1) \sim (m - 1) + m$

$\Rightarrow$ $(f(m) + m - 1) \sim (2m - 1)$

(iii) $(f(m) + n, n + m) \; \forall \, n < m$

If d is odd

$(m + d + 1) \sim (m + 1)$

$\Rightarrow$ $(m + d + 1)$ and $(m + 1)$ have opposite parity

$\Rightarrow$ $2 \mid$ only one of them

$\Rightarrow$ $d \in$ even

Let p be any prime $p < m$.

Case (A) :

If $p \mid m$

Take $n = p$

$\Rightarrow$ Using (iii), $p \mid (n + m)$

$\Rightarrow$ $p \mid f(m) + p$

$\Rightarrow$ $p \mid f(m)$

Case-(B) :

If $p \nmid m$

Let $m \equiv r \bmod (P), \; r \in \{1, \ldots, p - 1\}$

Let $n = p - r$

$\Rightarrow$ $p \mid (m + p - r)$

$\Rightarrow$ $p \mid (m + n)$

Using (iii)

$p \mid f(m) + p - r$

$\Rightarrow$ $p \mid f(m) + n$ and using $p \mid (n + m)$

$p \mid (f(m) + n) - (n + m)$

$\Rightarrow$ $p \mid f(m) - m$

$\Rightarrow$ $p \mid d$

$\Rightarrow$ $\forall p < m, \; p \mid d$ and d even

$\Rightarrow$ $d = 0$

$\Rightarrow$ $f(m) = m \; \forall \; m \in N_0$

Q5. Three lines l_1, l_2, l_3 form as acute angled triangle T in the plane. Point P lies in the interior of T. Let τ_i denote the transformation of the plane such that the image $\tau_i(X)$ of any point X in the plane is the reflection of X in l_i, for each $i \in \{1, 2, 3\}$. Denote by P_{ijk} the point $\tau_k(\tau_j(\tau_i(P)))$ for each permutation (i, j, k) of $(1, 2, 3)$. Prove that P_{123}, P_{132}, P_{213}, P_{312}, P_{321} are concyclic if and only if P coincides with the orthocentre of T.

S5.

Let H be any point inside an acute angel $\triangle ABC$.

Lemma : H_{21} is image of H_{12} about the line HC.

Since $\angle HCA = 2\beta + \alpha$

$\Rightarrow$ $\angle CAH_{12} = 2\beta + \alpha$

$\Rightarrow$ $\angle H_1CH_{12} = 3(\alpha + \beta)$

$\Rightarrow$ $\angle H_1CH_{21} = \dfrac{3}{2}(\alpha + \beta)$

Similarly, $\Rightarrow \angle H_2CH_{21} = \dfrac{3}{2}(\alpha + \beta)$

$\Rightarrow$ H_{21} is image of H_{12}.

If H is orthocentre

$\Rightarrow$ These set of lines

$HA = A'H_1 = AH_1$ are same line and similarly

$HB = B'H_2 = BH_2$ are same line

$HC = C'H_3 = CH_3$ are same set of lines S, where A', B' and C' are image of A, B, C with respect to line BC, AB, AC respectively.

Since these lines are $\perp$ and their intersection point is H

$\Rightarrow$ H will be orthocentre due to intersection of $\perp$ lines.

If H is not orthocentre

$\Rightarrow$ These set of lines

$HA \neq A'H \neq AH_1$ and so on

$\Rightarrow$ and AH is perpendicular but not other set of lines.

6 of these bisectors meets at 3 distinct points outside $\triangle ABC$ hence all of 6 points cannot be cyclic.

Since circumcircle of any 3 consecutive points

H_{123}, H_{213}, …, H_{132} is the point H is centre of entire circle as it in intersection point of perpendicular bisector of chords.

Q6. Two decks A and B of 40 cards each are placed on a table at noon. Every minute thereafter, we pick the top cards $a \in A$ and $b \in B$ and perform a duel.

For any two cards $a \in A$ and $b \in B$ each time a and b duel, the outcomes remains the same and is independent of all other duels. A duel has three possible outcomes:

- If a card wins, it is placed back at the top of its deck and the losing card is placed at the bottom of its deck.
- If a and b are evenly matched, they are both removed from their respective decks,
- If a and b do not interact with each other, then both are placed at the bottom of their respective decks.

The process ends when both decks are empty. A process is called a game if it ends. Prove that the maximum time a game can last equals 356 hours.

S6. We are given two decks A and B of 40 cards each. Each minute the top card a of A and the top card b of B duel, with fixed outcome for the pair (a, b), independent of past duels. The possible outcomes are :

(i) a wins $\rightarrow$ a goes to top of A, b goes to bottom of B.

(ii) b wins $\rightarrow$ b goes to top of B, a goes to bottom of A.

(iii) evenly matched $\rightarrow$ both cards are removed.

(iv) No interaction $\rightarrow$ both go to the bottom of their own decks.

The gam ends when both decks are empty. We must determine the maximum possible duration of a game. At each duel:

If a card goes to bottom, that card is **not destroyed** and will eventually come back to the top after all other cards in its deck have been cycled through.

If evenly matched: both destroyed.

No interaction: each goes to bottom (so cards cycle through decks without destruction).

Lemma-1 : If we wants the game to last as long as possible, every card must be destroyed exactly once, and destruction must happen only when a card meets a specific opponent from the other deck for the first time.

Proof. A card can only be removed by an "evenly matched" outcome. If a card were to survive after meeting its destruction partner, that partnership could be used to destroy it earlier, shortening the number of preceding non-destruction duels involving it. Hence, to maximize total duels, each card's destruction should be postponed until it has met all other cards from the opposite deck that are still present.

For a given m (number of cards currently in each deck), it is possible to arrange the outcomes of duels so that before any destruction occurs at this stage, every pair (a, b) with $b \neq \pi(a)$ (among the remaining cards) meets exactly one.

Theorem. The maximum number of duels is 21360, corresponding to 356 hours.

Proof. Before the k-th destruction round ($k = 1, \ldots, 40$), when there are $m = 41-k$ cards

k cards in each deck, we can have exactly $m(m - 1)$ non-destruction duels. Afterwards, one destruction duel removes two cards.

Thus, total non-destruction duels $= \sum_{m=1}^{40} m(m-1)$.

We compute

$$\sum_{m=1}^{40} m(m-1) = \sum_{m=1}^{40} (m^2 - m) = \frac{40.41.81}{6} - \frac{40.41}{2} = 22140 - 820 = 21320$$

Destruction duels = 40.

Hence, total duels = 21320 + 40 = 21360.

Since each duel takes one minute, this is $\frac{21360}{60} = 356$ hours.

No arrangement can exceed this number, because if a destruction occurs earlier, the term $m(m - 1)$ for that m would be replaced by a smaller number and the sum is maximized by visiting all possible non-destruction pairs at each stage before any destruction.

❏   ❏   ❏