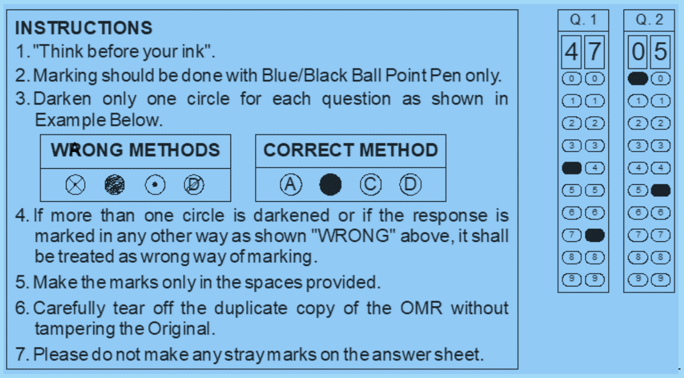
Answers & Solutions for IOQM – 2025-26 (28 Sept 2025)

INSTRUCTIONS TO CANDIDATES

1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen.** Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The registration number and date of birth will be your login credentials for accessing your score.
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two-digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 & 30 carry 5 marks each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the Candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

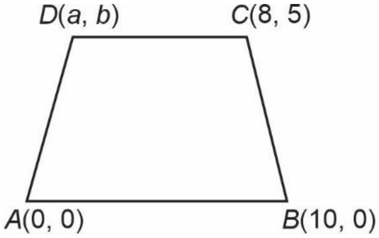
Note:

1. $\gcd\left(a, \sqrt{b}\right)$ denotes the greatest common divisor of integers *a* and *b*.
2. For a positive real number *m*, $\sqrt{m}$ denotes the positive square root of *m*. For example, $\sqrt{4} = +2$.
3. Unless otherwise stated all numbers are written in decimal notation.

1. Let *ABCD* be a quadrilateral in the *xy*-plane with *AB* parallel to *CD* and *AD* = *BC*. Suppose *A* = (0, 0), *B* = (10, 0), *C* = (8, 5) and *D* = (*a*, *b*). Determine the value of a^2b

Answer (20)

Sol.



$m_{AB} = m_{CD}$

$0 = \dfrac{5-b}{8-a}$

$\Rightarrow \quad b = 5$

$AD^2 = BC^2$

$a^2 + b^2 = 4 + 25$

$\Rightarrow \quad a^2 = 4$

$\therefore \quad a^2b = 20$

2. A function is defined on the set of positive integers such that if n is an odd integer, $f(n) = n - 1$ and if n is an even integer, $f(n) = n^2 - 1$. Determine the sum of all possible values of n such that $f(f(n)) = 99$.

Answer (11)

Sol. $f(n) = \begin{cases} n-1 & n \in \text{odd} \\ n^2 - 1 & n \in \text{even} \end{cases}$

$f(f(n)) = 99$

$n \in$ odd

$f(f(n)) = f(n-1) = (n-1)^2 - 1 = 99$

$= (n-1)^2 = 100$

$\Rightarrow \quad n - 1 = \pm 10$

$\Rightarrow \quad n = 11$ or -9

$\because \quad n \in I^+$

$\Rightarrow \quad n = 11$

3. Find the number of positive integers n less than or equal to 100 such that n is not divisible by any prime number other than 2 or 3.

Answer (20)

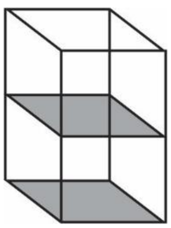
Sol. Numbers will be the form of 2^a3^b ($a, b \geq 0$)

$\Rightarrow \quad 2^a.3^b \le 100$

$$\left.\begin{array}{l} b=0 \Rightarrow 2^a \le 100 \Rightarrow N = 1,2,4,8,16,32,64 \\ b=1 \Rightarrow 3.2^a \le 100 \Rightarrow N = 3,16,12,24,48,72,96 \\ b=2 \Rightarrow 2^a \le 100 \Rightarrow N = 9,18,36,54 \\ b=3 \Rightarrow 37.2^a \le 100 \Rightarrow N = 27 \\ b=4 \Rightarrow 81.2^a \le 100 \Rightarrow N = 81 \end{array}\right\} \text{20 numbers}$$

4. The six faces of a cubical die are numbered with 2^0, 2^1, 2^2, 2^3, 2^4, 2^5 in such a way that the product of the numbers on any pair of opposite faces is 2^5. The such dice are stacked on top of another. If N is the greatest possible sum of the 9 visible numbers (for all such arrangements of dice), find the sum of the squares of the digits of N

Answer (11)



Sol.

Hidden number be $2^0 + 2^a + 2^{5-a}$

Total = $(2^0 + 2^1 + \ldots + 2^5) \times 2$

Hidden number should be minimum

$\Rightarrow \quad a = 3$

$[\because \quad 2^2 + 2^3 < 2^0 + 2^5]$

$\because \quad$ Hidden number = $2^0 + 3^2 + 2^2$

= 13

Sum of numbers = $(2^6 - 1) \times 2 - 13$

= 113

= $1^2 + 1^2 + 32$

= 11

5. Let N be the coefficient of x^{2025} in the expansion of $(x + 1)(x^2 + 3)(x^4 + 5)(x^8 + 7) \ldots (x^{1024} + 21)$.

What is the remainder when N is divided by 100?

Answer (35)

Sol. Coefficient of x^{2025} in

$(x + 1)(x^2 + 3)(x^4 + 5)(x^4 + 5)(x^8 + 7) \dots (x^{1024} + 21)$

$\Rightarrow$ coefficient of x^{24} in $(x^2 + 3)(x^4 + 5)(x^8 + 7) \dots (x^{1024} + 21)$

$2025 = 1024 + 512 + 256 + 128 + 64 + 32 + 8 + 1.$

$\therefore$ Coefficient $= N = 3 \times 5 \times 9 = 135$

Remainder when N is divided by 100 is 35

6. Find the largest integer n such that a square of side length n is contained in a circular disc of area 1000

Answer (25)

Sol. Area = 1000

$\Rightarrow \quad \pi r^2 = 1000$

$\Rightarrow \quad r = \sqrt{\dfrac{1000}{\pi}}$

Square will fit if diagonal of square at most circle digit

$\Rightarrow \quad n\sqrt{2} \le 2r$

$\Rightarrow \quad n \le r\sqrt{2}$

$\Rightarrow \quad n \le \sqrt{\dfrac{2000}{\pi}} \approx 25.23$

Largest $n = 25$

7. The sum of four distinct prime numbers is 240. If none of the four primes is greater than 70, what is the smallest of the four number?

Answer (53)

Sol. Let primes be P_1, P_2, P_3, P_4

WLOG

$P_1 < P_2 < P_3 < P_4 \le 70$

$P_1 + P_2 + P_3 + P_4 = 240$

$\Rightarrow$ Largest prime be ≤ 70 is 67

$\Rightarrow \quad P_4 = 67$

Now, P_3 has to be < 67

$\Rightarrow \quad P_3 = 61$

Similarly, $P_2 = 59$

Now $P_1 = 53$

$\Rightarrow$ Smallest prime = 53

8. Consider a 2 × 3 rectangle made of 6 unit squares. In how many ways can we fill up the six cells using the numbers 1, 2, 3, 4, 5, 6 one in each cell, such that any two numbers in adjacent cells (that is in cells that share a common side) are coprime to each other?

Answer (16)

Sol.

	X	6
		Y

6 is coprime with 1 and 5

$\therefore$ 6 has to come in corner

$\therefore$ 2, 3, 4 cannot come at place *X* and *Y*

$\therefore$ *X*, *Y* should be fill with 1 and 5

a	1	6
3	*b*	5

2 and 4 must come at *a* and *b*

$\therefore$ Total ways ${}^4C_1.2!.2! = 16$

9. Find the largest positive integer n for which the inequality $\sum_{k=1}^{2n}(-1)^k k^2 < 100$ holds

Answer (6)

Sol. $\sum_{k=1}^{2n}(-1)^k k^2 < 100$

$$S = -1+4-9+16... = \sum_{j=1}^{n}(2j)^2 - (2j-1)^2$$

$$= \sum_{j=1}^{n}4j^2 - 4j^2 - 1 + 4i = \sum_{j=1}^{n}4j - 1$$

$= 2n(n + 1) – n$

$= 2n^2 + n$

$\Rightarrow$ $2n^2 + n < 100$

$2n^2 + n – 100 < 0$

$\Rightarrow$ $n \in (–7.32, 6.82)$

$\Rightarrow$ $n = 6$

10. How many positive integers $n \leq 100$ are divisible by all positive integers i such that $i^3 \leq n$?

Answer (26)

Sol. $n \leq 100$, divisible by all i such that $i^3 \leq n$

$\Rightarrow$ $n = 1$

$\Rightarrow$ $i^3 \leq 1$

$\Rightarrow$ $i = 1$

$\Rightarrow$ $n = 1$ works (1 value)

For all $n \in \{2, \ldots, 7\}$

$i^3 \leq n$

$\Rightarrow$ $i = 1$

$\Rightarrow$ all works (6 values)

For $n \in \{8, \ldots, 26\}$

All $i \mid n$, $8 \leq i^3 \leq 26$

$\Rightarrow$ $i = 1, 2, 2! \mid n$

$\Rightarrow$ n must be even

$\Rightarrow$ $\{8, 10, \ldots, 26\}$ (10 values)

For $n \in \{27, \ldots, 63\}$

All $i \mid n$, $i^3 \leq 63$

$\Rightarrow$ $i = 1, 2, 3$, lcm $(1, 2, 3) = 3!$

$\Rightarrow$ $3! \mid n$

$\Rightarrow$ n is multiple of 6

$\Rightarrow$ $n \in \{30, 36, \ldots 60\}$ (6 values)

For $n \in \{64, \ldots, 100\}$

All $i \mid n$, $i^3 \leq 100$

$\Rightarrow$ $i = 1, 2, 3, 4$, lcm of $(1, 2, 3, 4) \mid$ n

$\Rightarrow$ $12 \mid n$

$\Rightarrow$ n is multiple of 12

$\Rightarrow$ $\{72, 84, 96\} \Rightarrow$ (2 values)

$\Rightarrow$ Total 26 values

11. Let m be a positive integer satisfying the equation $5(2m+1)(2m+3)(2m+5) = \overline{ababab}$

Where a and b represent different digits and $\overline{ababab}$ is a six digit number. What is the value of $m + a + b$?

Answer (24)

Sol. $\overline{ababab} = b\left(10^0 + 10^2 + 10^4\right) + a\left(10^1 + 10^3 + 10^5\right)$

$= (10101)(10a + b)$

$= 3 \times 3367\left(\overline{ab}\right)$

$5(2m + 1)(2m + 3)(2m + 5) = 3\cdot 7\cdot 13\cdot 37\left(\overline{ab}\right) \quad (\because\ b = 5)$

$= 3.7.13.37(10a + 5)$

$5(2m + 1)(2m + 3)(2m + 5) = 3.7.13.37.5(2a + 1)$

$\Rightarrow \quad 5(2m + 1)(2m + 3)(2m + 5) = 3.7.13.37\,(2a + 1)$

$\therefore \quad 2m + 1 = 7 \times 5 = 35$

$2m + 3 = 37$

$2m + 5 = 39$

$\therefore \quad m = 17$

$2a + 1 = 5$

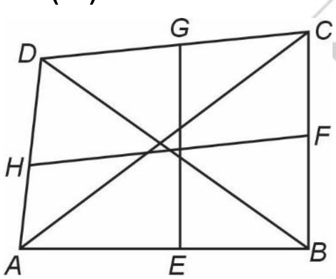
$a = 2$

$b = 5$

$\therefore \quad m + a + b = 5 + 2 + 17 = 24$

12. In a convex quadrilateral $ABCD$, the lengths of the diagonals are 12 and 16 and the line segments joining the midpoints of the opposite sides area of equal length. What is the maximum possible area of the quadrilateral $ABCD$?

Answer (96)

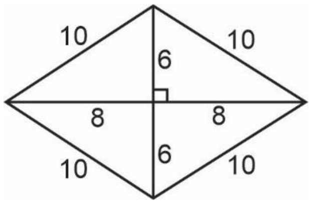


Sol.

$HF = GE$

$\text{Area} = \frac{1}{2} d_1 d_2 \sin\theta$

$\text{Area}_{max} \Rightarrow \theta = 90°$



$$\text{Area} = \left(\frac{1}{2} \times 6 \times 8\right) \times 4$$

= 96 sq. unit

13. Find the numbers of ordered pairs (m, n) where m and n are positive integers less than or equal to 20000 such that $m^2 + n^4$ is a power of 2

Answer (8)

Sol. $m, n \in N$ and $n, m \leq 20000$

Such that $m^2 + n^4 = 2^c$, let c be such power $c \in N$

$\Rightarrow$ Clearly (1, 1) satisfies $1^2 + 1^4 = 2^c$

$\Rightarrow$ $c = 1$

If m, n are both odd then $m = 2x + 1$, $n = 2y + 1$

$\Rightarrow$ $(2x + 1)^2 + (2y + 1)^4 = 2^c$

$\Rightarrow$ $(4x^2 + 4x + 1) + (4y^2 + 4y + 1)^2 = 2^c$

$\Rightarrow$ $4(x^2 + x) + 4(4(y^2 + y)^2 + 2(y^2 + 1)) + 2 = 2^c$

If $c \geq 2$

$\Rightarrow$ 4 | 2 hence contradiction

$\Rightarrow$ Only $c = 1$ is possible for (m, n) both odds.

Now, if (m, n) are opposite parity, no solution

If m and n are even

$\Rightarrow$ let $m = 2^x p$

$n = 2^y q$

Where p and q are odd number

$\Rightarrow$ $(2x)^2 p^2 + (2^y q)^4 = 2^c$

$\Rightarrow$ $(2^{2x} p^2 + 2^{4y} q^4) = 2^c$

If $2x > 4y$

$\Rightarrow \quad 2^{4y}\left[\underbrace{2^{2x-4y}.p^2}_{\text{even}}+\underbrace{q^4}_{\text{odd}}\right]=2^c$

That is not possible, similarly, $2x < 4y$ not possible

$\Rightarrow \quad 2x = 4y$

$\Rightarrow \quad m = 2^{2y}p,\ n = 2^y q$

$\Rightarrow \quad 2^{4y}p^2 + 2^{4y}q^4 = 2^c$

$\Rightarrow \quad 2^{4y}(p^2 + q^4) = 2^c$, again$(p, q) \equiv (1, 1)$

$\Rightarrow \quad (4^k, 2^k)$ type will be solution

$\Rightarrow \quad 2^{2k} < 20000$

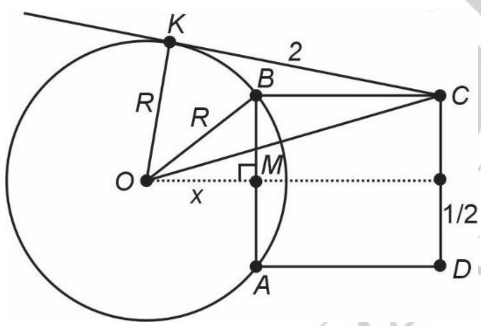
$\Rightarrow \quad 4^k < 20000$

$\Rightarrow \quad k \le 7$

$\Rightarrow \quad 8$

14. The side AB of a square $ABCD$ is 1 and it is also a cord of circle S. the side CD does not intersect S. the length of the tangent CK, drawn from C to S at the point K is 2. If d is the diameter of S, then calculate d^2

Answer (10)



Sol.

$OM \perp AB$ and M is mid-point of AB

$CK \perp OK$

$OC^2 = R^2 + 2^2$

Also, $(OC)^2 = (x+1)^2 + \left(\frac{1}{2}\right)^2$

ΔOBM

$\Rightarrow \quad R^2 = x^2 + \left(\frac{1}{2}\right)^2$

$\Rightarrow \quad (x+1)^2 + \frac{1}{4} = R^2 + 2^2 = x^2 + \frac{1}{4} + 4$

$\Rightarrow \quad x^2 + 2x + 1 = x^2 + 4$

$\Rightarrow \quad x = \frac{3}{2}$

$\Rightarrow \quad R^2 = \frac{9}{4} + \frac{1}{4} = \frac{10}{4}$

$4R^2 = (\text{Diameter})^2 = 10$

15. If a, b, c, d are positive integers such that $17(abcd + ab + ad + cd + 1) = 20(bcd + b + d)$, find $a^2 + b^2 + c^2 + d^2$

Answer (31)

Sol. $17(abcd + ab + ad + cd + 1) = 20(bcd + b + b)$

$17[(ab + 1)(cd + 1) + da] = 20[b(cd + 1) + d]$

$(cd + 1)\,[17ab + 17 - 20b] = 20d - 17ad$

$(cd + 1)[17(ab + 1) - 20b] = d(20 - 17a)$

If $20 - 17a \geq 1$

$\Rightarrow \quad a \leq 1$

$\Rightarrow \quad a = 1$

$\Rightarrow \quad (cd + 1)(17 - 3\text{b}) = 3\text{d}$

$\Rightarrow \quad 17 - 3b \geq 0$

$\Rightarrow \quad b \leq 5$

$b = 1, 2, 3, 4, 5$

For $b = 5$

$(cd + 1)^2 = 3d$

$\Rightarrow \quad d = 2$

$cd + 1 = 3$

$c = 1$

$a = 1$, $b = 5$, $c = 1$

$d = 2$

For $b < 5$

Let $b = 4$

$(cd + 1)^5 = 3d$

$d = 5$

$5c + 1 = 3$ contradict

$\Rightarrow$ $a^2 + b^2 + c^2 + d^2 = 1 + 25 + 1 + 4$

$= 31$

16. Find the number of ordered pairs (m, n) where m and n are positive integers such that $1 \leq m < n \leq 50$ and the product mn is a perfect square

Answer (44)

Sol. $m, n \in N$ such that

$1 \leq m < n \leq 50$, mn is perfect square

Let $n = x^2y$, where y is square free or product of distinct primes

And $m = p^2q$, where q is square free or in another words product of distinct primes

$\Rightarrow$ $mn = (x^2y)(p^2q) \in$ perfect squares

When (yq) is perfect square

$\Rightarrow$ both have same primes as product

$\Rightarrow$ From fundamental theorem of arithmetic y and q have same prime factorization $\Rightarrow y = q$. Let $y = q = k$

$\Rightarrow$ $n = kx^2$

$m = ky^2$ such that $m < n$

$\Rightarrow$ $y < x$

Let $k = 1$

$\Rightarrow$ $x^2 \leq 50$

$\Rightarrow$ $x \in \{1, \ldots, 7\}$

x, y have ${}^7C_2$ choices = 21

let $k = 2$

$\Rightarrow$ $2x^2 \leq 50$

$\Rightarrow$ $x \in \{1, \ldots, 5\}$

x, y have ${}^5C_2$ choices = 10

Let $k = 3$

$\Rightarrow$ $3x^2 \leq 50$

$\Rightarrow$ $x \in \{1, 2, 3, 4, 5\}$

$\Rightarrow$ x, y have ${}^4C_2$ choices = 6

Let $k = 4$

$\Rightarrow$ $4x^2 \leq 50$

$\Rightarrow$ $x \in \{1, 2, 3\}$

$\Rightarrow$ x, y have ${}^3C_2$ choices = 3

Let $k = 5$

$\Rightarrow$ $5x^2 \le 50$

$\Rightarrow$ $x \in \{1, 2, 3\}$ again 3

Let $k = 6$

$\Rightarrow$ $6x^2 \le 50$

$\Rightarrow$ $x \in \{1, 2\} \Rightarrow {}^2C_2 = 1$

Let $k = 7$

$\Rightarrow$ $7x^2 \le 50$

$\Rightarrow$ $x \in \{1, 2\}$

We got same (x, y)

21 + 10 + 6 + 3 + 3 + 1 = 44.

17. If

$$1-\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7}}}}}}=\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{1}{x_3+\cfrac{1}{x_4+\cfrac{1}{x_5+\cfrac{1}{x_6+\cfrac{1}{x_7}}}}}}}$$

Where $x_1, x_2, \ldots, x_7$ are positive integers, find $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

Answer (27)

Sol. Let $A = 2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7}}}}}$, $B = [x_i;....]$

$A = [2; 3, 4, 5, 6, 7]$

$B = [x_1; x_2, x_3, x_4, x_5, x_6, x_7]$

$\Rightarrow$ $1-\frac{1}{A}=\frac{1}{B}$

$\Rightarrow \quad \dfrac{A-1}{A} = \dfrac{1}{B}$

$\Rightarrow \quad B = \dfrac{A}{A-1} = 1 + \dfrac{A}{A-1} - 1$

$= 1 + \dfrac{1}{A-1}$

$A - 1 = [1; 3, 4, 5, 6, 7]$

$$\Rightarrow \quad B = 1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{6 + \cfrac{1}{7}}}}}}$$

$\Rightarrow \quad x_1 = 1,\ x_2 = 1,\ x_i = i\ \forall i \in \{3, 4, 5, 6, 7\}$

$\Rightarrow$ Total sum 27

18. There are 100 cards in a box which are numbered from 1 to 100. While being blindfolded, Mainak is going to draw one or more cards from the box. After that, he will remove his blindfold and multiply together the numbers on these cards. Mainak wants the product of the numbers on the cards drawn to be a multiple of 6. How many cards does he need to draw to make sure that this will happen?

Answer (68)

Sol. Number of numbers from 1 to 100 which are multiple of 2 = 50

Multiple of 3 = 33

Largest set would contain all even + all other odd number which are not divisible by 3

= 50 + (50 – 33)

= 50 +17

= 67

$\therefore$ number of cards required = 67 +1 = 68

19. How many four digit numbers $\overline{abcd}$, with non-zero digits a, b, c, d in base 10, are there such that $a + c = bd$ and $b + d = ac$?

Answer (09)

Sol. $\overline{abcd}$, $a, b, c, d \neq 0$

$a + c = bd$

Clearly, $a + c \leq 18$

$\Rightarrow \quad bd \leq 18$

$\Rightarrow \quad ac \leq 18$

$\Rightarrow \quad abcd \leq (9 \times 9 \times 4) \leq (182)$

Solving for c and d in terms of a, b

We get $c = \dfrac{a+b^2}{ab-1}$, $d = \dfrac{b+a^2}{ab-1}$

Now, lets make cases on a

$a = 1$

$\Rightarrow \quad b - 1 \mid b^2 + 1$

$\Rightarrow \quad b - 1 \mid (b^2 - 1) + 2$

$\Rightarrow \quad b - 1 \mid 2$

$\Rightarrow \quad b - 1 = 1, 2$

$\Rightarrow \quad b = 2, 3$

If $b = 2$

$\Rightarrow \quad c = \dfrac{1+4}{2-1} = S,\ d = \dfrac{2+1}{2-1} = 3$

If $b = 3$

$\Rightarrow \quad c = \dfrac{1+9}{3-1} = S,\ d = \dfrac{3+1}{3-1} = 2$ (two pairs)

Similarly,

$a = 2$

$\Rightarrow \quad 2b - 1 \mid (b^2 + 2)$

$\Rightarrow \quad (2b - 1) \mid 4b^2 + 8$

$\Rightarrow \quad (2b - 1) \mid (2b - 1)^2 + 4b + 7$

$\Rightarrow \quad (2b - 1) \mid 2(2b - 1) + 9$

$\Rightarrow \quad 2b - 1 \mid 9$

$\Rightarrow \quad b \in \{1, 2, 5\}$ (3 pairs)

$a = 3$, $3b - 1 \mid 28$

$\Rightarrow \quad b \in \{1, 5\}$ (2 pairs)

$a = 4, 6, 7, \ldots, 9$ not possible as $a \geq 10$

But by symmetric of (a, b) and (b, d)

$a = 5$

⇒ $5b - 1 \mid 126$

⇒ $b \in \{2, 3\}$ (2 pairs)

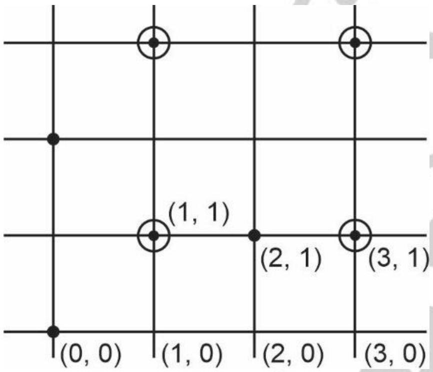
⇒ Total 9 pairs

(5, 3, 1, 2) and their permutation = 2(2!)(2!) and (2, 2, 2, 2, 2) = 9

20. In the plane let the positive end of the x-axis be directed towards East and the positive end of the y-axis be directed towards North. Suppose you are at (0, 0) and you want to go to (7, 12). At every move you are allowed to move unit length towards East or unit length towards North from your current position but you are not allowed to visit any point (h, k) where both h, k are odd. Find the number of such paths n.

Answer (84)

Sol.



Notice that, plane has to take two jumps in same direction otherwise it would jump onto both odd.

⇒ How many, (NN), (EE) block to reach (7, 12) but notice (7, 12) ≡ (6, 12) as there is only 1 way to go (7, 12) which is from (6, 12)

⇒ 3(NN) and 6(EE) blocks to be arranged

⇒ $\dfrac{(3+6)!}{3!6!} = {}^9C_3 = 84$

21. Three girls G_1, G_2, G_3 each read four stories S_1, S_2, S_3, S_4 and discuss which ones they like. No story is liked by all the three. For each of the three pairs of the girls, there is at least one story which is liked by the pair and not liked by the third. Let n be the number of ways in which this is possible. Find the sum of the squares of the digits of n.

Answer (14)

Sol. Let the girls be G_1, G_2, G_3

For each book we can make a set of girls who like it.

The possible set for book is

ϕ, $\{G_1\}$, $\{G_2\}$, $\{G_3\}$, $\{G_1G_2\}$, $\{G_2G_3\}$, $\{G_1G_3\}$… 7 ways.

Assign these sets to 4 stories such that 3 pair $\{G_1, G_2\}$, $\{G_2, G_3\}$, $\{G_1, G_3\}$ each occur at least once

$\therefore$ by principle of inclusion exclusion

$7^4 - {}^3C_1 \cdot 6^4 + {}^3C_2 \cdot 5^4 - 4^4$

$= 132$

$\therefore \; 1^2 + 3^2 + 2^2 = 14$

22. Let $f : R \to R$ be a function satisfying $4f(3 - x) + 3f(x) = x^2$ for any real x. Find the value of $f(27) - f(25)$ to the nearest integer

Answer (08)

Sol. $4f(3 - x) + 3f(x) = x^2$ …(1)

Replace x by $3 - x$

$4f(x) + 3f(3 - x) = (3 - x)^2$ …(2)

4(2) –3 (1) we get

$7f(x) = 4(3 - x)^2 - 3x^2$

$= x^2 - 24x + 36$

$7[f(27) - f(25)] = 27^2 - 25^2 - 24(27 - 25)$

$= 2 \times 52 - 48$

$\Rightarrow \quad f(27) - f(25) = \dfrac{4(26-12)}{7} = 8$

23. If a and b are positive integers satisfying $4^a + 4a^2 + 4 = b^2$, what is the maximum possible value of $a + b$?

Answer (22)

Sol. $4^a + 4a^2 + 4$ is a perfect square for $a \in \mathrm{N}$.

$(2^a)^2 < 4^a + 4a^2 + 4 < (2^a + 1)^2$

Then $4^a + 4a^2 + 4$ will not be possible square

$\Rightarrow \; 4^a + 4a^2 + 4 < 4^a + 2 \cdot 2^a + 1$

$4a^2 + 3 < 2^{a+1}$

Till a = 6, $4(6)^2 + 3 > 2^7$

But if $a > 7 \Rightarrow 4a^2 + 3 < 2^{a+1}$

$\Rightarrow$ no such perfect square exist.

$\Rightarrow$ check only 1,2,3,4,5,6 only.

Taking mod 3

$1^a + a^2 + 1 \equiv b^2 (\text{mod} 3)$

$\Rightarrow a^2 \equiv b^2 + 1 (\text{mod} 3)$

$\Rightarrow b \equiv 0 \ (\text{mod} 3)$ and $2 \mid b$

$\Rightarrow b \equiv 0 \text{ mod} 6$ and a $\not\equiv 0 \text{ mod} 3$

Check b = 6,12,18,24…

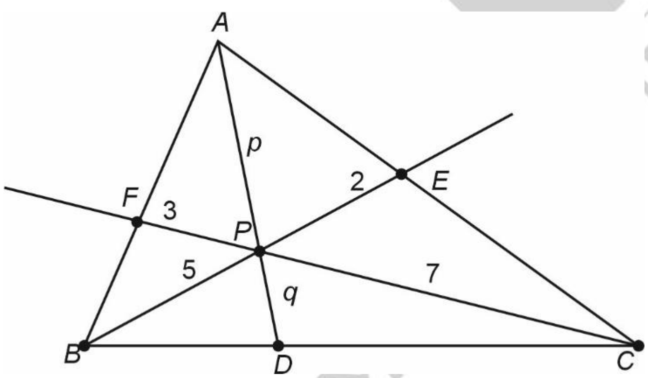
For b = 6, a= 2 $\Rightarrow$ a + b = 8

For b = 18, a = 4 $\Rightarrow$ a + b = 22

24. Let P be a point in the interior of a triangle ABC and let AP, BP, CP meet the sides BC, CA, AB in D, E, F respectively. If $\frac{BP}{PE} = \frac{5}{2}, \frac{CP}{PE} = \frac{7}{3}$ and $\frac{AP}{PD} = \frac{p}{q}$ where p, q are natural numbers and gcd(p, q) = 1, find p + q.

Answer (70)

Sol. Using the theorem that if cevians AD, BE and CF are concurrent at p then



$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1$$

$$\frac{q}{p+q} + \frac{2}{7} + \frac{3}{10} = 1$$

$$\Rightarrow \quad \frac{q}{p+q} = 1 - \left(\frac{20+21}{70}\right) = \frac{70-41}{70} = \frac{29}{70}$$

$$1-\frac{q}{p+q}=1-\left(\frac{70-41}{70}\right)=\frac{41}{70}$$

$$\frac{p}{p+q}=\frac{41}{70}\Rightarrow\frac{p}{q}=\frac{\frac{41}{70}}{\frac{29}{70}}=\frac{41}{29}$$

$\Rightarrow \quad p + q = 70$

25. How many natural numbers $n \le 105$ are there such that $7 \mid 2^n - n^2$?

Answer (30)

Sol. $n \in N$ and $n \le 105$

$7 \mid 2^n - n^2$

$\Rightarrow \quad 2^n \equiv n^2 \pmod 7$

n mod 7	0	1	2	3	4	5	6
n^2 mod 7	0	1	4	2	2	4	1

and $gcd(2, 7) = 1$,

there order of 2 modulo 7 is 3, $ord_7(2) = 3 \Rightarrow 23 \equiv 1 \bmod 7$

$\Rightarrow$ period is 3 or observe from

n mod 7	0	1	2	3	4	5	6
2^n mod 7	1	2	4	1	2	4	1

$\Rightarrow \quad 2^n \equiv 1 \pmod 7$, when $3 \mid n$

$n^2 \equiv 1 \pmod 7$, when $n \equiv 1, 6$ mod

$\Rightarrow \quad n \equiv 0 \bmod 3 \Rightarrow n^2 \equiv 0 \pmod 3$

$n \equiv 1, 6 \bmod 7 \Rightarrow n^2 \equiv 1 \pmod 7$

$n \equiv 6, 15 \bmod 21$

$\Rightarrow$ since $n \le 105$, we get 10 sol

Similarly,

$2^n \equiv 2 \bmod 7 \;\Rightarrow n \equiv 1 \pmod 3$

$n^2 \equiv 2 \bmod 7 \Rightarrow n \equiv 3, 4 (\bmod 7)$

again using C.R.T.

$n \equiv 4, 10 \ (\bmod 21)$

again 10 sol

Finally,

$2^n \equiv 4 \bmod 7 \quad \Rightarrow n \equiv 2 \bmod 3$

$n^2 \equiv 4 \ (\bmod 7) \Rightarrow n \equiv 2, 5 (\bmod 7)$

Using, C.R.T $\Rightarrow n \equiv 2, 5 \bmod 21$

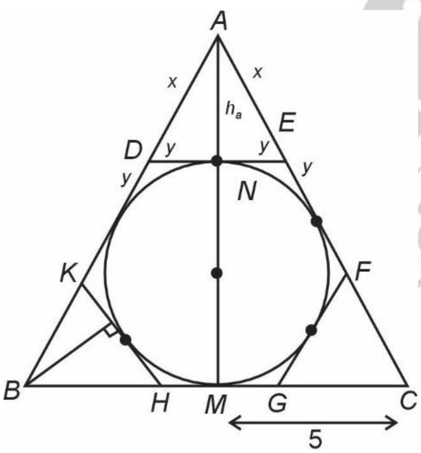
Again 10 sol

$\Rightarrow$ Total 30

26. Let ABC be an isosceles triangle with sides 13, 13 and 10. The tangents to the incircle, drawn parallel to the sides intersect the sides in points D, E, F, G, H, K which form a hexagon. If the area of the hexagon $DEFGHK$ is $m + \frac{n}{l}$, where m, n, l are positive integers with $n < l$ and $\gcd(n, l) = 1$, what is $m + n + l$?

Answer (55)

Sol.



$$AM = \sqrt{13^2 - 5^2} = 12 = 20 + ha$$

$$r = \frac{\Delta}{5} = \frac{\frac{1}{2} \times 10 \times 12}{\left(\frac{13 + 13 + 10}{2}\right)} = \frac{60}{\frac{36}{2}} = \frac{10}{3}$$

$h_a = 12 - \frac{20}{3} = \frac{16}{3}$

$\Delta ADE \sim \Delta ABC$

$\Rightarrow \quad \frac{\Delta ADE}{\Delta ABC} = \frac{h_a^2}{AM^2} = \frac{\left(\frac{16}{3}\right)^2}{12^2}$

$\Rightarrow \quad \Delta(ADE) = \frac{60 \times 16^2}{9 \times 12 \times 12} = \frac{5 \times 16^2}{9 \times 12}$

$\Delta BKH \sim \Delta BAC, KH || AC$

$\Rightarrow \frac{\Delta BKH}{\Delta BAC} = \frac{b_h^2}{(b_h + 2r)^2} \Rightarrow \Delta BHK = \frac{60 \times b_h^2}{\left(b_h + \frac{20}{3}\right)^2}$ ...(i)

Let $KH = b$

$\Delta BKH = \frac{1}{2} b b_h$, also using similar Δ

$\frac{\Delta BKH}{\Delta BAC} = \frac{b^2}{13^2} \Rightarrow \Delta BKH = \frac{60 \times b^2}{169} = \frac{1}{2} b b_h$

$\Rightarrow h_b = \frac{120}{169} b$ ...(ii)

By (i) and (ii) $h_b = \frac{100}{39}$

$\Rightarrow \quad$ Required area is $60 - \left(\frac{2 \times 60 \times \left(\frac{100}{39}\right)^2}{\left(\frac{100}{39} + \frac{20}{7}\right)^2} + \frac{16^2 \times 5}{9 \times 12} \right) = 38 + \frac{8}{9}$

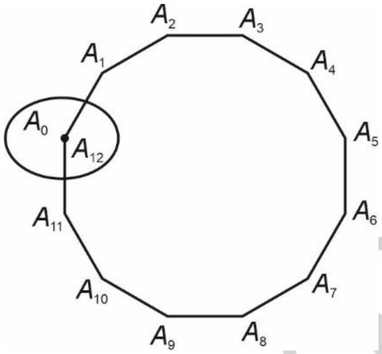
$\Rightarrow \quad 55$

27. The vertices of a regular dodecagon (*n* polygon with 12 sides) are coloured either blue or red. Let *N* be the number of all possible colourings such that no three points of the same colour form the vertices of an equilateral

triangle and no four points of the same colour from the vertices of a square. If N can be written as $N = 100p + q$ where p, q are two positive integers less than 100, find $p + q$

Answer (15)

Sol.



$T_1 = A_0 \ A_4 \ A_8$

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$T_1 = A_0 \ A_4 \ A_8$

and squares:

$S_1 : A_0 A_3 A_6 A_9$

$S_2 : A_1 A_4 A_7 A_{10}$

$S_3 : A_2 A_5 A_8 A_{11}$

Let's first arrange triangles such that all triangles doesn't follow mono chromatic

$\Rightarrow (2^3 - 2)$ ways to remove cases when same red or blue colour is used. $\Rightarrow$ 6 ways for each triangles $\Rightarrow (6^4)$ ways for all Δ.

Let's remove cases where squares which will be formed does follows monochromatism (only one colour).

$\Rightarrow$ at least 1 square is mono chromatic, $\left({}^3C_1\right)$ for one out of S_1, S_2, S_3 and $\left({}^2C_1\right)$ for colour. Notice that for a particular square each vertex lie in exactly one triangle $\Rightarrow$ other 2 vertices must satisfy non monochromatic $\Rightarrow$ $(2^2 - 1)$, subtracting when the square chosen colour was used to make it mono chromatic, for each triangle

$\Rightarrow \left({}^3C_1\right)\left({}^2C_1\right)\left(2^2 - 1\right)^4$

Similarly,

At least 2 squares are mono chromatic

(A) both squares uses same colour

$$\Rightarrow \underset{\text{Square,}}{\left({}^3C_1\right)} \ \underset{\text{Colour,}}{\left({}^2C_1\right)} \ \underset{\text{triangle fixed}}{(1)^4}$$

(B) both squares uses different colour

$$\Rightarrow \underset{\text{Square,}}{\left({}^3C_2\right)} \ \underset{\text{arrangement of colour}}{(2!)} \ \underset{\text{remaining vertex of 2 choices}}{[2]^4}$$

Finally, all squares are mono chromatic obviously all of them cannot use same colour other wise triangle is monochromatic

⇒ 2 colours are used. {*RRB*, *BBR*}

$$\Rightarrow \left({}^3C_3\right).\left({}^3C_2\right)\left({}^2C_1\right)[1]^4$$

⇒ using inclusion and exclusion

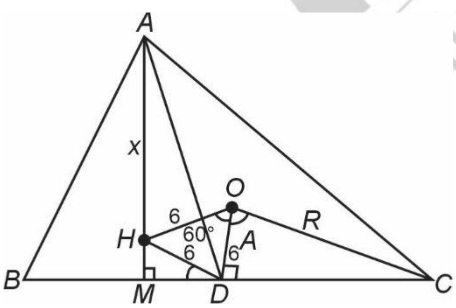
$$\left(2^3-2\right)^4 - {}^3C_1\cdot{}^2C_1\left(2^2-1\right)^4 + {}^3C_2\left[{}^2C_1 + 2!(2)^4\right] - {}^3C_3\left[{}^3C_2\cdot{}^2C_1\right] = 906$$

⇒ 15

28. Let *ABC* be a triangle, *D* be the mid-point of side *BC*, *O* be the circumcenter and *H* be the orthocenter. If the triangle *ODH* is equilateral with side length equal to 6 and the area of the triangle *ABC* can be written as $a\sqrt{b}$, where a, b are positive integers and b is not divisible by the square of any prime, find *a* + *b*

Answer (92)

Sol.



$$\sin(\angle HDM) = \frac{HM}{6} = \frac{1}{6} \qquad (\because \ \angle HDM = 30°)$$

⇒ $HM = 3$

$$\cos(\angle DOC) = \frac{6}{R} = \cos A$$

$$\Rightarrow \quad R = \frac{6}{\cos A}$$

$$\tan A = \frac{DC}{6} = \frac{\frac{BC}{2}}{6} = \frac{BC}{12}$$

Now, $AH = 20D$

$\Rightarrow$ $AH = 12$ ($AH = x$

$\Rightarrow$ $AM = (x + HM) = 15$ (Height of ΔABC)

($\because$ AD intersects HO in 2 : 1 ratio due to Euler line ratio)

Distance between orthocentre and cirucmcentre

$OH \Rightarrow OH^2 = 9R^2 - (a^2 + b^2 + c^2)$

Solving these we get

$$R^2 = \frac{36}{\frac{1}{3}} = 108$$

$$\Rightarrow \quad R = 6\sqrt{3}$$

$$\Rightarrow \quad \cos A = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan A = \sqrt{2}$$

$$\Rightarrow \quad \frac{BC}{12} = \sqrt{2}$$

$$\Rightarrow \quad BC = 12\sqrt{2}$$

$$\Rightarrow \quad \text{Area} = \frac{1}{2} \times 12\sqrt{2} \times 15 = 90\sqrt{2}$$

$\Rightarrow$ $90 + 2 = 92$

29. Consider the collection M of all ordered pairs (a, b) of positive integers a and b which satisfy

$ab = 400 + 11.\text{lcm}(a, b) + 7.\text{gcd}(a, b)$.

What is the smallest possible value of $a + b$?

Answer (98)

Sol. Let $\gcd(a, b) = d$

$\Rightarrow$ $a = dx$, $b = dy$, $\gcd(x, y) = 1$

⇒ Substituting using gcd(a, b).lcm(a, b) = ab

⇒ $(dx)(dy) = 406 + \frac{11(dx)(dy)}{d} = 7d$

⇒ $d^2xy = 406 + 11dxy + 7d$

⇒ $d(dxy - 11xy - 7) = 406 = 2 \times 7 \times 298$

$d \mid 2 \times 7 \times 29$ … (i)

Also, $xy(d - 11) - 7 = \frac{406}{d}$

⇒ $xy = \left(\frac{406}{d} + 7\right)\frac{1}{(d-11)} = \frac{(406 + 7d)}{d(d-1)}$

⇒ $d \geq 1$

$xy \geq 1$

⇒ $d(d - 11) \leq 406 + 7d$

⇒ $d^2 - 11d - 7d \leq 406$

⇒ $(d - 9)2 \leq 406 + 81$, $d \in N$

⇒ $(d - 9)^2 \leq 484$

⇒ $d \leq 31$

⇒ $12 \leq d \leq 31$

Using (i)

$d \mid 2 \times 7 \times 29$

⇒ $d \in \{2, 7, 14, 58 \ldots$

⇒ $d = 14$

⇒ $xy = \left(\frac{406}{14} + 7\right) \times \frac{1}{3} = 12$, $\gcd(x, y) = 1$

⇒ $a + b = d(x + y) = 14(x + y)_{min} = 14(3 + 4) = 98$

30. There are 10 members in a delegation. No two of them have the same height. Let N be the number of ways in which they can stand in a line for a photograph such that

The leftmost person is the shortest

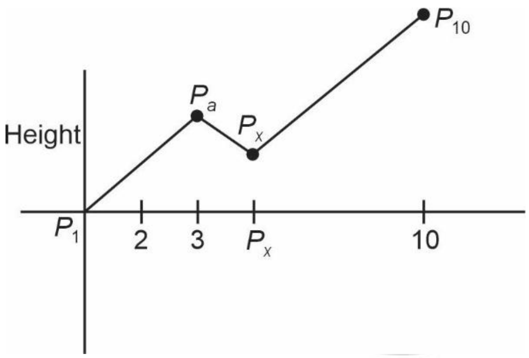
The rightmost person is the tallest and

In the line between the shortest and tallest person, there is exactly one person who is shorter than both of his immediate neighbours.

If N can be written as $100a + b$ where a and b are positive integers less than 100, find $a + b$.

Answer (49)

Sol. Let person's are $P_1, P_2, \ldots, P_{10}$ such that $P_i < P_i + 1$, clearly, P_1 is left most and P_{10} is right most.



Let the person be P_2 can be placed at place = 3, … 9

If P_2 at i, there will be $(i - 2)$ people will be there before $P_2 \Rightarrow ({}^7C_{i-2})$ and their increasing order is fixed

$\Rightarrow \quad \sum_{i=3}^{9} {}^7C_{i-2} = (2^7 - 1)$

Lets place P^3 be the person, 2 can't be after 3

$\Rightarrow \quad$ Lets pick P_a such that $P_a > P_x$

$\Rightarrow \quad P_1P_2$ consecutive

Now 3 will act just like 2 did

$\Rightarrow \quad \sum_{i=3}^{8} {}^6C_{i-2} = (2^6 - 1)$

Similarly for other numbers

$\Rightarrow \quad (2^7 - 1) + (2^6 - 1) + (2^5 - 1) + (2^4 - 1) + (2^3 - 1) + (2^2 - 1) + (2^1 - 1) + (2^0 - 1)$, notice for P_9 it can't have two bigger neighbour

$\Rightarrow \quad 247 = 2 \times 100 + 47$

$\Rightarrow \quad a + b = 49$

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