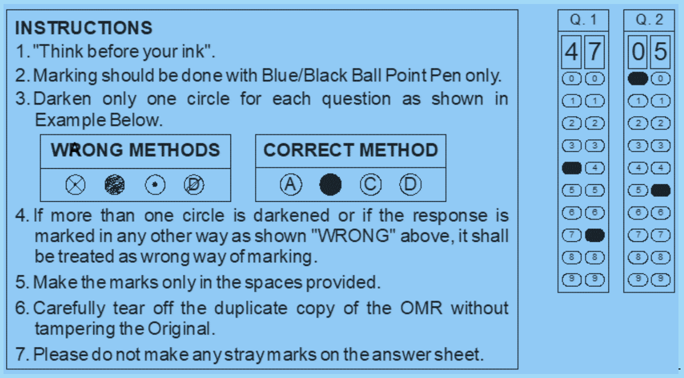

# Answers & Solutions
# for
# IOQM – 2025-26

# INSTRUCTIONS TO CANDIDATES

1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen.** Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The registration number and date of birth will be your login credentials for accessing your score.
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two-digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.




6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 & 30 carry 5 marks each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the Candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

**Note:**

1. $\gcd\left(a, \sqrt{b}\right)$ denotes the greatest common divisor of integers *a* and *b*.
2. For a positive real number *m*, $\sqrt{m}$ denotes the positive square root of *m*. For example, $\sqrt{4} = +2$.
3. Unless otherwise stated all numbers are written in decimal notation.

1. If 60% of a number *x* is 40, then what is *x*% of 60?

**Answer (40)**

**Sol.** $\therefore$ 60% of *x* = 40

Then $\frac{60x}{100} = 40$

$\therefore \quad x = \frac{400}{6}$

$\therefore \quad x\% \text{ of } 60 = 60 \times \frac{400}{600}$

$= 40$

2. Find the number of positive integers $n$ less than or equal to 100, which are divisible by 3 but are no divisible by 2

**Answer (17)**

**Sol.** Total number of positive integers less than or equal to 100 = 100

The numbers divisible by 3 = 33

The numbers divisible by 2 and 3 i.e. divisible by 6 = 16

$\therefore$ The numbers divisible by 2 and 3 i.e., divisible by 6 = 16

$\therefore$ The numbers divisible by 3 but not by 2 and less then or equal to 6 = 33 – 16

= 17

3. The area of an integer-sided rectangle is 20. What is the minimum possible value of its perimeter?

**Answer (18)**

**Sol.** Let l and b are sides and P is perimeter of the rectangle.

$lb = 20$

$P = 2(l + b)$

Possible integer side pair (1, 20), (2, 10), (4, 5)

For (1, 20)

$P = 2(1 + 20) = 42$

For (2, 10)

$P = 2(2 + 10) = 24$

For (4, 5)

$P = 2(4 + 5) = 18$

$\therefore$ Smallest perimeter = 18

4. How many isosceles integer-sided triangles are there with perimeter 23?

**Answer (6)**

**Sol.** Let the sides of triangle be $a$, $a$, $b$

Perimeter = $2a + b = 23$

$b = 23 - 2a$

For a valid triangle, $2a > b$

$\Rightarrow \quad 2a > 23 - 2a$

$\Rightarrow \quad 4a > 23$

$\Rightarrow \quad a \geq 6$

Also, $b > 0$

$\Rightarrow \quad 23 - 2a > 0$

$\Rightarrow \quad a \leq 11$

So, $6 \leq a \leq 11$

When, $a = 6$, $b = 11$

$\Rightarrow \quad 6 + 6 > 11$ (valid)

$a = 7$, $b = 9$

$\Rightarrow \quad 7 + 7 > 9$ (valid)

$a = 8$, $b = 7$

$\Rightarrow \quad 8 + 8 > 7$ (valid)

$a = 9$, $b = 5$

$\Rightarrow \quad 9 + 9 > 5$ (valid)

$a = 10$, $b = 3$

$\Rightarrow \quad 10 + 10 > 3$ (valid)

$a = 11$, $b = 1$

$\Rightarrow \quad 11 + 11 > 1$ (valid)

There are 6 such isosceles integer-sided triangle.

5. How many 3-digit numbers $\overline{abc}$ in base 10 are there with $a \neq 0$ and $c = a + b$?

**Answer (45)**

**Sol.** $\because$ $C$ is a digit $a + b \leq 9$

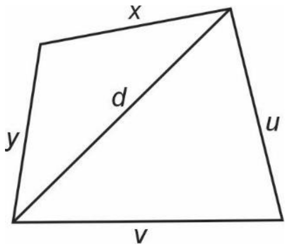
For each $a \in \{1, \ldots, 9\}$ can be 0, 1, …, 9 – a, which gives $10 - a$ choices.

Total count $\sum_{a=1}^{9}(10-a) = 9 + 8 + \ldots + 1 = 45$

6. Four sides and a diagonal of a quadrilateral are of lengths 10, 20, 28, 50, 75 not necessarily in that order. Which amongst them is the only possible length of the diagonal?

**Answer (28)**

**Sol.**




$|x - y| < d < x + y$

$|u - v| < d < u + v$

When $d = 75$

$x + y > 75$

$u + v > 75$

Sum of all 4 sides = 10 + 20 + 28 + 50

= 108 < 150

$\therefore$ 75 can't be diagonal

When $d = 50$

$|75 - x| < 50$

$\Rightarrow$ $x > 25$

Among 10, 20, 28 only 28 works

So, possible paring is (75, 28), (10, 20)

But 10 + 20 = 30 < 50

So, triangle can't be formed

When $d = 20$

$|75 - x| < d$

$\Rightarrow$ $x > 55$, when $d = 20$

And $x > 65$ when $d = 10$

So, 20, 10 are not possible

Now, $d = 28$, pairs are (10, 20) and (50, 75)

|10– 25| = 10

10 < 28 < 30

|50 – 75| < 28 < 50 + 75

$\therefore$ Diagonal = 28

7. The age of a person (in years) in 2025 is a perfect square. His age (in years) was also a perfect square in 2012. His age (in years) will be a perfect cube $m$ years after 2025. Determine the smallest value of $m$.

**Answer (15)**

**Sol.** Let his age in 2025 be $x$

$\Rightarrow$ his age in 2012 = $x - 13$

$x = l^2$

$x - 13 = m^2$

$l^2 - m^2 = 13$

$(l + m)(l - m) = 13$

$$(l, m) = (7, 6)\begin{cases} l + m = 13 \\ l - m = 1 \end{cases}$$

$$\text{Or } \left.\begin{matrix} l + m = 1 \\ l - m = 13 \end{matrix}\right\}(l,\ m) = 7, -6$$

$\Rightarrow$ age at 2025 is 49

Age at 2012 is 36

Next perfect cube after 49 is 64

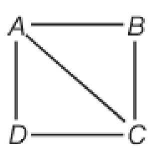
That will come after 64 – 49 years = m

$\Rightarrow$ $m = 15$.

8. A quadrilateral has four vertices $A$, $B$, $C$, $D$. We want to colour each vertex in one of the four colours red, blue, green or yellow, so that every side of the quadrilateral and the diagonal $AC$ have end points of different colours. In how many ways can we do this?

**Answer (48)**

**Sol.**




$A$ has 4 choices

$B$ has 3 choices (cannot be equal to $A$)

$C$ has 2 choices (cannot be equal to $A$ and $B$)

$D$ has 2 choices (cannot be equal to $A$ and $C$)

$= 4 \times 3 \times 2 \times 2 = 48$ ways.

9. The height and the base radius of a closed right circular cylinder are positive integers and its total surface area is numerically equal to its volume. If its volume is $k\pi$ where $k$ is a positive integer, what is the smallest possible value of $k$?

**Answer (54)**

**Sol.** Let radius of base and height of a right circular cylinder be $r$ and $h$ respectively

$\because$ Numerically total surface area = volume of cylinder

$\therefore$ $2\pi r^2 + 2\pi rh = \pi r^2 h$

Or $2r + 2h = rh$

Or $rh - 2r - 2h + 4 = 4$

Or $(r - 2)(h - 2) = 4$

$\therefore$ $r - 2 = 1$, then $h - 2 = 4$

or $r - 2 = 2$ then $h - 2 = 2$

or $r - 2 = -1$, then $h - 2 = -4$

or $r - 2 = -2$, then $h - 2 = -2$

$\therefore$ $(r, h) = (3, 6), (6, 3)$ or $(4, 4)$

$\therefore$ Minimum possible value of $r^2h = 54$

10. The sum of two real numbers is a positive integer $n$ and the sum of their squares is $n + 1012$. Find the maximum possible value of $n$.

**Answer (46)**

**Sol.** Let the two real numbers are $x$ and $y$

$\therefore$ $x + y = n$

And $x^2 + y^2 = n + 1012$

$\therefore$ $xy = \dfrac{(x+y)^2 - (x^2 + y^2)}{2}$

$\therefore$ $xy = \dfrac{n^2 - n - 1012}{2}$

$\therefore$ $x$ and $y$ are roots of equation

$$t^2 - nt + \frac{n^2 - n - 1012}{2} = 0$$

$\because$ roots are real

$\therefore$ Discriminant $(D) \geq 0$

Thus $n^2 - 4.1.\left(\dfrac{n^2 - n - 1012}{2}\right) \geq 0$

Or $n^2 - 2n - 2024 \leq 0$

Or $(n + 44)(n - 46) \leq 0$

$\therefore$ $n \in [-44, 46]$

$\therefore$ Maximum possible value of $n$ is 46.

11. Consider a fraction $\frac{a}{b} \neq \frac{3}{4}$, where $a$, $b$ are positive integers with $gcd(a, b) = 1$ and $b \leq 15$. If this fraction is chosen closest to $\frac{3}{4}$ amongst all such fractions, then what is the value of $a + b$?

**Answer (26)**

**Sol.** $\left|\frac{a}{b} - \frac{3}{4}\right|$ should be minimum for gcd($a$, $b$) = 1 and $b \leq 15$

$\Rightarrow$ gcd(a, b) = gcd(b, 4a – 3b) = 1

$\Rightarrow$ $m = \left|\frac{4a-3b}{4b}\right| = \frac{|4a-3b|}{4b}, \; b \in I^+$

$\Rightarrow$ For minimum $m$, $|4a - 3b|_{min}$ and $(4b)_{max}$

$\Rightarrow$ $|4a - 3b| = 1$, $b_{max}$

$\Rightarrow$ $4a - 3b = 1$ or $-1$

For $4a - 3b = 1$

(10, 13) for $k = 3$

$\Rightarrow$ $\frac{1}{4 \times 13} = \left(\frac{1}{52}\right) = m$

Solution are $(3k + 1, 4k + 1)$ for some $k \in$ I

For $4a - 3b = -1$

Solution are $(3k - 1, 4k - 1)$ for some $k \in$ I

For $k = 4$

(11, 15)

$\Rightarrow$ $\frac{1}{4 \times 15} = \frac{1}{60} m(\text{minimum})$

If $|4a - 3b| \neq 1$

$|4a - 3b| \geq 2$

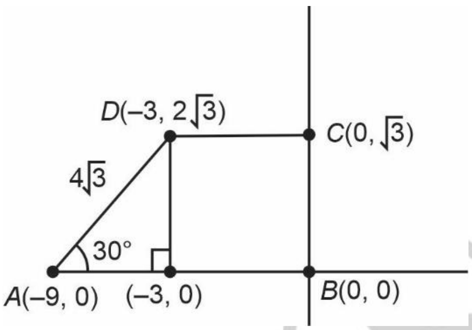
$\Rightarrow$ $\left|\frac{4a-3b}{4b}\right| \geq \frac{2}{4b} \geq \frac{1}{2b}$

$b \leq 15$

12. Three sides of a quadrilateral are $a = 4\sqrt{3}$, $b = 9$ and $c = \sqrt{3}$. The sides $a$ and $b$ enclose an angle of 30°, and the sides $b$ and $c$ enclose an angle of 90°. If the acute angle between the diagonals is $x$°, what is the value of $x$?

**Answer (60)**

**Sol.**




Slope of $AC = \dfrac{\sqrt{3}}{9} = \dfrac{1}{3\sqrt{3}}$

Slope of $BD = \dfrac{2\sqrt{3}}{-3} = \dfrac{-2}{\sqrt{3}}$

$$\tan\theta = \left|\frac{\frac{1}{3\sqrt{3}} + \frac{2}{\sqrt{3}}}{1 - \frac{2}{9}}\right|$$

$$= \left|\frac{\frac{7}{3\sqrt{3}}}{\frac{7}{9}}\right|$$

$\tan\theta = \sqrt{3}$

$\Rightarrow \quad \theta = 60°$.

13. Consider five-digit positive integer of the form $\overline{abcab}$ that are divisible by the two-digit number $\overline{ab}$ but not divisible by 13. What is the largest possible sum of the digits of such a number?

**Answer (33)**

**Sol.** Let $p = \overline{abcab}$

$q = \overline{ab}$

$13 \nmid p$ but $q \mid p$

$\Rightarrow \quad p = 10^4a + 10^3b + 10^2c + 10a + 10^0b$

$p = 10a(10^3 + 1) + 1000b + 10^2c + b$

$= 10a(1001) + 1001b + 100c$

$p = 1001[10a + b] + 100c$

$p = 13 \times 77 \times \overline{ab} + 100c$

$\overline{ab} \mid p$

$\Rightarrow \quad \overline{ab} \mid 100c$

But $13 \nmid p$

$\Rightarrow \quad 13 \nmid 100c$

$\Rightarrow \quad 13 \nmid c$

As $c \in \{0, 1, \ldots, 9\}$

Max possible value of $\overline{ab} = 900$

When $c = 9$

$\Rightarrow \quad p = 75975$

$\Rightarrow$ Answer is 33.

14. A function $f$ is defined on the set of integers such that for any two integers $m$ and $n$,

$$f(mn+1) = f(m)f(n) - f(n) - m + 2$$

Holds and $f(0) = 1$. Determine the largest positive integer $N$ such that $\sum_{k=1}^{N} f(k) < 100$

**Answer (12)**

**Sol.** $f(mn+1) = f(m)f(n) - f(n) - m + 2 \qquad m, n \in I \qquad \ldots\text{(i)}$

replace $m \leftrightarrow n$

$f(mn+1) = f(n)f(m) - f(m) - n + 2 \qquad m, n \in I \qquad \ldots\text{(ii)}$

(i) – (ii)

$-f(n) - m + 2 + f(m) + n - 2 = 0$

$(f(m) - m) = f(n) - n = \lambda$

$\Rightarrow \quad f(x) = \lambda + x$

Alos $f(0) = 1$

$\Rightarrow \lambda = 1$

$f(x) = x + 1$

Now $\sum_{k=1}^{N} (x+1) < 100$

$\frac{N(N+1)}{2} + N < 100$

$N^2 + 3N - 200 < 0$

$N6(-15.721, 12.721)$

$\Rightarrow$ largest integral value of $N$ is 12

15. There are six coupons numbered 1 to 6 and six envelopes, also numbered 1 to 6. The first two coupons ae placed together in any one envelope. Similarly, the third and the fourth are placed together in a different envelope, and the last two are placed together in yet another different envelope. How many ways can this be done if no coupon is placed in the envelope having the same number as the coupon?

**Answer (40)**

**Sol.** For the group $(i, j)$ assigned to envelope k we required $k \neq 1, k \neq j$

Let the group A: coupons (1, 2)

Let the group B: coupons (3, 4)

Let the group C: coupons (5, 6)

Let group A goes to envelope $a \Rightarrow a \neq 1, 2$

Let group B goes to envelope $a \Rightarrow b \neq 3, 4$

Let group C goes to envelope $c \Rightarrow c \neq 5, 6$

$\therefore \quad f\,;\{A, B, C\} \to \{1, , 3, 4, 5, 6\}$ such that

$f(A) \in \{3, 4, 5, 6\}$

$f(B) \in \{1, 2, 5, 6\}$

$f(C) \in \{1, 2, 3, 4\}$

Case-I: $f(A) = 3$

$f(B) = 1$ or $2 \Rightarrow f(\text{C}) \to 2$ possible values

*OR*

$f(B)$ = 5 or 6 $\Rightarrow f(C) \rightarrow$ 3 possible values

$\therefore$ 2 × 2 + 2 × 3 = 10 possible cases

Case-II: $f(A)$ = 4

$f(B)$ = 1 or 2 $\Rightarrow f(C) \rightarrow$ 2 possible values

$f(B)$ = 5 or 6 $\Rightarrow f(C) \rightarrow$ 3 possible values

$\therefore$ 2 × 2 + 3 × 2 = 10 possible cases

Case-III: $f(A)$ 5 or 6

$f(B)$ = 1 or 2 ; $f(C)$ = 3 possible cases

$f(B)$ = 1 possible case ; $f(C)$ = 4 possible case

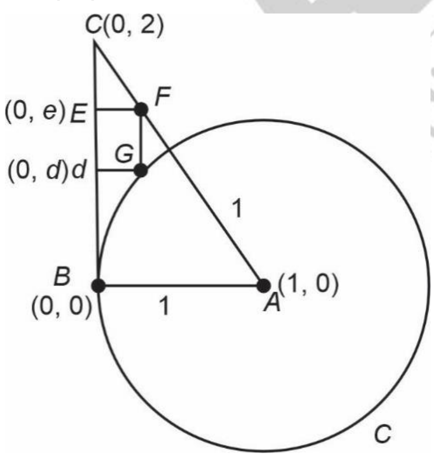
$\therefore$ 2(3 × 2 + 4) = 20 cases

$\therefore$ total 40 cases

16. In triangle $ABC$, $\angle B = 90°$, $AB$ = 1 and $BC$ = 2. On the side $BC$ there are two points $D$ and $E$ such that $E$ lies between $C$ and d and $DEFG$ is a square, where $F$ lies on $AC$ and $G$ lies on the circle through $B$ with centre $A$. If the area of $DEFG$ is $\frac{m}{n}$ where $m$ and $n$ are positive integers with gcd($m$, $n$) = 1, what is the value of $m + n$?

**Answer (29)**

**Sol.**




Let side = $S$, $e - d = S$

$F \equiv (S, e)$

$G \equiv (S, d)$

$AC \equiv \dfrac{x}{1} + \dfrac{y}{2} = 1$

$\Rightarrow$ Circle $C$ is $(x - 1)^2 + y^2 = 1$

$(8 - 1)^2 + d^2 = 1$

$F$ lie on $AC$ :

$\Rightarrow$ $2x + y - 2 = 0$

$2S + e - 2 = 0$

$2S + (d + S) - 2 = 0$

$\Rightarrow$ $3S + d = 2$

$(S - 1)^2 + (2 - 3S)^2 = 1$

$\Rightarrow$ $10S^2 - 2S - 12S + 1 + 4 = 1$

$\Rightarrow$ $10S^2 - 14S + 4 = 0$

$\Rightarrow$ $5S^2 - 7S + 2 = 0$

$\Rightarrow$ $5S^2 - 5S - 2S + 2 = 0$

$\Rightarrow$ $5S(S - 1) - 2(S - 1) = 0$

$\Rightarrow$ $S = 1$ and $S = \dfrac{2}{5}$ but $S \neq 1$

As $DG < AB = 1$

$\Rightarrow$ $S^2 = \dfrac{4}{25}$

$\Rightarrow$ 29

17. Let $f(x)$ and $g(x)$ be two polynomials of degree 2 such that $\dfrac{f(-2)}{g(-2)} = \dfrac{f(3)}{g(3)} = 4$. If $g(5) = 2$, $f(7) = 12$, $g(7) = -6$, what is the value of $f(5)$?

**Answer (22)**

**Sol.** $h(x) = f(x) - 4g(x)$ has two roots –2 and 3

Now $h(x)$ is also a quadratic equation

$\Rightarrow$ $h(x) = f(x) - 4g(x) = k(x + 2)(x - 3)$

$x = 7$

$f(7) - 4g(7) = k(9)(4)$

$\frac{12 + 4 \times 6}{9 \times 4} = k \ \Rightarrow k = 1$

$\Rightarrow \quad f(x) = 4g(x) + (x+2)(x-3)$

For $f(5)$ put $x = 5$

$f(5) = 4g(5) + 7 \times 2$

$= 4 \times 2 + 14$

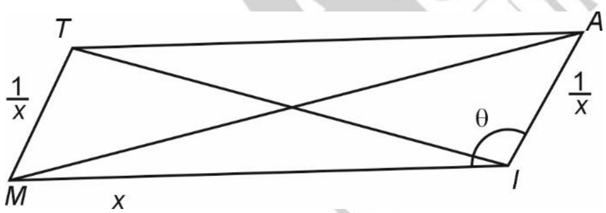
$= 8 + 14$

$= 22$

18. MTAI is a parallelogram of area $\frac{40}{41}$ square units such that $MI = 1/MT$. If d is the least possible length of the diagonal $MA$, and $d^2 = \frac{a}{b}$, where $a$, $b$ are positive integers with $gcd(a, b) = 1$, find $|a - b|$

**Answer (23)**

**Sol.** Let $x \geq 1$




Area of $MTAI = 2 \times \left(\frac{1}{2} x \times \frac{1}{x} \times \sin\theta\right) = \frac{40}{41}$

$\Rightarrow \sin\theta = \frac{40}{41}$ and $\sin(\pi - \theta) = \frac{40}{41}$

$\Rightarrow \cos\theta = \frac{x^2 + \frac{1}{x} - MA^2}{2 \times x \times \frac{1}{x}} = \pm\sqrt{1 - \left(\frac{40}{41}\right)^2} = \pm\frac{9}{41}$

$\Rightarrow\ x^2 + \frac{1}{x^2} - MA^2 = \pm\frac{18}{41}$

For $MA$ to be least $d$,

$x^2 + \frac{1}{x^2} - MA^2 = \frac{18}{41}$

$\Rightarrow\quad d^2 = x^2 + \frac{1}{x^2} - \frac{18}{41}$

Using $A.M. \ge G.M \qquad x^2 + \frac{1}{x^2} \ge 2$

$\Rightarrow d^2 \ge 2 - \frac{18}{41} = \frac{64}{41}$

$\Rightarrow$ minimum possible length of $MA^2 = \frac{64}{41} = \frac{a}{b}$

$d^2 = \frac{64}{41} = \frac{a}{b} \Rightarrow |a - b| = |64 - 41| = 23$

19. Let $f$ be the function defined by $f(n)$ = remainder when $n^n$ is divided by 7, for all positive integers $n$. Find the smallest positive integer $T$ such that $f(n + T) = f(n)$ for all positive integers $n$.

**Answer (42)**

**Sol.** $f(n) = r_n$

$\Rightarrow n^n = r_n \bmod (7)$

$(n+T)^{n+T} = n^n = r_n \bmod(7)\ \forall\ n \in I^+$

Let $n = 7$

$\Rightarrow\quad (7+T)^7 \,.\, (7+T)^T = 7^7 = 0 \quad (\text{mod } 7)$

$\Rightarrow\quad T^{7+T} \equiv 0 \bmod(7)$

$\Rightarrow\quad T \equiv 0 \bmod(7)$

Now, Let $7 \nmid n,\ T = 7k$

$\Rightarrow (n+7k)^{7k+n} \equiv n^n\,(\text{mod} 7)$

$\Rightarrow n^n \,.\, n^{7k} \equiv n^n\,(\text{mod} 7)$

Since $7 \nmid n \Rightarrow n^{7k} \equiv 1(\text{mod} 7)$

Since $7 \nmid n \Rightarrow n^6 \equiv 1 \pmod 7$

$\Rightarrow$ 6 is ord (7) $\Rightarrow$ $6 | 7k$

For minimum value of $7k$

For which $6 \mid 7k \Rightarrow k = 6$

$\Rightarrow T = 7k = 42$

20. Let $N$ be the number of nine-digit integers that can be obtained by permuting the digits of 223334444 and which have at least one 3 to the right of the right-most occurrence of 4. What is the remainder when $N$ is divided by 100?

**Answer (40)**

**Sol.** Required ways = Total – all 3's are to the left of rightmost 4

Total = $\dfrac{9!}{2!3!4!} = 1260$

All 3's are to the left of rightmost 4

Let the rightmost 4 is the position '$r$'

$r$ must be greater than or equal to 7

Case-I: $r = 7$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{3\,3's,\ 3\,4's}\ \underline{4}\,\underline{2}\ \underline{2}$$

${}^6C_3 \times 1 = 20$

Case-II: $r = 8$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{3\,3's,\ 3\,4's,\ 1\,2's}\ \underline{4}\ \_$$

$\dfrac{7!}{3!3!} = 140$

Case-III: $r = 9$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{3\,3's,\ 3\,4's,\ 2\,2's}\ \_\ \underline{4}$$

$\dfrac{8!}{3!3!2!} = 560$

Required = 1260 – (20 + 40 + 560) = 540

$\therefore$ answer 40

21. There are $m$ blue marbles and n red marbles on a table. Armaan and Babita play a game by taking turns. In each turn the player has to pick a marble of the colour of his/her choice. Armaan starts first, and the player who picks the last red marble wins. For how many choices of $(m, n)$ with $1 \le m, n \le 11$ can Armaan force a win?

**Answer (66)**

**Sol.** Let $(r, b)$ = (number of red, number of blue marbles) at any stage

If a layer removes red marbles

$(r, b) \to (r-1, b)$

If a player removes blue marble

$(r, b) \to (r, b-1)$

The player who removes the last red marble wins i.e. player who makes $r = 0$ wins

For $r = 1$

$(1, b)$ are all winning positions for Armaan $\to$ 11 cases

For $r = 2$

(2, 1) is winning $\left(\overset{A}{(2,1)} \to \overset{B}{(2,0)} \to \overset{A}{(1,0)} \to (0,0)\right)$

$\therefore$ (2, 2) is losing

(2, 3) is winning $\left(\overset{A}{(2,3)} \to \overset{B}{(2,2)} \to \overset{A}{(2,1)}\right)$

$\therefore$ for $(2, b)$ is winning when $b$ is odd number

$\therefore$ 6 cases

For r = 3

$\overset{A}{(3,1)} \to \overset{B}{(2,1)} \to$ losing

$\underset{B}{(3,0)} \to \underset{A}{(2,0)} \to \underset{B}{(1,0)} \to (0,0)$ losing

$\overset{A}{(3,2)} \to \overset{B}{(3,1)} \to$ (winning)

$\overset{A}{(3,3)} \to \overset{B}{(3,2)} \to$ losing

$\therefore$ $(3, b)$ is winning when b is even number

$\therefore$ 5 cases

Similarly, for $r \ge 2$

(1) when $r$ is even: $b$ should be odd

(2) when $r$ is odd: $b$ should be even

(3) for $r = 1$, 11 cases are there

$\therefore$ total ordered pairs of $(m, n)$

$= 5 \times 6 + 5 \times 5 + 11 = 66$ cases

22. For some real numbers m, n and positive integer a, the lost $(a + 1)n^2$, $m^2$, $a(n + 1)^2$ consists of three consecutive integers written in increasing order. What is the largest possible value of $m^2$?

**Answer (49)**

**Sol.** Let $K = (a + 1)n^2$ … (i)

$K + 1 = m^2$ … (ii)

$K + 2 = a(n + 1)^2$ … (iii)

Where $K \in I$

From (i) and (iii)

$(a + 1)n^2 + 2 = a(n + 1)^2$

$an^2 + n^2 + 2 = an^2 + a + 2an$

$n^2 + 2 - a - 2an = 0$

$\Rightarrow n^2 - 2an - (a - 2) = 0$

Must have integral roots as $n \in I$)

$\Rightarrow 4a^2 + 4(a - 2)$ is a perfect square

$\Rightarrow a^2 + a - 2$ is a perfect square

$\Rightarrow a^2 + a - 2 = l^2$

$\Rightarrow \left(a+\frac{1}{2}\right)^2 = l^2 + 2 + \frac{1}{4}$

$= (2a + 1)^2 = 4l^2 + 9$

$\Rightarrow (2a + 1)^2 - 4l^2 = 9$

$(2a + 1 - 2l)(2a + 1 + 2l) = 9$

$(a, l) = (2, 2)$ or $(-3, -2) = 2a - 1 - 2l = \pm 1$ and $2a - 1 + 2l = \pm 9$

Or $(a, l) = (2, -2)$ or $(-3, 2) \leftarrow 2a - 1 - 2l = \pm 9$ and $2a - 1 + 2l = \pm 1$

Or $(a, l) = (1, 0)$ or $[-2, 0) \leftarrow 2a - 1 - 2i = \pm 3$ and $2a - 1 + 2l = \pm 3$

$\Rightarrow a = 2$ or $-3$ or $1$ or $-2$

$(a + 1)n^2 + 2 = a(n + 1)^2$

For $a = 3$

$4n^2 + 2 = 3(n + 1)^2$

No integral values of $n$

For $a = 2$

$3n^2 + 2 = 2(n + 1)^2$

$\Rightarrow n = 0$ or $4$

For $a = -2$

$n = -2$

Now $m_{\max}^2 = (a+1)n^2 + 1$

$= 3 \times 16 + 1$

$= 49$

23. Let $P(x) = x^{2025}$, $Q(x) = x^4 + x^3 + 2x^2 + x + 1$. Let $R(x)$ be the polynomial remainder when the polynomial $P(x)$ is divided by the polynomial $Q(x)$. Find $R(3)$.

**Answer (53)**

**Sol.** Using polynomial Euclidian algorithm

$P(x) = f(x)\, Q(x) + R(x)$ for some $f(x)$ such that degree $(R(x))$ < degree $(Q(x)) = 4$

$\Rightarrow$ let $R(x) = ax^3 + bx^2 + cx + d$

$x^{2025} = f(x)\, [(x^2 + 1)\,(x^2 + x + 1)] + ax^3 + bx^2 + cx + d$

put $x = -i$, i, $\omega$, $\omega^2$ to find $a, b, c, d$

put $x = i$

$i = -ai - b + ci + d \Rightarrow b = d, -a + c = 1$

put $x = -i$

$-i = ai - b - ci + d \Rightarrow b = d,\ a - c = -1$

Put $x = \omega$

$1 = a + b\omega^2 + c\omega + d$

$(a + d) + b(-1 - \omega) + c\omega = 1$

$\Rightarrow a + d - b = 1,\ \omega(-b + c) = 0$

$\Rightarrow b = c \Rightarrow a = 1,\ b = c = d = 2$

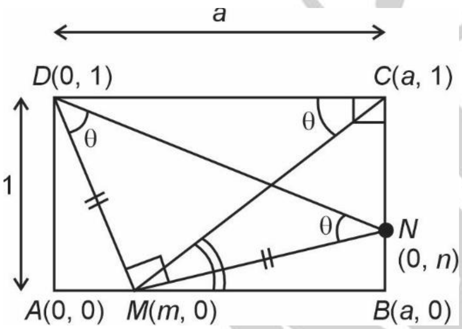
$R(x) = x^3 + 2x^2 + 2x + 2$

$\Rightarrow R(3) = 27 + 18 + 6 + 2 = 53$

24. Let $ABCD$ be a rectangle and let $m$, $N$ be points lying on sides $AB$ and $BC$, respectively. Assume that $MC = CD$ and $MD = MN$, and that points $C$, $D$, $M$, $N$ lie on a circle. If $\left(\frac{AB}{BC}\right)^2 = \frac{m}{n}$ where $m$ and $n$ are positive integers with $\gcd(m, n) = 1$, what is the value of $m + n$?

**Answer (03)**

**Sol.**




Let $\frac{AB}{BC} = a$

$\angle DMN = 90^\circ$, as $\angle NCD = 90^\circ$

As $MNCD$ lie on circle

$\Rightarrow \theta = 45^\circ$

$\Rightarrow \angle MCD = 45^\circ$, on

Same chord $DM$

$\angle CMB = \theta$, transversal line $DC \parallel MB$

$\Rightarrow \quad BM = CB(\theta = 45^\circ)$

$\Rightarrow \quad a - m = 1$

Equation of circle

$(x - 0)(x - a) + (y - 1)(y - n) = 0$

$MC = a$

$\Rightarrow \quad (m - a)^2 + 1 = a^2$

$\Rightarrow a^2 = 2 = \dfrac{m}{h}$

$\Rightarrow m + n = 3$

25. For how many numbers $n$ in the set {1, 2, 3, ……, 37} can we split the $2n$ numbers 1, 2, ……, $2n$ into $n$ pairs $\{a_i, b_i\}$, $1 \le i \le n$, such that b $\prod_{i=1}^{n}(a_i + b_i)$ is a square?

**Answer (36)**

**Sol.** Some rough work:

$n = 1$

$\Rightarrow \quad 2n \in \{1, 2\}$ let $P = \prod_{i=1}^{n}(a_i + b_i)$

for $n = 1$, $P = (1 + 2) = 3$ not a perfect square

for $n = 2$

$\Rightarrow \quad 2n \in \{1, 2, 3, 4\}$

pairing can be (1 + 4), (2 + 3)

$\Rightarrow P = (1 + 4)(2 + 3) = 25 \Rightarrow n = 2$ work observation for $n \in$ even

The pairing is obvious:

$(1, 2n), (2, 2n - 1), (3, 2n - 2) \ldots (n, n + 1)$

$\Rightarrow P = (2n + 1)^n$, since $n \in$ even

$P$ is a perfect square

$n = 3$

$\Rightarrow \quad 2n \in \{1, 2, 3, 4, 5, 6\}$

$(1 + 5)(2 + 4)(3 + 6) = 6 \times 6 \times 9 \Rightarrow$ perfect square for $n \in$ odd $\ge 3$

Pairing can be now think of first 6 as shown above and rest can pair as sown with even cases

$\Rightarrow n \in$ odd $\ge 3$

$(1 + 5)(2 + 4)(3 + 6)(7 + 2n)(8, 2n - 1) \ldots\ldots (6 + j + (2n - j + 1) \;\; \forall j = 1, \ldots\ldots n - 3$

$\Rightarrow (6 \times 6 \times 9)[2n + 7]^{n-3} \Rightarrow$ again perfect square

$\Rightarrow n = 2, 3, \ldots.., 37$ works

$\Rightarrow$ 36 values of $n$ works

26. A regular polygon with $n \geq 5$ vertices is said to be colourful if it possible to colour the vertices using at most 6 colours such that each vertex is coloured with exactly one colour, and such that any 5 consecutive vertices have different colours. Find the largest number $n$ for which a regular polygon with $n$ vertices is not colourful?

**Answer (19)**

**Sol.** Let the colors be *a, b, c, d, e, f*

Denote by $S_1$, the sequence *a, b, c, d, e* and the sequence $S_2$ be *a, b, c, d, e, f*. If $n$ is representable in the form $5x + 6y$, for $x, y \geq 0$, then $x$ satisfies the conditions of the problem.

We may place $x$-consecutive $S_1$ sequences followed by $y$-consecutive $S_2$ sequences, around the polygon.

Let $y = 0, 1, 2, 3, 4$, we find that $n$ may equal any number of the form $5x$, $5x + 6$, $5x + 12$, $5x + 18$, $5x + 24$. The only numbers greater than 4 not of this form are: 7, 8, 9, 13, 14 and 19.

Now, lets show that none of these numbers have the required property.

Assume for a contradiction that a color exist for $n$ equal to one of 7, 8, 9, 13, 14 & 19

$\therefore$ There exist a number $k$ such that

$6k < n < 6(k + 1)$. By P.H.P, atleast $K + 1$ vertices of the $n$-gon have the same color.

Between any two of these vertices are atleast 4others because any 5 consecutive vertices have different colors.

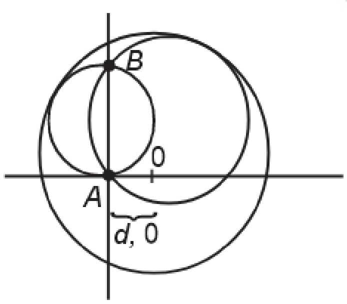
Hence, there are atleast $5(K + 1)$ vertices and $n \geq 5(K+1)$.

However this inequality fails for $n = 7, 8, 9, 13, 14, 19$ a contradiction.

27. Let $S$ be a circle of radius 10 with centre $O$. Suppose $S_1$ and $S_2$ are two circles which touch $S$ internally and intersect each other at two distinct points $A$ and $B$. If $\angle OAB = 90^\circ$ what is the sum of the radii of $S_1$ and $S_2$?

**Answer (10)**

**Sol.**




radius circle with centre $O$ is $r_1$

radius circle with centre $O_2$ is $r_1$

$OA \perp AB$

$\Rightarrow O_1O_2 \parallel AO$

$O0_1 = 10 - r_1$

$O0_2 = 10 - r_2$

$A(0, 0)$ $O(d, 0)$

$\Rightarrow O_1(x, h)$ $O_2(x_2, h)$

$A$ lies on both circles

$$x_1^2 + h^2 = r_1^2$$

$$x_2^2 + h^2 = r_2^2$$

Also

$$(d - x_1)^2 + h^2 = (10 - r_1)^2 \text{....(iii)}$$

$$\underline{(d - x_2)^2 + h^2 = (10 - r_2)^2 \text{....(iv)}}$$

$$2dx_1 = d^2 - 100 + 20r_1$$

Similarly,

$$2dx_2 = d^2 - 100 + 20r_2$$

$$\Rightarrow x_2 - x_1 = \frac{10(r_2 - r_1)}{d}$$

$$x_2^2 - x_1^2 = r_2^2 - r_1^2$$

$$\frac{10}{d}(x_2 + x_1) = r_1 + r_2 \text{...(v)}$$

(iii) + (iv)

$$\Rightarrow (x_1 + x_2) = \frac{d^2 - 100}{d} + \frac{10(r_1 + r_2)}{d} \quad \ldots \text{(vi)}$$

By (v) & (vi)

$(d^2 – 100) (r_1 + r_2 – 10) = 0$

$\Rightarrow r_1 + r_2 = 10$

28. Consider a sequence of real numbers of finite length. Consecutive four term averages of this sequence are strictly increasing, but consecutive seven term averages are strictly decreasing. What is the maximum possible length of such a sequence?

**Answer (10)**

**Sol.** Consider a sequence of 11 real numbers, $<a_i> \forall i = 1, \ldots 11$

Using the increasing sequence of average of 4 consecutive we get

$$\frac{a_i + a_{i+1} + a_{i+2} + a_{i+3}}{4} < \frac{a_{i+1} + a_{i+2} + a_{i+3} + a_{i+4}}{4}$$

$\Rightarrow \quad a_i < a_i + 4$

Using the decreasing, sequence of average of 7 consecutive we get

$$\frac{a_i + a_{i+1} + \ldots + a_{i+6}}{4} < \frac{a_{i+1} + a_{i+2} \ldots + a_{i+7}}{4}$$

$\Rightarrow \quad a_i < a_i + 7$

$$\Rightarrow \left.\begin{matrix} a_1 < a_5 \\ a_2 < a_6 \\ a_3 < a_7 \\ a_4 < a_8 \\ a_5 < a_9 \\ a_6 < a_{10} \\ a_7 < a_{11} \end{matrix}\right\} \text{from 4 consecutive average } (C_1)$$

$$\left.\begin{matrix} a_1 > a_8 \\ a_2 > a_9 \\ a_3 > a_{10} \\ a_4 > a_{11} \end{matrix}\right\} \text{from 7 consecutive average } (C_2)$$

$\Rightarrow \quad$ Adding $C_1$

$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 < a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11}$

$\Rightarrow$ $a_1 + a_2 + a_3 + a_4 < a_8 + a_9 + a_{10} + a_{11}$

$\Rightarrow$ Contradicting C2

$\Rightarrow$ $n \leq 10$

$\Rightarrow$ For $n = 10$

For $n = 10$ sequence can be

3, 6, 9, 1, 4, 7, 10, 2, 5, 8

$\therefore$ Largest $n$ is 10.

29. Find the number of ordered triples ($a, b, c$) of positive integers such that $1 \leq a, b, c \leq 50$ which satisfy the relation

$$\frac{lcm(a,c)+lcm(b,c)}{a+b}=\frac{26c}{27}.$$

Here, by lcm($x, y$) we mean the LCM, that is least common multiple of $x$ and $y$.

**Answer (40)**

**Sol.** Let $gcd(a, c) = da$

$gcd(b, c) = db$

$$\frac{\frac{ac}{\gcd(a,c)}+\frac{bc}{\gcd(b,c)}}{a+b}=\frac{26}{27}c$$

$\Rightarrow$ Let $K_a=\frac{a}{d_a}, K_b=\frac{b}{d_b}$

$$\Rightarrow \frac{K_a+K_b}{a+b}=\frac{26}{27}$$

$\Rightarrow$ let $a + b = 27$ m

$K_a + K_b = 26$m

Since $gcd(26, 27) = 1$

Since $a + b \in \{2, \ldots, 100\} \Rightarrow m = 1, 2$ or 3

$\Rightarrow a + b - K_a - K_b = m$

$$\Rightarrow a\left(1-\frac{1}{d_a}\right)+b\left(1-\frac{1}{d_b}\right)=m$$

If $d_a \geq 2, d_b \geq 2 \Rightarrow a\left(1-\frac{1}{d_a}\right)+b\left(1-\frac{1}{d_b}\right) \geq \frac{a+b}{2}=m$

$m \geq \frac{27\text{ m}}{2}$ absurd

$\Rightarrow$ at least one of $d_a$ and $d_b = 1$

Case (A) $d_a = 1$

$$\Rightarrow b\left(1-\frac{1}{d_b}\right) = m \Rightarrow b-m = \frac{b}{d_b}$$

$$\Rightarrow d_b = \frac{b}{b-m} \Rightarrow b-m \mid b \Rightarrow b-m \mid (b-m+m)$$

$\Rightarrow b - m \mid m$

$\Rightarrow$ A. 1 $m = 1$, $b - 1|1 \Rightarrow b - 1 = 1 \Rightarrow b = 2$

$\Rightarrow a + b = 27 \Rightarrow a = 25$

$d_a = 1 \Rightarrow gcd(25, c) = 1$ $\phi(25) = 25\left(1-\frac{1}{5}\right) = 20$

(A. 2) $m = 2 \Rightarrow b - 2 \mid 2$

$\Rightarrow b - 2 = -2, -1, 1, 2$

$b = 0, 1, 3, 4$

$\Rightarrow a + b = 54 \Rightarrow a = 54, 53, 51, 50$

$\Rightarrow b = 4, a = 50$

Since $2|a$ & $2|b \Rightarrow 2\nmid c \Rightarrow \gcd(50, c) = 1, \gcd(4, c) = \frac{4}{4-2} = 2$

$\Rightarrow 2|c$, contradiction

(A. 3) $m = 3$, $b - 3|3$

$b - 3 = -3, -1, 1, 3$

$a + b = 81 \Rightarrow (a, b) = \{(81, 0), (79, 2), (77, 4), (75, 6)\}$

no such values of a in [1, 50]

$\Rightarrow$ 20 cases

(B) $d_b = 1$ gcd$(b, c) = 1$

$\Rightarrow a - m \mid m$

(B. 1) $m = 1 \Rightarrow a = 2$, $b = 25$ again

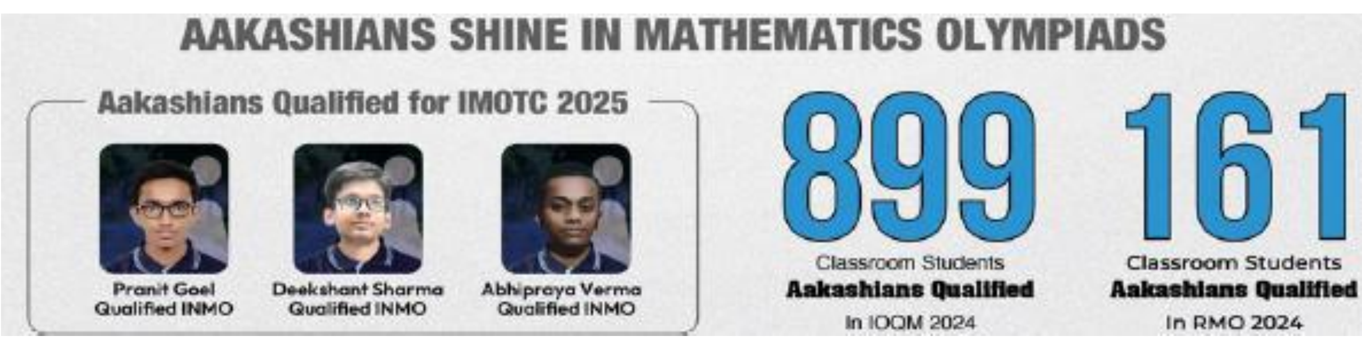
gcd$(25, c) = 1$

gcd$(2, c) = 1$ 20 pair

Similarly, $m = 2$ and $m = 3$ will not work

(20 + 20) ordered pairs $\Rightarrow$ 40





30. Assume $a$ is a positive integer which is not a perfect square. Let $x$, $y$ be non-negative integers such that $\sqrt{x-\sqrt{x+a}} = \sqrt{a} - y$. What is the largest possible value of a such that a < 100?

**Answer (91)**

**Sol.** $\sqrt{x-\sqrt{x+a}} = \sqrt{a} - y$

$\Rightarrow \sqrt{a} - y \geq 0 \Rightarrow y \leq \sqrt{a}$

$x - \sqrt{x+a} = a + y^2 - 2\sqrt{a}y$

$\sqrt{x+a} = \underbrace{x - a - y^2 + 2y\sqrt{a}}_{Int + K\sqrt{a}}$

$\Rightarrow y = 0$

$\Rightarrow \sqrt{x} + a = x - a$

Let $x + a = t^2$

$t = t^2 - 2a$

$\Rightarrow a = \dfrac{t^2 - t}{2}$

Now $a < 100$

$\Rightarrow t^2 - t < 200$

$t \in (-13.65, 14.65)$

So, for mas value of a $t = 14$

$\Rightarrow a = 91$

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