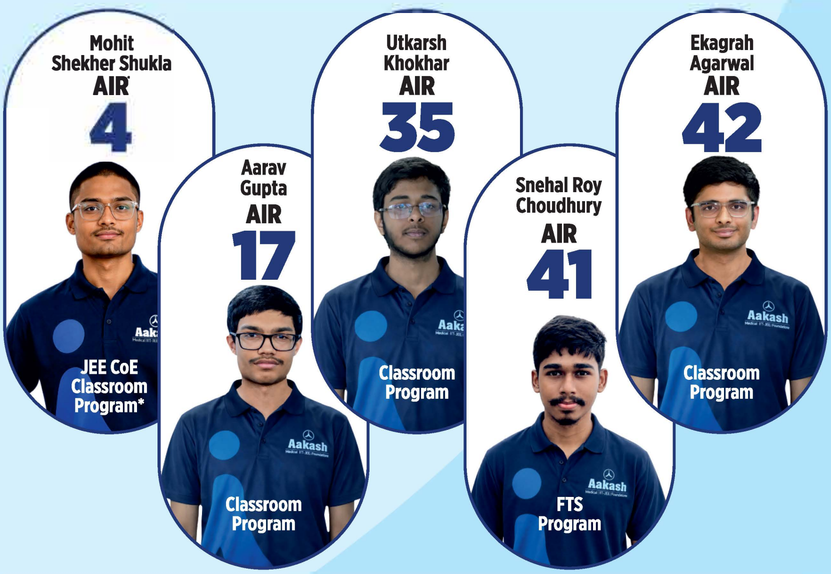
Aakash
Medical | IIT-JEE | Foundations

Answers & Solutions
for
IOQM 2026-27



*Fully sponsored student from March 2022 to June 2025

Top Ranks in JEE (Advanced) 2026

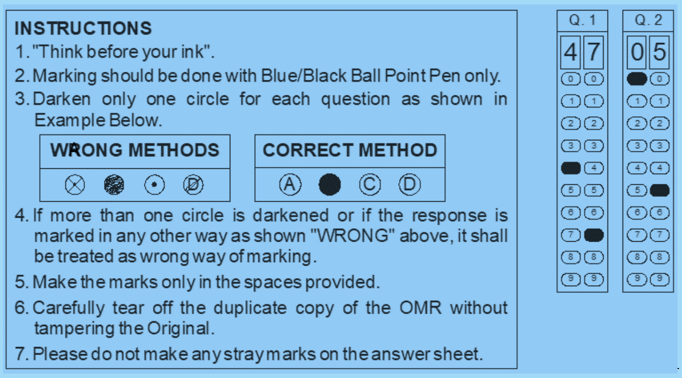
11	26	57
Ranks in Top 100 AIR*	Ranks in Top 200 AIR*	Ranks in Top 500 AIR*

*Includes students of classroom, digital & distance across all categories.

Trust Aakash

INSTRUCTIONS TO CANDIDATES

1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen.** Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The registration number and date of birth will be your login credentials for accessing your score.
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two-digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 & 30 carry 5 marks each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the Candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

Note:

1. $\gcd\left(a, \sqrt{b}\right)$ denotes the greatest common divisor of integers *a* and *b*.
2. For a positive real number *m*, $\sqrt{m}$ denotes the positive square root of *m*. For example, $\sqrt{4} = +2$.
3. Unless otherwise stated all numbers are written in decimal notation.

1. The central of a 9 × 9 chessboard is white. How many white squares are there on the board? (The squares of the chessboard are coloured alternately black and white)

Answer (41)

Sol. Total no. of squares = 9 × 9 = 81 squares

Since the total no. of squares (81) is an odd no., the colour can't be split perfectly even.

One colour will have one more square than the other.

(i) One colour will have 41 sq.

(ii) The other colour will have 40 sq.

On any grid with odd dimension (like 3 × 3, 5 × 5, 9 × 9), the four corners & the exact central sq. always share the same colour. This dominant colour always claim the extra sq.

White sq : 41

Black sq. : 40

2. An integer M is divisible by 4 but not by 256. What is the number of distinct possible remainders when M is divided by 256?

Answer (63)

Sol. When any integer is divided by 256, the possible remainders range from 0 to 255.

the first condition (divisible by 4):

Since M is a multiple of 4, its remainder when divided by 256 also be a multiple of 4.

The multiples of 4 between 0 and 255 are:

0, 4, 8, 12, ..., 252

To find how many numbers are in this list:

$$\frac{252}{4} + 1 = 63 + 1 = 64 \text{ possible remainders}$$

the second condition (not divisible by 256):

Since M is not divisible by 256, the remainder cannot be 0.

Removing 0 from our list leaves the remainders:

4, 8, 12, ..., 252

the final count:

64 – 1 = 63.

3. If $x_1, x_2, \ldots, x_{49}$ are non-zero integers such that $\sum_{i=1}^{49} x_i = 0$, then what is the minimum possible value of $\sum_{i=1}^{49} x_i^2$

Answer (52)

Sol. Analyse the constraints :

We have 49 non-zero integers $x_1, x_2, \ldots, x_{49}$

Their sum is zero : $\sum_{i=1}^{49} x_i = 0$

We want to minimize the sum of their squares : $S = \sum_{i=1}^{49} x_i^2$

Since xi are non-zero integers, the minimum possible value for any x_i^2 is 1 (when x_i = 1 or –1)

If all 49 elements were ±1, then the sum of squares would be 49.

However, for their sum to be 0, the number of +1s must equal the number of –1s.

This requires the total number of elements to be even, but we have 49 elements (an odd number). Therefore, it is impossible for all elements to be ±1.

Minimize deviation :

To keep the sum of squares as small as possible, we should deviate from ±1 be the smallest amount possible.

The next smallest possible square for a non-zero integer is $2^2 = 4$ (where $x_i = \pm 2$). Changing a single element from ±1 to ±2 increases the sum of squares by 4 – 1 = 3.

Let's test a configuration where 48 elements are ±1 and 1 element is –2

Let there be 25 elements equal to +1.

Let there be 23 elements equal to –1.

Let there be 1 element equal to –2.

Total number of elements : 25 + 23 + 1 = 49

Sum of elements : 25(1) + 23(–1) + 1(–2) = 25 – 23 – 2 = 0

Sum of squares : $25(1)^2 + 23(-1)^2 + 1(-2)^2 = 25 + 23 + 4 = 52$

4. All six digits of thee 2-digit numbers are different. If *N* is the largest possible sum of three such numbers, what is the sum of digits of *N*?

Answer (12)

Sol. Let the three nos. be $10a_1 + b_1$, $10a_2 + b_2$, $10a_3 + b_3$ where a_1, a_2, a_3 are the tens digits and b_1, b_2, b_3 are the units digits.

All six digits $(a_1, a_2, a_3, b_1, b_2, b_3)$ must be different.

Total sum, $N = 10(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)$

To make N as large as possible, we must assign the largest possible single-digit nos. $(8,9,7,6,5,4)$ to the places with the highest weight (the tens place):

$\{a_1, a_2, a_3\} = \{9,8,7\}$

$\{b_1, b_2, b_3\} = \{6,5,4\}$

$a_1 + a_2 + a_3 = 9 + 8 + 7 = 24$

$b_1 + b_2 + b_3 = 6 + 5 + 4 = 15$

$\therefore N = 10(24) + 15 = 255$

Sum of digits of $N = 2 + 5 + 5 = 12$.

5. In triangle *ABC* we are given that ∠*CAB* = 80°. Let the perpendicular bisector of *BC* meet the circumcircle of triangle *ABC* in *N*, where we assume that *A* and *N* lie on the same side of the chord *BC*. Then what is the measure of ∠*NBC* in degrees?

Answer (50)

Sol. The points A and N both lie on the circumcircle of ΔABC and are on the same side of the chord BC. According to the inscribed angle theorem, angles subtended by the same chord at the circumference on the same side are equal:

$\angle BNC = \angle BAC = 80^{\circ}$

The point N lies on the perpendicular bisector of BC. By definition, any point on the perpendicular bisector of a line segment is equidistant from the endpoints of that segment. Therefore:

$NB = NC$

Since $NB = NC$, ΔNBC is an isosceles triangle, which means its base angles are equal:

$\angle NBC = \angle NCB$

The sum of the angles in ΔNBC is 180°:

$\angle BNC + \angle NBC + \angle NCB = 180^{\circ}$

$80^{\circ} + 2\angle NBC = 180^{\circ}$

$2\angle NBC = 100^{\circ}$

$\angle NBC = 50°$

6. Let N be the smallest possible integer whose digits add up to 2026. What is the leading digit of $N + 1$?

Answer (02)

Sol. To make N as small as possible, we must minimize its total number of digits. We to make N as small as possible, we must minimize its total number of digits. We achieve this by using the largest possible digit, 9 as many times as possible.

$\frac{2026}{9} = 225$ with a remainder of 1

This means N must consist of one 1 and 225 9s, giving it a total of 226 digits.

To make a number with a fixed set of digits as small as possible, the smallest digit must be placed at the highest place value (the leading position).

. $N = 1\underbrace{999...9}_{225 \text{ times}}$

Adding 1 to a number ending in a string of 9s causes a carry-over effect that turns all the 9s into 0s and increments the leading digit by 1 :

$N + 1 = 2\underbrace{000...0}_{225 \text{ times}} = 2 \times 10^{225}$

Thus, the leading digit of $N + 1$ is 2.

7. Find the number of positive integers n satisfying all the following conditions:

(a) The digits of n lie in the set {1, 2, 4, 8}. (Digits may be repeated)

(b) The sum of the digits is 14.

(c) If 1 occurs ass a digit, then it can occur only immediately to the right of 8.

Answer (31)

Sol. Condition (c) states that if 1 occurs as a digit, it can only occur immediately to the right of 8. This means :

A 1 can never be the first digit

A 1 can never follow another 1

Every 1 must b e paired with the 8 preceding it, forming a combined block of 81.

Thus, we can treat the number as being built from the following independent components (or "block")

2 (contributes 2 to the digit sum)

4 (contributes 4 to the digit sum)

8 (contributes 8 to the digit sum)

81 (contributes 8 + 1 = 9 to the digit sum)

We need the total sum of the digits to be 14. Let n_2, n_4, n_8 and n_9 be the number of times the blocks 2, 4, 8 and 81 are used respectively :

$2n_2 + 4n_4 + 8n_8 + 9n_9 = 14$

Let's test the possible values for n_9 :

If $n_9 = 1$: The equation becomes $2n_2 + 4n_4 + 8n_8 = 14 - 9 = 5$. Since $2n_2 + 4n_4 + 8n_8$ must be an even number, it can never equal the odd number 5. Thus $n_9 = 1$ is impossible.

If $n_9 \geq 2$: The sum would be at least $9 \times 2 = 18$, which exceeds 14.

Because the block 81 cannot be used at all, the digit 1 can never appear in any valid integer. The problem simplifies to finding the combinations of the digits {2, 4, 8} that sum up to 14.

Dividing the entire equation $2n_2 + 4n_4 + 8n_8 = 14$ by 2, we get :

$n_2 + 2n_4 + 4n_8 = 7$

We can find all valid non-negative integer solutions for (n_2, n_4, n_8) and calculate the number of unique arrangements for each

Case-1 : $n_8 = 1$

(i) $n_2 + 2n_4 = 3$

(ii) If $n_4 = 1$

$\Rightarrow$ $n_2 = 1$: digits are {8, 4, 2}

$\Rightarrow$ $\frac{3!}{1!1!1!} = 6$ arrangements.

(iii) If $n_4 = 0$

$\Rightarrow$ $n_2 = 1$: Digits are {8, 4, 2}

$\Rightarrow$ $\frac{4!}{1!3!} = 4$ arrangements.

Case-2 : $n_8 = 0$

(i) $n_2 + 2n_4 = 7$

(ii) If $n_4 = 3$

$\Rightarrow$ $n_2 = 1$: digits are {4, 4, 4, 2}

$\Rightarrow$ $\frac{4!}{3!1!} = 4$ arrangements.

(iii) If $n_4 = 2$

$\Rightarrow$ $n_2 = 3$: digits are {4, 4, 2, 2, 2}

$\Rightarrow$ $\dfrac{5!}{2!3!} = 10$ arrangements.

(iv) If $n_4 = 1$

$\Rightarrow$ $n_2 = 5$: digits are {4, 2, 2, 2, 2, 2}

$\Rightarrow$ $\dfrac{6!}{1!5!} = 6$ arrangements.

(v) If $n_4 = 0$

$\Rightarrow$ $n_2 = 7$: digits are {2, 2, 2, 2, 2, 2, 2}

$\Rightarrow$ $\dfrac{7!}{7!} = 1$ arrangement.

Total count

Summing up the arrangements from all valid cases :

Total = 6 + 4 + 4 + 4 + 10 + 6 + 1 = 31.

8. Find the number of 2-digit positive integers n such that $n = 25 + (a \times b)$, where a and b are the two digits of n.

Answer (03)

Sol. Let the 2-digit number be $n = 10a + b$, where :

a is the tens digit ($a \in \{1, 21, \ldots, 9\}$)

b is the units digit ($b \in \{0, 1, \ldots, 9\}$)

According to the problem $n = 26 + (a \times b)$. substituting $n = 10a + b$

$10a + b = 26 + ab$

$10a - 26 = ab - b$

$10a - 26 = b(a - 1)$

$$b = \frac{10a - 26}{a - 1}$$

$$b = \frac{10(a-1) - 16}{a - 1} = 10 - \frac{16}{a - 1}$$

Since b must be a whole number $(a - 1)$ must be a factor of 16. Since a is a single digit $(1 \le a \le 9)$, the value of $(a - 1)$ can only be between 0 and 8. The positive factors of 16 less than of equal to 8 are 1, 2, 4 and 8

If $a - 1 = 1$: $a = 2$

$\Rightarrow$ $b = 10 - \dfrac{16}{1} = -6$ (invalid, be cannot be negative)

If $a - 1 = 2$: $a = 3$

$\Rightarrow$ $b = 10 - \dfrac{16}{1} = 2$

$\Rightarrow$ $n = 32$

If $a - 1 = 4 : a = 5$

$\Rightarrow \quad b = 10 - \frac{16}{4} = 6$

$\Rightarrow \quad n = 56$

If $a - 1 = 8 : a = 9$

$\Rightarrow \quad b = 10 - \frac{16}{8} = 8$

$\Rightarrow \quad n = 98$

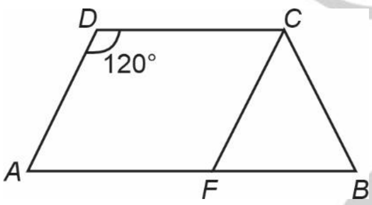
Conclusion

There are exactly 3 such numbers : 32, 56 and 98.

9. In trapezium *ABCD*, it is given that *AB* is parallel to *CD*. Assume that $AB = 3CD$, $CD = DA$ and $\angle CDA = 120°$. If the largest angle of *ABCD* is $x°$ and the smallest angle is $y°$, what is the value of $\frac{x}{y}$?

Answer (05)

Sol.



Let $CD = a$

$AD = CD = a$

and $AB = 3a$

$\angle DAB + \angle CDA = 180^\circ \quad [\because AB \parallel CD]$

$\Rightarrow \angle DAB = 180^\circ - 120^\circ = 60^\circ$

Let $CF \parallel DA$

$AF = CD = a$

$FC = DA = a$

and $\angle AFC = \angle CDA = 120^\circ$

$BF = AB - AF = 3a - a = 2a$

$\angle CFB = 180^\circ - 120^\circ$

$= 60^\circ$

In ΔFCB

$CB^2 = FC^2 + FB^2 - 2(FC)(FB)\cos 60^\circ$

$$CB^2 = a^2 + 4a^2 - 2(a)(2a)\frac{1}{2}$$

$$\Rightarrow CB^2 = 3a^2$$

$$\Rightarrow CB = \sqrt{3}a$$

and $FC^2 + CB^2 = a^2 + 3a^2$

$$= 4a^2 = FB^2$$

$\Rightarrow \Delta FCB$ is right angle triangle

$$\Rightarrow \angle FCB = 90^\circ$$

$$\sin \angle FBC = \frac{FC}{FB} = \frac{1}{2}$$

$$\Rightarrow \angle ABC = 30^\circ$$

$$\Rightarrow \angle BCD = 180^\circ - \angle ABC \quad [\because AB \parallel CD]$$

$$= 180^\circ - 30^\circ$$

$$= 150^\circ$$

largest angle is $x^\circ = 150^\circ$

smallest angle is $y^\circ = 30^\circ$

$$\Rightarrow \frac{x}{y} = \frac{150}{30} = 5$$

10. What is the number of integers in the set {0, …, 20} which can be expressed as the sum of two square integers?

Answer (13)

Sol. Using squares up to 20:

$0^2, 1^2, 2^2, 3^2, 4^2$ = 0, 1, 4, 9, 16

Possible sums:

0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20

So, the integers in {0, 1, …, 20} that can be expressed as a sum of two square integers are:

0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20

Count: 13.

11. A 7 × 7 board is divided into 49 unit squares. We place checkers on the board, at most one per square. Find the largest number of checkers that can be placed on the unit squares so that each row, as well as each column, contains an even number of checkers.

Answer (42)

Sol. Maximum checkers per line: Since the board is 7 × 7, each row and column contains 7 squares. For the number of checkers in each row or column to be even, the maximum number possible per line is 6

Maximum theoretical total: With 7 rows and at most 6 checkers per row, the maximum total number of checkers cannot exceed:

7 × 6 = 42.

12. Let A be a 3-digit number with distinct nonzero digits and B be the number obtained by reversing the digits of A. Determine the largest possible prime factor of $|A - B|$.

Answer (11)

Sol. Let $A = 100a + 10b + c$

where a, b, c are distinct nonzero digits. Then

$B = 100c + 10b + a$

so, $|A - B| = |99(a - c)| = 99|a - c|$

Since a, c are distinct nonzero digits, $|a - c| \in \{1, 2, ..., 8\}$. Thus

$|A - B| = 99d = 3^2 \cdot 11 \cdot d,\ d \le 8$

The prime factors of d are at most 7, so no prime factor of $|A - B|$ can exceed 11. Also 11 can occur as the largest prime factor; for example A = 123:

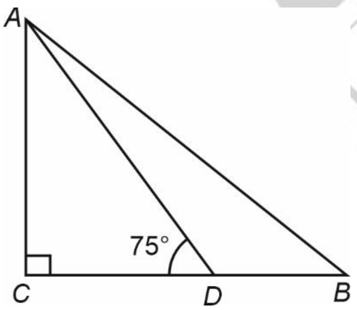
$|123 - 321| = 198 = 2 \cdot 3^2 \cdot 11$

Therefore, the largest possible prime factor is 11.

13. In an isosceles triangle ABC, with $\angle ACB = 90°$. The point D is on the side BC such that $\angle ADC = 75°$. If the area of triangle ADC is 81, what is the length of segment BD?

Answer (18)

Sol. Let $AC = BC = x$ and $CD = y$



Now, In ΔACD

$$\tan 75° = \frac{AC}{CD}$$

$$\frac{x}{y} = 2 + \sqrt{3}$$

$$\Rightarrow x = (2 + \sqrt{3})y$$

Now, or $(\Delta ADC) = 81$

$$\frac{1}{2}\times x \times y = 81$$

$xy = 162$

$$\left(2+\sqrt{3}\right)y^2 = 162$$

$$y^2 = 162\left(2-\sqrt{3}\right)$$

$$y = 9\left(\sqrt{3}-1\right)$$

$$\Rightarrow x = 9\left(2+\sqrt{3}\right)\left(\sqrt{3}-1\right)$$

$$= 9\left(\sqrt{3}+1\right)$$

$$\because BD = BC - CD = x - y$$

$$= 9\sqrt{3}+9-9\sqrt{3}+9$$

$= 18.$

14. Let a, b, c, d be positive integers such that $a^2 + b^2 - cd^2 = 2026$. Find the minimum possible value of $a + b + c + d$.

Answer (51)

Sol. Lower bound for $a + b$

Since, $c, d \geq 1$, we have $cd^2 \geq 1$, which gives:

$$a^2 + b^2 = 2026 + cd^2 \geq 2027$$

For positive integers $a, b \geq 1$:

If $a + b \leq 46$, the maximum possible value for $a^2 + b^2$ occurs at $\{a, b\} = \{45, 1\}$, where:

$$a^2 + b^2 = 45^2 + 1^2 = 2025 + 1 = 2026$$

This contradicts $a^2 + b^2 \geq 2027$

Therefore, we must have

$$a + b \geq 47$$

Lower Bound for $c + d$

A positive integer can be written as the sum of two squares $a^2 + b^2$ if and only if all of its prime factors of the form $4k + 3$ appear with an even exponent

1. If $c + d = 2$:

$$(c, d) = (1, 1) \Rightarrow cd^2 = 1 \Rightarrow a^2 + b^2 = 2027$$

2027 is a prime congruent to 3(mod 4), so it cannot be expressed as the sum of two squares.

2. If $c + d = 3$

$(c, d) = (1, 2) \Rightarrow cd^2 = 4 \Rightarrow a^2 + b^2 = 2030.$

$$2030 = 2 \times 5 \times 7 \times 29$$

The prime factor $7 \equiv 3 (\text{mod} 4)$ has an odd exponent (1), so 2030 cannot be a sum of two squares.

$$(c, d) = (2, 1) \Rightarrow cd^2 = 2 \Rightarrow a^2 + b^2 = 2028$$

$$2028 = 2^2 \times 3 \times 13^2$$

The prime factor $3 \equiv 3$ (mod4) has an odd exponent (1), so 2028 cannot be a sum of two squares.

Since neither $c + d = 2$ nor $c + d = 3$ yields a valid solution, we must have:

$c + d \geq 4$

Combining the two lower bounds:

$$a + b + c + d \geq 47 + 4 = 51$$

We can achieve this lower bound with the positive integers:

$a = 45$

$b = 2$

$c = 3$

$d = 1$

Checking the equation:

$$a^2 + b^2 - cd^2 = 45^2 + 2^2 - 3(1^2) = 2025 + 4 - 3 = 2026$$

The sum is

$$a + b + c + d = 45 + 2 + 3 + 1 = 51$$

The minimum possible value of $a + b + c + d$ is 51.

15. Find the number of non-constant polynomials $P(x)$, with real coefficients, such that $P(x^2) = P(P(x))$

Answer (02)

Sol. Let $d = \deg P$. Since P is nonconstant, $d \geq 1$.

$$\deg P(x^2) = 2d, \deg P(P(x)) = d^2$$

So, $2d = d^2 \Rightarrow d = 2$

Thus P is quadratic:

$$P(x) = ax^2 + bx + c, a \neq 0$$

Now compare coefficients in

$$P(x^2) = P(P(x))$$

We have

$$P(x^2) = ax^4 + bx^2 + c$$

And

$$P(P(x)) = a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c$$

Expanding and equating coefficients:

For $x^4 : a^3 = a \Rightarrow a = \pm 1$

For $x^3 : 2a^2b = 0 \Rightarrow b = 0$

For $x^2 : ab^2 + 2a^2c + ab = b \Rightarrow 2c = 0 \Rightarrow c = 0$

So,

$$P(x) = ax^2, a = \pm 1$$

Both $P(x) = x^2$ and $P(x) = -x^2$ work

So, the number of non-constant polynomials satisfying the equation is 2.

16. Find the number of ordered triples (x, y, z) of positive integers such that $1 \le x, y, z \le 8$ and $|x - y| + |y - z| + |z - x| = 8$

Answer (96)

Sol. Given: $|x - y| + |y - z| + |z - x| = 8$

$\Rightarrow 1 \le x, y, z \le 8$

$\Rightarrow$ Let $M = \max(x, y, z)$ and $m = \min(x, y, z)$

$\Rightarrow$ Identify: $|x - y| + |y - x| + |z - x| = 2(M - m)$

$\Rightarrow 2(M - m) = 8$

$\Rightarrow$ M – m = 4

$\Rightarrow (m, M) \in \{(1, 5), (2, 6), (3, 7), (4, 8)\}$

$\Rightarrow$ 4 valid pairs for (m, M)

$\Rightarrow$ Let k be the remaining third variable in the triplet

$\Rightarrow m \le k \le M$

$\Rightarrow$ Case 1: k = m $\Rightarrow$ Elements $\{m, m, M\} \Rightarrow \frac{3!}{2!} = 3$ permutations

$\Rightarrow$ Case 2: k = M $\Rightarrow$ Elements $\{m, M, M\} \Rightarrow \frac{3!}{2!} = 3$ permutations

$\Rightarrow$ Case 3: m < k < M $\Rightarrow$ k has $(M - m - 1) = 3$ possible values

$\Rightarrow$ Elements {m, k, M} $\Rightarrow 3 \times 3! = 18$ permutations

$\Rightarrow$ Total permutations per pair = 3 + 3 + 18 = 24

$\Rightarrow$ Total ordered triplets (x, y, z) = 4 × 24 = 96.

17. Four points A, B, C and D lie on a straight line, in this order. A point E, not on the line, satisfies $\angle AEB = \angle BEC = \angle CED = 45°$. Let F and G be the mid-points of AC and BD, respectively. If $\angle FEG = x°$, what is the value of x?

Answer (90)

Sol. Let the line containing A, B, C and D be the x-axis and let E be at $(0, h)$, where $h > 1$

Let the angles of the rays EA, EB, EC, ED with the perpendicular from E to the line be α, $\alpha + 45°$, $\alpha + 90°$, and $\alpha + 135°$

$$x_A = h\tan\alpha$$

$$x_B = h\tan(\alpha + 45°)$$

$$x_C = h\tan(\alpha + 90°) = -h\cot\alpha$$

$$x_D = h\tan(\alpha + 135°) = -h\cot(\alpha + 45°)$$

Let h = 1 without loss of generally

F is the midpoint of $AC \Rightarrow x_F = \dfrac{\tan\alpha - \cot\alpha}{2} = -\cot(2\alpha)$

G is the midpoint of $BD \Rightarrow x_G = \dfrac{\tan(\alpha + 45°) - \cot(\alpha + 45°)}{2} = \tan(2\alpha)$

$$E = (0, 1), F = (-\cot(2\alpha), 0), G = (\tan(2\alpha), 0)$$

Slope of $EG = \dfrac{0-1}{\tan(2\alpha) - 0} = -\cot(2\alpha)$

Slope of EF × Slope of EG = $\tan(2\alpha) . (-\cot(2\alpha)) = -1$

$$EF \perp EG \Rightarrow \angle FEG = 90°$$

x = 90.

18. Let M be the smallest positive integer with the following two properties :

(a) The leading digit of M is equal to 3.

(b) If N is the number obtained by moving this leading 3 to the units place, and shifting all the other digits one place to the left, then $N = \dfrac{M}{4}$.

What is the sum of the digits of M?

Answer (27)

Sol. Let M have k digits and let its leading digit be 3. Write

$$M = 3 \cdot 10^{k-1} + x,$$

where x is the integer formed by the remaining k – 1 digits. Moving the leading digit 3 to the units place gives

$$N = 10x + 3$$

Since $N = M/4$, we have

$$4(10x + 3) = 3 \cdot 10^{k-1} + x$$

Hence,

$$40x + 12 = 3 \cdot 10^{k-1} + x \Rightarrow 39x = 3 \cdot 10^{k-1} - 12$$

Dividing by 3,

$$13x = 10^{k-1} - 4$$

Thus

$$x = \frac{10^{k-1} - 4}{13}$$

For x to be an integer, we require

$$10^{k-1} \equiv 4 \pmod{13}$$

Powers of 10 modulo 13 are

$10^1 \equiv 10, 10^2 \equiv 9, 10^3 \equiv 12, 10^4 \equiv 3, 10^5 \equiv 4, 10^6 \equiv 1$

Thus $10^{k-1} \equiv 4$ occurs when $k-1 \equiv 5 \pmod 6$. The smallest positive integer is k – 1 = 5, so k = 6. Therefore,

$$x = \frac{10^5 - 4}{13} = \frac{100000 - 4}{13} = \frac{99996}{13} = 7692$$

Hence,

$$M = 3 \cdot 10^5 + 7692 = 307692$$

Checking:

$$\frac{M}{4} = \frac{307692}{4} = 76923,$$

which is exactly M with the leading digit 3 moved to the units place.

Therefore, the sum of the digits of M is

3 + 0 + 7 + 6 + 9 + 2 = 27.

19. A sequence $a_1, a_2, a_3, \ldots$ of real numbers satisfies $\frac{a_{n+3} - a_{n+2}}{a_n - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_n + a_{n+1}}$ for all $n \geq 1$. Suppose $a_{55} = 6$, $a_{66} = 2$ and $a_{77} = 1$. Let N denote the sum $a_1^2 + a_2^2 + \ldots + a_{2026}^2$. What is the sum of the digits of N?

Answer (15)

Sol. From the condition,

$$\frac{a_{n+3} - a_{n+2}}{a_n - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_n + a_{n+1}}$$

Cross-multiplying gives

$$\left(a_{n+3} - a_{n+2}\right)\left(a_n + a_{n+1}\right) = \left(a_{n+3} + a_{n+2}\right)\left(a_n a_{n+1}\right)$$

Expanding both sides:

$$a_{n+3}a_n + a_{n+3}a_{n+1} - a_{n+2}a_n - a_{n+2}a_{n+1} = a_{n+3}a_n - a_{n+3}a_{n+1} + a_{n+2}a_n - a_{n+2}a_{n+1}$$

Cancelling common terms,

$$a_{n+3}a_{n+1} - a_{n+2}a_n = -a_{n+3}a_{n+1} + a_{n+2}a_n,$$

which gives

$$2a_{n+3}a_{n+1} = 2a_{n+2}a_n$$

Hence,

$$a_{n+1}a_{n+3} = a_na_{n+2}$$

From (1), replacing n by n + 1,

$$a_{n+2}a_{n+4} = a_{n+1}a_{n+3}$$

Using (1) again, $a_{n+2}a_{n+4} = a_na_{n+2}$

Assuming nonzero terms, we get

$$a_{n+4} = a_n$$

Thus, the sequence is periodic with period 4

Now,

$$55 \equiv 3 \pmod 4),\ 66 \equiv 2 \pmod 4),\ 77 \equiv 1 \pmod 4)$$

Given $a_{55} = 6, a_{66} = 2, a_{77} = 1$,we have

$$a_3 = 6, a_2 = 2, a_1 = 1$$

Using $a_1a_3 = a_2a_4$ from (1) with n = 1,

$$1 \cdot 6 = 2a_4 \Rightarrow a_4 = 3$$

Thus, the repeating block is

(1, 2, 6, 3)

The sum of squares over one period is

$$1^2 + 2^2 + 6^2 + 3^2 = 1 + 4 + 36 + 9 = 50$$

Since 2026 = 4 · 506 + 2, we have

$$N = 506 \cdot 50 + 1^2 + 2^2 = 25300 + 5 = 25305$$

The sum of the digits of N is

$$2 + 5 + 3 + 0 + 5 = 15.$$

20. Let P be a regular polygon with 8 vertices. By a labelling of P we mean an assignment of integers 1, 2, …, 8 to the vertices in some order. A labelling is considered good if the path consisting of line segments 1 to 2, 2 to 3, … and 7 to 8 does not self-intersect. If N is the number of good labellings, what is the remainder when N is divided by 100?

Answer (12)

Sol. We have a regular octagon with 8 vertices. We assign the numbers 1, 2, 3, …., 8 to the vertices and connect them in the order of their labels: from vertex 1 to 2, from 2 to 3, and so on, up to 7 to 8. We want this chain of 7 segment to not cross itself.

Consider the process starting at the vertex labelled 1:

- The remaining vertices to be labelled 2, 3, …, 8 lie on a circle

- If we jump to a vertex in the interior of the remaining unvisited block, the unvisited vertices are divided into two separated groups. To visit vertices in both groups later on, the path would eventually have to cross the chord already drawn, resulting in a self-intersections.
- Therefore, to avoid self-intersections, at each step the next vertex must be chosen as either the clockwise-most (rightmost) or counterclockwise-most (leftmost) available vertex among the remaining unvisited ones along the perimeter.

Fix the position of vertex 1:

- There are 7 remaining vertices to visit in the sequence 2, 3, … 8
- For each of the first 6 choices (selecting vertices 2 through 7), there are exactly 2 valid choices: the leftmost or the rightmost remaining vertex along the circle.
- One the first 7 vertices are assigned, the final vertex (labelled 8) is uniquely determined (1 choice)

Thus, for a fixed position of vertex 1, the number of valid paths is:

$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$

Since the vertex labelled 1 can be placed at any of the 8 vertices of the octagon the total number of good labellings is:

$N = 8 \times 64 = 512$

Dividing N by 100:

$512 = 5 \times 100 + 12$

The remainder when N is divided by 100 is 12.

21. Complex numbers x, y, z satisfy the following system of equations :

$x^2 + y^2 + z = xy$

$x + y^2 + z^2 = yz$

$x^2 + y + z^2 = xz$

Determine the sum of all distinct possible values of $|(x^2 - y)(y^2 - z)(z^2 - x)|$

Answer (12)

Sol.

Subtracting the second equation from the first equation:

$$\left(x^2 - x\right) + \left(z - z^2\right) = y\left(x - z\right) \Rightarrow \left(x - z\right)\left(x + z - y - 1\right) = 0$$

By cyclic symmetry, we obtain three cases for the system:

Case 1: $x = y = z$

The system reduces to $x^2 + x^2 + x = x^2 \Rightarrow x^2 + x = 0$, giving two sub-cases:

1. $x = y = z = 0$: $\left|\left(x^2 - y\right)\left(y^2 - z\right)\left(z^2 - x\right)\right| = \left|(0)(0)(0)\right| = 0$

2. $x = y = z = -1$: $\left|\left(x^2 - y\right)\left(y^2 - z\right)\left(z^2 - x\right)\right| = \left|\left((-1)^2 - (-1)\right)^3\right| = \left|2^3\right| = 8$

Case 2: Exactly two variables are equal

Without loss of generality, let $x = y \neq z$. From $x = z - y - 1$, we get $z = 1$. Substituting $z = 1$ and $y = x$ into the original equations yields:

$x^2 + 1 = 0 \Rightarrow x = \pm i$

For $(x, y, z) = (i, i, 1)$:

$x^2 - y = -1 - i, \quad y^2 - z = -2, \quad z^2 - x = 1 - i$

Evaluating the absolute value of the product:

$\left|(x^2 - y)(y^2 - z)(z^2 - x)\right| = \left|(-1-i)(-2)(1-i)\right| = \left|2(1+i)(1-i)\right| = |4| = 4$

Case 3: ***x, y, z*** **are all distinct**

If no two variables are equal, then $x + z - y = 1$, $y + x - z = 1$, and $z + y - x = 1$.

Adding these gives $x + y + z = 3 \Rightarrow x = y = z = 1$, which contradicts that all three variables are distinct.

The distinct possible values are 0, 4, and 8. The required sum is: 0 + 4 + 8 = 12

22. Let N be the number of distinct 8-digit numbers obtained by arranging the six numbers 0, 1, 2, 3, 10, 23 where the first digit of the 8 digit number is not zero. Find the sum of the digits of N.

Answer (21)

Sol. We arrange the six blocks {0, 1, 2, 3, 10, 23} to form an 8-digit number, with the condition that the first block is not 0.

Let A be the set of valid permutations where the block 1 is immediately followed by the block 0 (creating the sequence [1, 0]). Let B be the set of valid permutations where the block 2 is immediately followed by the block 3 (creating the sequence [2, 3]).

Total Valid Permutations There are 6 blocks. The block 0 cannot be first, so the first block has 5 choices, and the remaining 5 blocks can be ordered in 5! ways:

Total = 5 × 5! = 600

Sizes of Subsets A, B, and $A \cap B$

- **Size of** ***A*:** Treat [1, 0] as one fused block. The blocks are

$[1, 0], 2, 3, 10, 23$

None of these blocks starts with 0, so every permutation is valid: |A| = 5! = 120

Size of B: Treat [2, 3] as one fused block. The blocks are [2, 3], 0, 1, 10, 23

The block 0 cannot be first, so there are 4 choices for the first position, and the remaining 4 blocks permute freely |B| = 4 × 4! = 96

Size of $A \cap B$**:** Fuse both [1, 0] and [2, 3]. The blocks are [1, 0], [2, 3], 10, 23

None starts with 0, hence $|A \cap B| = 4! = 24$

Counting Distinct Numbers Overcounting occurs precisely when 1 is immediately followed by 0 (swapping the adjacent pair [1, 0] with the block 10 produces the same 8-digit string) and/or when 2 is immediately followed by 3 (swapping [2, 3] with the block 23 produces the same string

Neither adjacency occurs

$600-(|A|+|B|-|A\cap B|)=600-(120+96-24)=408$ strings

Each corresponds to a single permutation, contributing 408 distinct numbers

Only 1 followed by 0 occurs: $|A|-|A\cap B|=120-24=96$ permutations

These pair up into orbits of size 2, contributing $\frac{96}{2}=48$ distinct numbers.

Only 2 followed by 3 occurs: $|B|-|A\cap B|=96-24=72$ permutations

These pair up into orbits of size 2, contributing $\frac{72}{2}=36$ distinct numbers

Both adjacencies occur: $|A\cap B|=24$ permutations

These group into orbits of size 4, contributing $\frac{24}{4}=6$ distinct numbers.

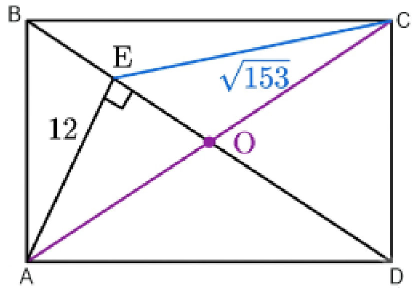
Summing these disjoint cases, the total number of distinct 8-digit numbers is N = 408 + 48 + 36 + 6 = 498

The sum of the digits of N is 4 + 9 + 8 = 21.

23. Let *ABCD* be a rectangle and let *E* be a point on *BD* such that *AE* is perpendicular to *BD*. If *AE* = 12 and $CE=\sqrt{193}$, compute the area of the rectangle *ABCD*.

Answer (300) Bonus

Sol.



Let *O* be $AC\cap BD$

$\Rightarrow AO=OC=\frac{1}{2}AC=\frac{1}{2}BD$

For ΔAEC, By Apollonius theorem $AE^2+CE^2=2(EO^2+AO^2)$

also, ΔAEO, is right angled at E.

$\Rightarrow AO^2=AE^2+EO^2$

$\Rightarrow 4AO^2=3AE^2+CE^2=625$

$\Rightarrow AO=\frac{25}{2}\Rightarrow AC=BD=25$

Area of $\Delta ABD=\frac{1}{2}\times AE\times BD=\frac{1}{2}\times 12\times 25$

$\Rightarrow$ Area of rectangle $ABCD = 2\times\frac{1}{2}\times 12\times 25$

= 300

24. Let $E = \{p^4 + p^2 - 2 \mid p$ is a prime, $p > 3\}$. What is the largest positive integer that divides all the numbers in E?

Answer (72)

Sol. Let $p > 3$ be a prime. We factor the expression: $p^4 + p^2 - 2 = (p^2+2)(p^2-1) = (p^2+2)(p-1)(p+1)$

Since p is an odd prime, $p - 1$ and $p + 1$ are consecutive even integers, making one divisible by 2 and the other by 4. Thus, $8|(p^2 - 1)$.

Since p is not divisible by 3, $p \equiv \pm 1 \pmod 3$, so $p^2 \equiv 1 \pmod 3$. This implies $3*\left(p^2-1\right)$ and $3*\left(p^2+2\right)$, so $9*\left(p^4+p^2-2\right)$.

Since gcd(8, 9) = 1, 72 divides $p^4 + p^2 - 2$ for all primes $p > 3$.

Testing $p = 5$: $5^4 + 5^2 - 2 = 648 = 72 \times 9$

Thus, the largest positive integer that divides all elements in E is 72.

25. A 1 × 5 rectangle is divided into five 1 × 1 squares by drawing four line segments parallel to the shorter side of the rectangle. Each of the resulting sixteen unit-length line segments is coloured red, blue or green. A 1 × 1 square is called colourful if all the three colours are used in colouring its sides. If N is the number of ways of colouring such that all the five 1 × 1 squares are colourful, find the remainder when N is divided by 100.

Answer (96)

Sol. Let squares be S_1, S_2, S_3, S_4, S_5.

Segments of S_i are left v_{i-1}, top t_i, bottom b_i, right v_i

Colors available = 3 (Red, Blue, Green)

First vertical segment v_0 can be assigned any of the 3 colors.

Assume v_{i-1} is assigned color C_1

For S_i to use all 3 colors, segments $\{t_i, b_i, v_i\}$ must include remaining colors C_2 and C_3

Total unrestricted ways to color $\{t_i, b_i, v_i\} = 3^3 = 27$

Ways missing color $C_2 = 2^3 = 8$

Ways missing color $C_3 = 2^3 = 8$

Ways missing both C_2 and $C_3 = 1^3 = 1$

Valid ways to color $\{t_i, b_i, v_i\}$ (via Inclusion Exclusion) = 27 – 8 – 8 + 1 = 12

Valid ways for $S_i = 12$

Each square S_i has exactly 12 valid colorings independent of the color of v_{i-1}

$\Rightarrow N = (\text{Ways for } v_0)\times(\text{Ways for } S_1)\times(\text{Ways for } S_2)\times(\text{Ways for } S_3)\times(\text{Ways for } S_4)\times(\text{Ways for } S_5)$

$\Rightarrow N = 3 \times 12 \times 12 \times 12 \times 12 \times 12$

$\Rightarrow N = 3 \times 12^5$

$\Rightarrow N = 3 \times 248832$

$\Rightarrow N = 74696$

Remainder when divided by 100 $\Rightarrow$ N (mod 100)

$\Rightarrow 746496 \equiv 96 \pmod{100} \Rightarrow 96$

26. There are n points in the plane, no three of which are collinear. Every pair of points is joined by a segment which is coloured red or blue such that the following conditions hold:

(a) If A, B, C are three points such that AB is red and BC is blue, then AC is red.

(b) For any point A, three are exactly three points B, C, D such that AB, AC, AD are red.

Find the sum of all possible values of n.

Answer (10)

Sol. A 3-regular graph is a graph where every single vertex connects to exactly three edges.

Let the graph be K_n with vertices V and edges colored Red or Blue

$\Rightarrow$ Condition (a): $R(A,B)$ and $B(B,C) \Rightarrow R(A,C)$

$\Rightarrow$ Triangles with exactly 1 Red and 2 Blue edges (RBB) are forbidden

$\Rightarrow$ Condition (b): Every vertex has exactly 3 Red edges

$\Rightarrow$ Red edges form a 3-regular graph on n vertices

$\Rightarrow$ Sum of degrees $= 3n = 2|E_R|$

$\Rightarrow$ n must be an even positive integer

$\Rightarrow$ Choose any vertex $v \in V$

$\Rightarrow$ Let $N_R(v)$ be vertices connected to v by red edges

$\Rightarrow |N_R(v)| = 3$

$\Rightarrow$ Let $N_B(v)$ be vertices connected to v by Blue edges

$\Rightarrow |N_B(v)| = n - 1 - 3 = n - 4$

$\Rightarrow$ Choose $u \in N_R(v)$ and $w \in N_B(v)$

$\Rightarrow$ Edge uv is Red and edge vw is Blue

$\Rightarrow$ Apply Condition (a) to Δuvw

$\Rightarrow$ Edge uw must be Red.

$\Rightarrow$ Every edge between $N_R(v)$ and $N_B(v)$ is Red

$\Rightarrow$ Vertex $u \in N_R(v)$ connects to v (1 Red edge) and all $w \in N_B(v)$ ($|N_B(v)|$ Red edges)

$\Rightarrow$ Total Red edges for $u \geq 1 + |N_B(v)|$

$\Rightarrow$ By Condition (b), u has exactly 3 Red edges

$\Rightarrow 1 + |N_B(v)| \leq 3$

$\Rightarrow |N_B(v)| \leq 2$

$\Rightarrow n - 4 \leq 2$

$\Rightarrow n \leq 6$

$\Rightarrow$ Possible even values for n are 4 and 6

Test $n = 4$

$\Rightarrow d_R(v) = 3 \Rightarrow$ All edges are Red

$\Rightarrow$ No Blue edges exist, making Condition (a) vacuously true

$\Rightarrow n = 4$ is a valid solution

Test $n = 6$

$\Rightarrow |N_B(v)| = 2$

$\Rightarrow$ Red edges form a complete bipartite graph $K_{3,3}$

$\Rightarrow$ Blue edges form two disjoint K_3 triangles

$\Rightarrow$ Every triangle is either BBB (within partition) or RRB (across partitions)

$\Rightarrow$ No RBB triangles exist, satisfying Condition (a)

$\Rightarrow n = 6$ is a valid solution

$\Rightarrow$ Sum of all possible values of $n = 4 + 6$

$\Rightarrow$ 10.

27. The lengths of the sides of a convex quadrilateral are $\sqrt{a}$, $\sqrt{a+3}$, $\sqrt{a+2}$ and $\sqrt{2a+5}$, in this order. The length of each diagonal is $\sqrt{2a+5}$. If $\theta°$ is the difference between the largest angle and the second largest angle of the quadrilateral then determine the value of θ.

Answer (30)

Sol. Let the vertices of the convex quadrilateral in order be A, B, C, D. The given side lengths are

$AB=\sqrt{a},\quad BC=\sqrt{a+3},\quad CD=\sqrt{a+2},\quad DA=\sqrt{2a+5}$ and the diagonal lengths are

$AC=\sqrt{2a+5},\quad BD=\sqrt{2a+5}$

In ΔBCD, we observe that $BC^2+CD^2=(a+3)+(a+2)=2a+5=BD^2$

By the converse of the Pythagorean theorem, ΔBCD is a right-angled triangle with $\angle BCD$ = 90°.

We place the configuration on a Cartesian coordinate plane parameterized by *a*.

* Let D = (0, 0) be the origin.

* Place $C=\left(0,\sqrt{a+2}\right)$ on the positive *y*-axis.

* Since $\angle BCD$ = 90°., vertex *B* is horizontally aligned with *C*:

$B=\left(\sqrt{a+3},\sqrt{a+2}\right)$

Let vertex A = (x, y). Since $DA^2 = AC^2 = 2a + 5$, we have

$x^2+y^2=2a+5$ (i)

$$x^2+\left(y-\sqrt{a+2}\right)^2=2a+5 \quad \text{(ii)}$$

Subtracting (i) from (ii) yields

$2y\sqrt{a+2}-(a+2)=0 \Rightarrow y=\dfrac{\sqrt{a+2}}{2}$

Substituting *y* into (ii) gives

$x^2=(2a+5)-\dfrac{a+2}{4}=\dfrac{7a+18}{4}$

Using the side length $AB=\sqrt{a}$,

$\left(x-\sqrt{a+3}\right)^2+\left(y-\sqrt{a+2}\right)^2=a \Rightarrow \left(x-\sqrt{a+3}\right)^2+\dfrac{a+2}{4}=a,$

which simplifies to

$\left(x-\sqrt{a+3}\right)^2=\dfrac{3a-2}{4}.$

Expanding and substituting $x^2=\dfrac{7a+18}{4}$,

$\dfrac{7a+18}{4}-2x\sqrt{a+3}+(a+3)=\dfrac{3a-2}{4} \Rightarrow x\sqrt{a+3}=a+4$

Squaring both sides:

$$(a+4)^2 = x^2(a+3) = \left(\frac{7a+18}{4}\right)(a+3)$$

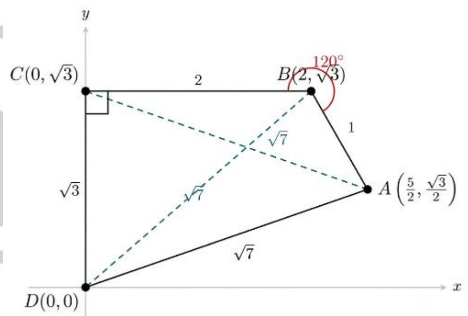
$$4\left(a^2+8a+16\right) = 7a^2+39a+54 \Rightarrow 3a^2+7a-10=0$$

Factoring gives $(a - 1)(3a + 10) = 0$. Since side lengths must be positive real numbers ($a > 0$), we obtain $a = 1$

Substituting $a = 1$, the lengths simplify to

$$AB = 1, \quad BC = 2, \quad CD = \sqrt{3}, \quad DA = \sqrt{7}, \quad AC = \sqrt{7}, \quad BD = \sqrt{7}$$

and the corresponding coordinates are $D(0, 0)$, $C\left(0,\sqrt{3}\right)$, $B\left(2,\sqrt{3}\right)$, and $A\left(\frac{5}{2},\frac{\sqrt{3}}{2}\right)$



Now we evaluate the interior angles of $ABCD$: In ΔABC, applying the Law of Cosines gives

$$\cos(\angle ABC) = \frac{AB^2+BC^2-AC^2}{2\cdot AB\cdot BC} = \frac{1^2+2^2-(\sqrt{7})^2}{2\cdot 1\cdot 2} = -\frac{1}{2} \Rightarrow \angle ABC = 120^\circ$$

As established from right-angled ΔBCD

$\angle BCD = 90°$

In ΔDAB, since $AD = BD = \sqrt{7}$,

$$\cos\left(\angle DAB\right) = \frac{AB^2+AD^2-BD^2}{2\cdot AB\cdot AD} = \frac{1}{2\sqrt{7}} \Rightarrow \angle DAB \approx 79.11^\circ$$

In ΔCDA, since $DA = AC = \sqrt{7}$,

$$\cos\left(\angle CDA\right) = \frac{CD^2+DA^2-AC^2}{2\cdot CD\cdot DA} = \frac{3}{2\sqrt{21}} = \frac{\sqrt{21}}{14} \Rightarrow \angle CDA \approx 70.89^\circ$$

Arranging the four angles in descending order yields

$$\angle B\left(120^\circ\right) > \angle C\left(90^\circ\right) > \angle A\left(\approx 79.11^\circ\right) > \angle D\left(\approx 70.89^\circ\right)$$

The largest angle is 120° and the second largest is 90°, so the difference is

$\theta = 120 - 90 = 30$

28. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be strictly increasing sequences of positive integers such that

(a) $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$

(b) $b_n = 2b_{n-1}$ for all $n \geq 2$

(c) $a_{10} = b_{10} < 2026$

Find the sum of all possible values of $a_1 + b_1$

Answer (33)

Sol. Let $a_1 = x$ and $a_2 = y$ where $1 \leq x < y$ ensures that the sequence of positive integers is strictly increasing.

By the Fibonacci linear recurrence,

$a_{10} = F_8x + F_9y = 21x + 34y$

For the geometric progression b_n,

$b_{10} = 2^9 b_1 = 512b_1$

Given that $a_{10} = b_{10} < 2026$, we have

$21x + 34y = 512b_1 < 2026$,

which restricts the positive integer b_1 to $b_1 \in \{1,2,3\}$

Case $b_1 = 1$: $21x + 34y = 512$

Taking this modulo 21: $13y \equiv 512 \equiv 8 \pmod{21} \Rightarrow y \equiv 7 \pmod{21}$

For $y = 7$, $21x = 512 - 34(7) = 274$, which is not divisible by 21. For $y \geq 28$, $34y > 512$. Thus, no positive integer solution exists

Case $b_1 = 2$: $21x + 34y = 1024$

Module 21:

$13y \equiv 1024 \equiv 16 \pmod{21} \Rightarrow y \equiv 19 \pmod{21}$

Taking $y = 19$ yields

$21x = 1024 – 34 (19) = 378 \Rightarrow x = 18$

Since $1 \leq 18 < 19$, this forms a valid strictly increasing sequence, giving

$a_1 + b_1 = 18 + 2 = 20$

Case $b_1 = 3$: $21x + 34y = 1536$

Modulo 21: $13y \equiv 1536 \equiv 3 \pmod{21} \Rightarrow y \equiv 18 \pmod{21}$

Testing candidates for y: If $y = 18$: $21x = 1536 – 34(18) = 924 \Rightarrow x = 44$. Here $x > y$, violating the strictly increasing condition

If $y = 39$:

$21x = 1536 – 34(39) = 210 \Rightarrow x = 10$

Here $1 \leq 10 < 39$, which is valid. This gives

$a_1 + b_1 = 10 + 3 = 13$

The possible values of $a_1 + b_1$ are 13 and 20, and their sum is 13 + 20 = 33.

29. Let $n = \dfrac{4^{31} - 1}{3}$. Find the remainder when 2^{n-1} is divided by n.

Answer (01)

Sol. From the given relation:

$$3n = 2^{62} - 1$$

$$2^{62} \equiv 3n + 1 \equiv 1 \pmod{n}$$

To evaluate n (mod62), consider the moduli 2 and 31:

Expressing n as a geometric series:

$$n = 1 + 4 + 4^2 + \ldots + 4^{30} \equiv 1 \pmod{2}$$

By Fermat's Little Theorem, $4^{30} \equiv 1 \pmod{31}$, giving:

$$3n = 4^{31} - 1 \equiv 4 - 1 \equiv 3 \pmod{31} n \equiv 1 \pmod{31}$$

Combining both congruences via the Chinese Remainder Theorem:

$$n \equiv 1 \pmod{62} n - 1 = 62k \quad (k \in \mathbb{Z})$$

Therefore:

$$\frac{2^{n-1} - 1}{n} = \frac{\left(2^{62}\right)^k - 1}{n} \equiv \frac{1^k - 1}{n} \equiv 0 \pmod{n} 2^{n-1} \equiv 1 \pmod{n}$$

The remainder is 1.

30. In triangle ABC, it is given that $\angle CAB = 50°$ and $\angle ABC = 70°$. Points D and E are chosen on sides BC and AC respectively such that $\angle ABE = \angle DAB = 30°$. If $\angle DEB = x°$, what is the value of x?

Answer (40)

Sol. In ΔABD

$$\angle BAD = 30^\circ, \quad \angle ABD = 70^\circ \Rightarrow \angle ADB = 80^\circ$$

By the sine rule,

$$BD = \frac{AB\sin 30^\circ}{\sin 80^\circ} \qquad \text{(i)}$$

In ΔABE:

$$\angle ABE = 30^\circ, \quad \angle BAE = 50^\circ \Rightarrow \angle AEB = 100^\circ$$

Again by the sine rule,

$$AE = \frac{AB\sin 30^\circ}{\sin 100^\circ} = \frac{AB\sin 30^\circ}{\sin 80^\circ} = BD$$

and also

$$BE = \frac{AB\sin 50^\circ}{\sin 100^\circ} = \frac{AB\sin 50^\circ}{\sin 80^\circ}$$

From (i) and (ii)

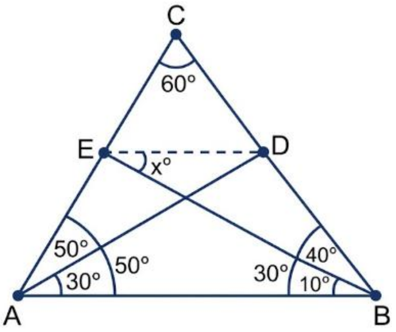
$$\frac{BD}{BE}=\frac{\sin 30^\circ}{\sin 50^\circ}=\frac{1}{2\sin 50^\circ}=\frac{\sin 40^\circ}{\sin 80^\circ}=\frac{\sin 40^\circ}{\sin 100^\circ} \qquad \text{(ii)}$$

Now $\angle EBD = 40^\circ$ because $\angle ABC = 70^\circ$ and $\angle ABE = 30^\circ$ Thus in ΔEBD we have

$$\frac{BD}{BE}=\frac{\sin 40^\circ}{\sin 100^\circ}$$

Which implies $\angle DEB = 40^\circ$ by the sine rule

Therefore x = 40



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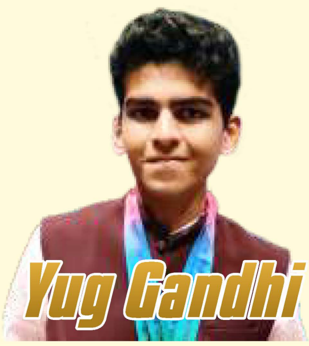

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