Time : 4.5 Hrs.   **39th Indian National Mathematical Olympiad-2025**   **January 19, 2025**

Instruction :

- **Calculators (in any form) and protractors are not allowed. Rulers and compasses are allowed.**

- **All questions carry equal marks. Maximum marks: 102.**

- **No marks will be awarded for stating an answer without justification.**

- **Answer to each question should start on a new page. Clearly indicate the questions number.**

1. Consider the sequence defined by $a_1 = 2$, $a_2 = 3$ and $a_{2k+2} = 2 + a_k + a_{k+1}$ and $a_{2k+1} = 2 + 2a_k$

    For all integers $k \geq 1$. Determine all positive integers n such that $\dfrac{a_n}{n}$ is an integer.

2. Let $n \geq 2$ be a positive integer. The integers $1, 2, \ldots, n$ are written on a board. In a move, Alice can pick two

    integers written on the board $a \neq b$ such that $a + b$ is an even number, erase both a and b from the board and

    write the number $\dfrac{a+b}{2}$ on the board instead. Find all n for which Alice can make a sequence of moves so that

    she ends up with only one number remaining on the board.

    Note : When $n = 3$, Alice changes (1, 2, 3) to (2, 2) and can't make any further moves.

3. Euclid has a tool called splitter which can only do the following two types of operations:

    Given three non-collinear marked points X, Y, Z, it can draw the line which forms the interior angle bisector of

    $\angle XYZ$.

    It can mark the intersection point of two previously drawn non-parallel lines.

    Suppose Euclid is only given three non-collinear marked points A, B, C in the plane. Prove that Euclid can use

    the several times to draw the centre of the circle passing through A, B and C.

4. Let $n \geq 3$ be a positive integer. Find the largest real number t_n as a function of n such that the inequality

    $\max(|a_1 + a_2|, |a_2 + a_3|, \ldots, |a_{n-1} + a_n|, |a_n + a_1|) \geq t_n \cdot \max(|a_1|, |a_2|, \ldots, |a_n|)$ holds for all real numbers

    $a_1, a_2, \ldots, a_n$.

5. Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have a least one coin. Griphook stops only when the cannot make any more moves. What is the maximum and minimum number of coins that he could have collected?

6. Let $b \geq 2$ be a positive integer. Anu has an infinite collection of notes with exactly $b - 1$ copies of a note worth $b^k - 1$ rupees, for every integer $k \geq 1$. A positive integer n is called payable if Anu can pay exactly $n^2 + 1$ rupees by using some collection of her notes. Prove that if there is a payable number, there are infinitely many payable numbers.

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Solution

1. Solution

   Observation :

   Terms of the sequence are

   2, 3, 6, 7, 8, 11, 14, 15, 16, 17, 18, 21, 24, 27, 30, 31, 32, … $\left(\dfrac{a_n}{n}\right)$ are integer for n = 1, 3, 7, 15, …

   Also, $a_1 = 2(1)$, $a_3 = 2(3)$, $a_7 = 2(7)$, $a_{15} = 2(15)$ …

   Notice that $n < a_n \leq 2n$ and equality seems to occur at only $n = 2^k - 1$, $k \geq 1$.

   Claim-I

   $a_n = 2n$, $n = 2^k - 1$, $k \geq 1$

   $a_{2^k-1} = 2(2^k - 1)$

   Base case : $k = 1$

   $a_{2^1-1} = a_1 = 2(2^1 - 1) = 2$, correct

   Let $a_{2^k-1} = 2(2^k - 1)$

   $\Rightarrow$ Now, $a_{2^{k+1}-1} = a_{2 \cdot 2^k - 1}$

   Using given relation

   $a_{2k+1} = 2(1 + a_k)$

   $\Rightarrow$ $a_{2k-1} = 2(1 + a_{k-1})$, $k \geq 2$

   $\Rightarrow$ $a_{2 \cdot 2^k - 1} = 2\left(1 + a_{2^k - 1}\right) = 2\left(1 + \left(2^k - 1\right) \cdot 2\right) = 2\left(2^{k+1} - 1\right)$

   Hence proved.

   Claim-II

   $n < a_n < 2n \; \forall \; n \in N - \{2^k - 1, \, k \geq 1\}$

Notice that terms are in arithmetic progression with common difference 1

$(a_1, a_2), (a_3, a_4, a_5), (a_7, a_8, a_9, a_{10}, a_{11})$

$\Rightarrow$ a_{2^k-1} to $a_{3 \cdot 2^k-1}$ are in arithmetic progression with common difference 1.

And next terms till $a_{2^{k+1}-1}$ are in arithmetic progression with common difference 3.

Since $a_{2^k-1} = 2(2^k-1)$

$a_{2^k} = 2(2^k-1) + 1 = 2^{k+1} - 1$

$a_{2^k+1} = 2^{k+1}$

$\Rightarrow$ $a_{3 \cdot 2^{k-1}} = 5 \cdot 2^{k-1} - 2 < 2(3 \cdot 2^k - 1)$

Also, $a_{2^{k+1}-1} = 2(2^{k+1}-1)$

$a_{2^{k+1}-2} = 2(2^{K+1}-1) - 3$

$= 2^{k+2} - 5 < 2(2^{k+1} - 2)$

Therefore, $a_n \in (n, 2n) \; \forall \; n \in N - \{2^{k-1}, k \geq 1\}$

Hence, only $n = 2^k - 1, k \in N$ will satisfy.

2. Solution

    Analyze small cases for n

    $n = 1$ (Trivial)

    $n = 2$ (Not possible as 1 + 2 = 3)

    $n = 3$ (Not possible already given in question)

    $n = 4$ 1, 2, 3, 4

Replace	Number written on Board
	1, 2, 3, 4
(1, 3)	2, 2, 4
(2, 4)	2, 3

$\therefore$ $n = 4$ is not possible

$n = 5$

Replace	Number written on Board
	1, 2, 3, 4, 5
(1, 3)	2, 2, 4, 5
(2, 4)	2, 3, 5
(3, 5)	2, 4

(2, 4)	3

$\therefore$ $n = 5$ is possible

$n = 6$

Replace	Numbers written on Board
	1, 2, 3, 4, 5, 6
(1, 3)	2, 2, 4, 5, 6
(2, 4)	2, 3, 5, 6
(3, 5)	2, 4, 6
(2, 4)	3, 6

$\therefore$ $n = 6$ is not possible (other combinations are also not possible)

$n = 7$

Claim-I : Sequence can be reduced to one number for every odd n greater than equal to 7.

	1, 2, 3, 4, 5, …, $2k - 1$
(1, 3)	2, 2, 4, 5, …, $2k - 1$
(2, 4)	2, 3, 5, …, $2k - 1$
(3, 5)	2, 4, 6, …, $2k - 1$
$\vdots$	2, $2k - 3$, $2k - 1$
(2, $2k - 2$)	2, $2k - 2$
	k

Claim-2 : For $n \geq 8$, the values of the form of $n = 4k$

Proof : Let numbers be 1, 2, …, $4k - 1$, $4k$

Now these numbers 1, 2, …, $4k - 1$ can be reduced to $2k$ (by claim 1) and $2k$, $4k$ can be replaced by $3k$.

Claim-II : For $n \geq 10$, the values of form of $n = 4k + 2$

The numbers we have 1, …, $4k - 1$, $4k$, $4k + 1$, $4k + 2$.

Now, 1, …, $4k - 1$ can be reduced to 2k.

$2k$, $4k$, $4k + 1$, $4k + 2$

For $k = 4m$

$2k$, $4k$, $4k + 1$, $4k + 2$

$\Rightarrow$ $3k$, $4k + 1$, $4k + 2$

$\Rightarrow$ $12m$, $16m + 1$, $16m + 2$

$\Rightarrow$ $14m + 1$, $16m + 1$

$\Rightarrow$ $15m + 1$

$\therefore$ $k = 4m$ is accepted

For $k = 4m + 2$

$2k, 4k, 4k + 1, 4k + 2$

$\Rightarrow$ $3k + 1, 4k, 4k + 1$

$\Rightarrow$ $12m + 7, 16m + 8, 16m + 9$

$\Rightarrow$ $14m + 8, 16m + 8$

$\Rightarrow$ $15m + 8$

$\therefore$ $k = 4m + 2$ is accepted

For $k = 4m + 3$

$2k, 4k, 4k + 1, 4k + 2$

Replacing $2k$ and $4k + 2$ by $3k + 1$

$3k + 1, 4k, 4k + 1$

$\Rightarrow$ $12m + 10, 16m + 12, 16m + 13$

$\Rightarrow$ $14m + 11, 16m + 13$

$\Rightarrow$ $15m + 12$

$\therefore$ $k = 4m + 3$ type numbers are accepted.

For $k = 4m + 1$

$2k, 4k, 4k + 1, 4k + 2$

Replacing $2k$ and $4k$ by $3k$

$3k, 4k + 1, 4k + 2$

$\Rightarrow$ $12m + 3, 16m + 5, 16m + 6$

$\Rightarrow$ $14m + 4, 16m + 6$

$\Rightarrow$ $15m + 5$

$\therefore$ $k = 4m + 1$ is accepted.

$\therefore$ $n = 4k$ and $4k + 2$ are also accepted

$\therefore$ It works for all $n \geq 7$.

$\therefore$ $n = 1, 5$ and $n \geq 7$ are accepted values.

3.   Solution

     The problem essentially is to get to perpendicular bisectors of sides from angle bisectors of vertices.

     Let's construct Incentre of the triangle ABC using angle bisectors of sides and their point of concurrency.
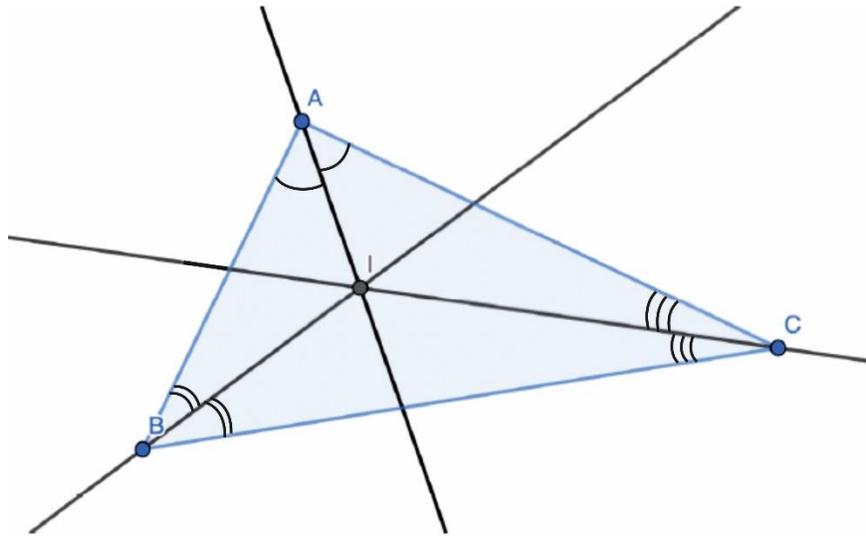
Now, let's consider the triangle BIC. to get ex-centre B and ex-centre C of triangle BIC we need to draw external angle bisectors. Therefore, let's take a point D = intersection point of angle bisector of the angle BAI and already drawn line CI (angle bisector of angle ACB)
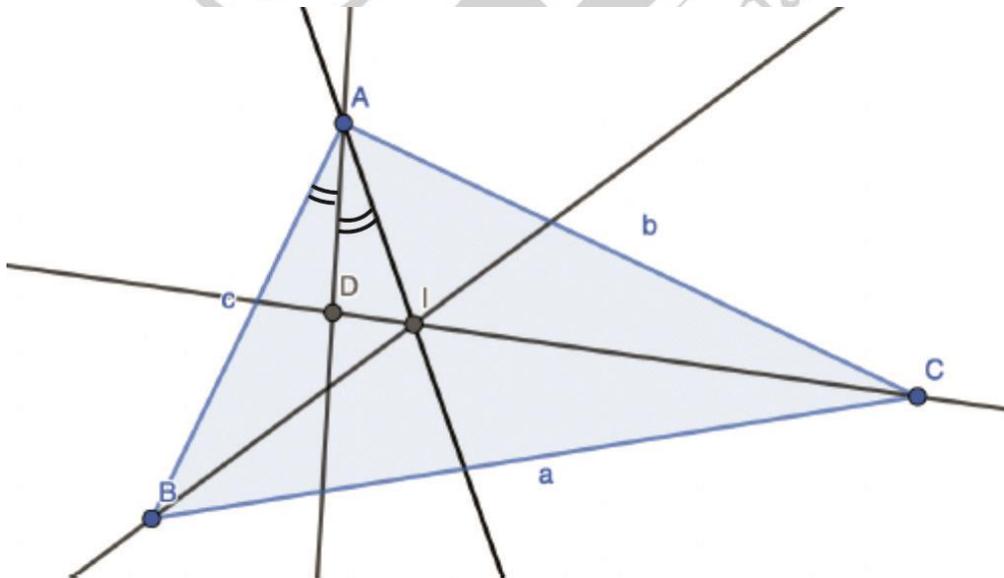
Now, let's draw ex-centres for B and C for the triangle BIC using angle DIB.
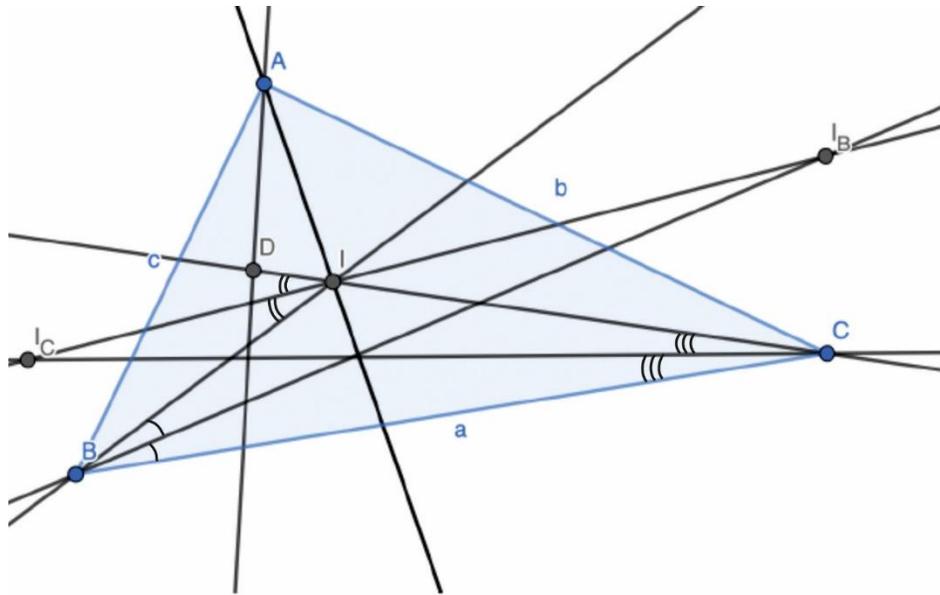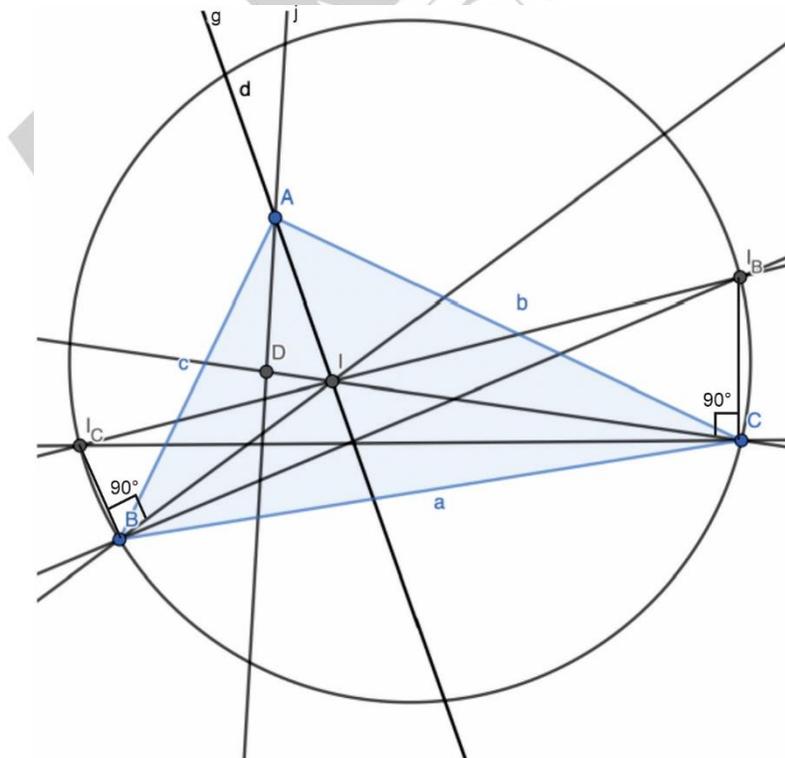
Lemma I:

The quadrilateral BCI_BI_C is cyclic with I_BI_C as diameter.

Proof : Since $\angle I_CBI_B = 90°$

$\angle I_CCI_B = 90°$

$\Rightarrow$ I_BI_C will be diameter and BCI_BI_C will be cyclic quadrilateral.

Lemma 2:

Angle bisector of I_BBI_C bisects the arc I_BI_C.

Therefore, let E be the intersection point of Angle bisector of I_BBI_C and the angle bisector of I_BCI_C.

Therefore, triangle EI_BI_C will be an isosceles triangle. Now, draw the angle bisector of the angle I_CEI_B. Let the point of intersection of this and II is F. Then EF is angle bisector as well as the perpendicular bisector of the largest chord (diameter) of the circle. Hence, EF is perpendicular bisector, hence F is the center of the circle. Therefore, FB = FC. Similarly, using the same construction for A and C we will get FC = FA. Therefore, F will the centre of the circumcircle of triangle ABC.
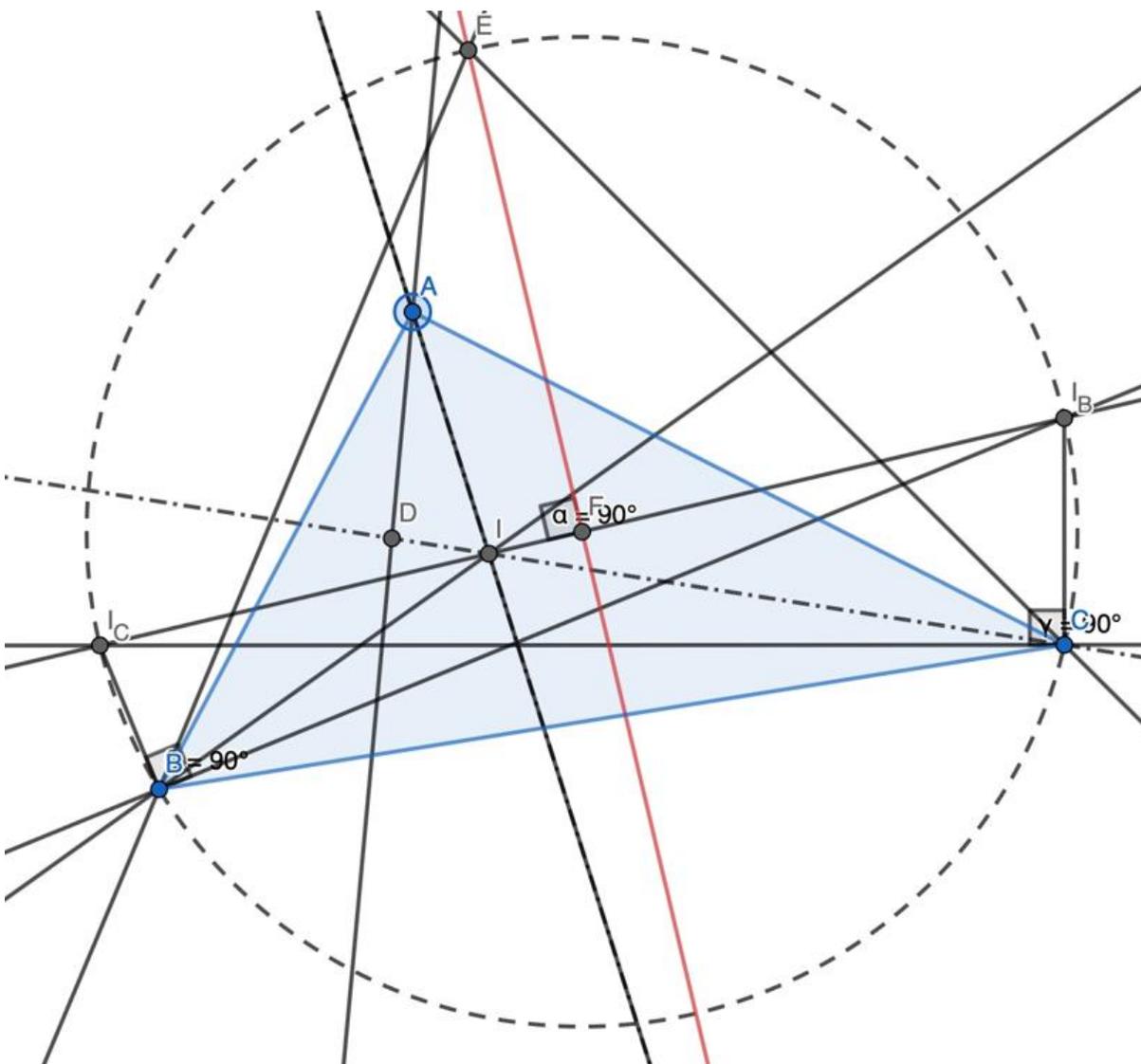
4. Solution

    We need to find the largest real number t_n such that the given inequality holds true for all real numbers a_i.

    Case-A : When n is even.

    Since the equality holds for every possible a_i.

    Let $\{a_i\}$ to be a sequence such that $|a_1 + a_2| > |a_2 + a_3| + \ldots |a_n + a_1|$ becomes zero.

    Therefore, an easy sequence will be alternate $-1, +1, -1, +1, \ldots$

    $a_i = (-1)^i \; \forall \, i \in \{1, 2, \ldots n\}$

$\Rightarrow$ $\max\{|a_1 + a_2|, |a_2 + a_3|, \ldots + |a_n + a_i|\} = \max\{0, 0, \ldots, 0\}$

$0 \geq t_n \{-1, 1, |1|, \ldots |-1|\}$

$0 \geq t_n$

$\Rightarrow$ The largest real value will be $t_n = 0$ and equality will hold.

Case-B :

When n is odd

Let $\max \{|a_1 + a_2| + |a_2 + a_3| + \ldots + |a_n + a_1|\} = M$

WLOG $|\underline{a}_1| \geq |a_2| \geq \ldots \geq |a_n|$

$\Rightarrow$ $|a_1 + a_2| \leq M$

$\Rightarrow$ $a_2 \in [-a, -M, -a_1 + M]$

$\Rightarrow$ $|(a_2 + a_3)| \leq M$

$\Rightarrow$ $-M \leq a_2 + a_3 \leq M$

$\Rightarrow$ $a_3 \in [-2M + a_1, 2M + a_1]$

$\Rightarrow$ By induction

   $a_n \in [-(n-1)M - a_1, (n-1)M - a_1], n \in$ even

   $a_n \in [-(n-1)M + a_1, (n-1)M + a_1], n \in$ odd

$\Rightarrow$ $(a_n + a_1) \in [-(n-1)M + 2a_1, (n-1)M + 2a_1]$

$\Rightarrow$ $|a_n + a_1| \leq M$

$\Rightarrow$ $(a_n + a_1) \in [-M, M]$

   $-(n-1)M + 2a_1 \leq M$ and $-M \leq (n-1)M + 2a_1$

   $2a_1 \leq nM$ and $-nm \leq 2a_1$

$\Rightarrow$ $2a_1 \in [-nM, nM]$

$\Rightarrow$ $|a_1| \leq \dfrac{nM}{2}$

   $\dfrac{2}{n}|a_1| \leq M$

$\Rightarrow$ $t_n = \dfrac{2}{n}$ $\forall$ n odd natural numbers.

Construction to show if equality occurs.

$\Rightarrow$ Let $|a_1| = d$, $|a_{k+2}| = (2k-1)d$

   $|a_2| = 3d$ $|a_{k+3}| = (2k-3)d$

   $|a| = (2k+1)d$ $|a_{2k+1}| = d$

Such that $a_i a_{i+1} < 0$

$\Rightarrow$ $a_1 = d, a_2 = -3d, a_3 = 5d, \ldots, a_n = d$

$\Rightarrow$ $|a_i + a_{i+1}| = 2d \forall k$

   $\text{Max}(|a_i|) = nd$
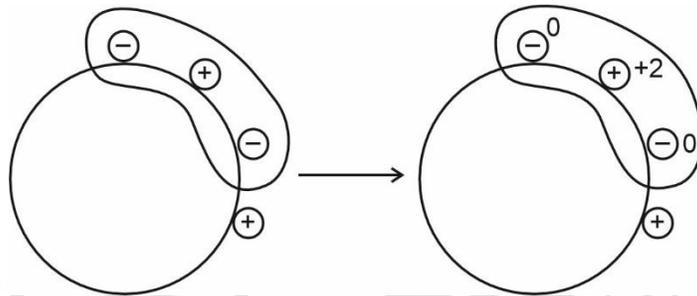
$\Rightarrow \quad (2d) = t_n(nd)$

$\Rightarrow \quad t_n \le \dfrac{2}{n}, \max(t_n) = \dfrac{2}{n}$

5. Solution

Claim-I : (1998) coins are the maximum possible. If we consider coins to have weights +1, −1 alternatively, then the weighted sum is (+1) + (−1) + (+1) + (−1) … = 0 (mod 3)

After a move on the vertex

$S_T = S \,(-1 + 1 - 1) \rightarrow S_T = S + (0 + 2 + 0)$

$S - 1 \equiv S_T \,(\text{mod } 3) \rightarrow (S + 2) \equiv S_T \,(\text{mod } 3)$

$\Rightarrow$ Weighted sum is invariant under mod 3.
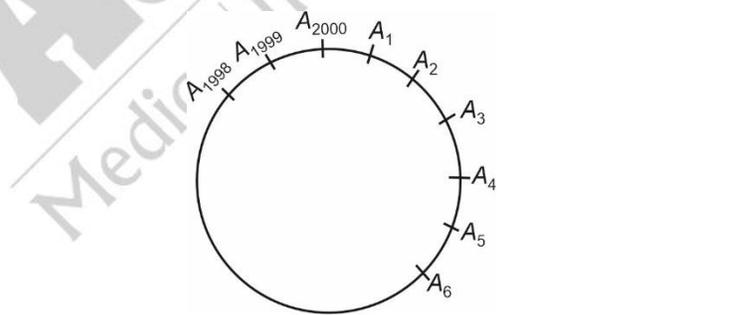
So it in the end 1999 coins are taken off then remaining coin can be ±1.

But initially the weighted sum is 0(mod 3)

Therefore $\pm 1 \not\equiv 0(\text{mod } 3)$.
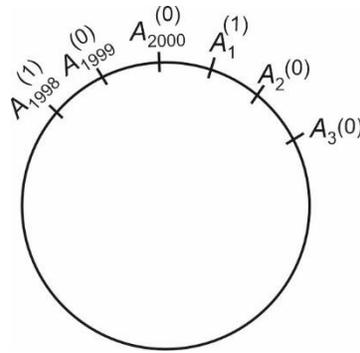
Hence (1999) is not possible.

Now, let see a construction which can maximize coins as 1998.

Apply on A_1

$\Rightarrow$ A_1 will have 2 coins and A_{2000} and A_2 will have 0 coins. Now apply on A_2, A_3, …, A_{1998}. After A_{1998} (0) the situation will be $A_1(1)$ and $A_{1998}(1)$
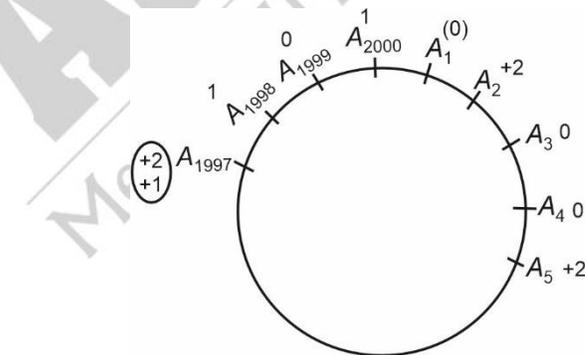
$\Rightarrow$ (1998) coins will be collected.

Claim-II

Minimum occurance when (A_{3k+2}) vertices S coins are picked then sequence of coins will be generated as 0, 2, 0, 0, 2, 0 … therefore after a circular round there will be not more moves available.
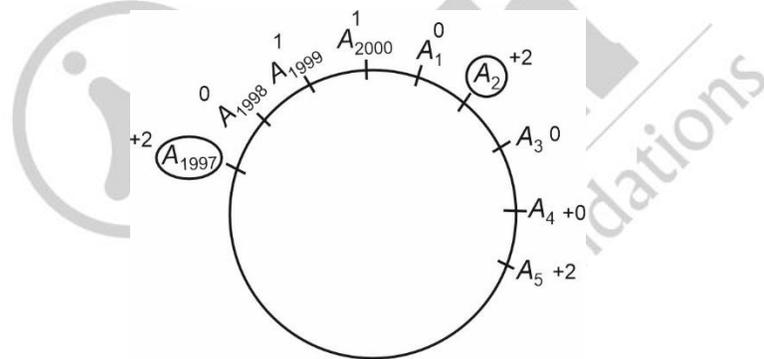
$\Rightarrow \quad 2 + (n-1)(3) = 1997 = n = \dfrac{1985}{3} + 1 = 666$

Now, apply on A_{1998}

Now, apply finally on A_{1999}, which will make A_{2000} to be 0 and A_{1998} zero.

Two sequence $A_1 A_2 … A_{2000}$ will be

{0, 2, 0, 0, 2, 0, 0, 2, …, 1, 0, 1, 0}

Therefore, no more move possible

$\Rightarrow \quad$ (666 + 1 + 1) moves minimum required

= 668.

6. Solution

We can use infinite descent to prove the result. Let (w_k) be number of notes used to form the *payable* amount of worth $(b^k - 1)$ $\forall$ $k \geq 1$

$\Rightarrow$ Let N_0 be the *payable*

$\Rightarrow$ $\left(N_0^2 + 1\right) = \sum_{k=1}^{\infty} w_k \left(b^k - 1\right)$

Since $(b - 1) \mid (b^k - 1)$ $\forall$ $k \geq 1$

$\Rightarrow$ $(b-1) \mid \left(N_0^2 + 1\right)$, since $w_k \neq 0$ $\forall$ $k \geq 1$

Claim :

If N_0 is *payable* then there exist infinitely many *payable* numbers.

Since $(b-1) \mid \left(N_0^2 + 1\right)$

$\Rightarrow$ $N_0^2 \equiv (-1) \bmod (b-1)$

Also, $b \equiv (1) \bmod (b - 1)$

$\Rightarrow$ $b^2 \equiv 1^2 \bmod (b - 1)$

$\Rightarrow$ $b^2 \equiv 1^2 \bmod (b - 1)$

$\Rightarrow$ $N_0^2 b^2 \equiv (-1) \bmod (b-1)$

$\Rightarrow$ $(N_0 b)^2 + 1 \equiv 0 \bmod (b - 1)$

Now, to make that any number which is of form $(\lambda^2 + 1)$ is payable when $(b - 1)$ divides it. Let $\{0, 1, 2, \ldots, (b - 1)\}$ be digits in base p system.

Consider a function $F(x) = a_1 x^1 + a_2 x^2 + \ldots + a_k x^k$ {let $w_{p+1} = 0$ $\forall$ $p \geq k$}

$F(b) - F(1) = \sum_{k=1}^{\infty} \left(b^k - 1\right) w_k$

Will be uniquely mapped. Using the unique representation in base b.

Also from here as well we can clearly see that $(b - 1) \mid F(b) - F(1)$

$\Rightarrow$ Therefore, this will be uniquely written in the form of $F(b) - F(1)$.

Hence, $(N_0 b)$ will be *payable*.

$\Rightarrow$ $(N_0 b)b$, $(N_0 b^2)b$ … so on will be *payable*. Hence if there exist a *payable* number, there are infinitely many *payable* numbers.

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