Chapter 11

Integral Calculus

*The word '**Integration**' literally means '**Summation**'. It is applied to almost all branches of science such as Physics, Chemistry, Biology, Engineering, Economics, Statistics, Algebra, trigonometry, Co-ordinate geometry, Geometry dynamics, and even to Social sciences, Quality control, pediatrics, Mensuration etc. whenever the ratio of change of a quantity is known we can always calculate the total change in a specified interval of time. Finding areas, volumes, lengths, moment of inertia, pressure, work, moments of force etc. are some of the innumerable applications of integration.*

INTEGRAL AS AN ANTIDERIVATIVE

The process of finding the antiderivative is called integration. Integration is the inverse process of differentiation.

$$\boxed{\begin{array}{l} \text{If } g'(x) = f(x) \text{ then} \\ \int f(x)dx = g(x) + c \end{array}}$$

i.e. $\int f(x)dx = g(x)$ iff $g'(x) = f(x)$

$$\int f(x)dx = g(x) + c \text{ as } (g(x) + c)' = f(x)$$

Here 'c' is called constant of integration

$f(x)$ is called integrand.

So $\int f(x)dx$ is not unique due to presence of 'c'.

Derivative of a function is unique, but integration is not unique

Geometrical Significance

The derivative of a function give the slope whereas, the integration of a bounded function gives a particular algebraic area.

Derivative of Integral, Integral of Derivative

$$\frac{d}{dx}\int f(x)dx = f(x)$$

$$\int\left\{\frac{d}{dx}(f(x))\right\}dx = f(x) + c$$

$$\int\left\{f(x)\phi'(x) + f'(x)\phi(x)\right\}dx = f(x).\phi(x) + c$$

$$\int\left\{\frac{f'(x).\phi(x) - f(x)\phi'(x)}{\{\phi(x)\}^2}\right\}dx = \frac{f(x)}{\phi(x)} + c$$

$$\int f'(g(x))g'(x)dx = fog(x) + c$$

Fundamental Rules of Integration

$$\int\{f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots\ldots\ldots \pm f_n(x)\}dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm \ldots\ldots\ldots \pm \int f_n(x)dx$$

$\int k f(x)dx = k\int f(x)dx$, where k is a constant $(k \neq 0)$

$$\int f'(ax + b)dx = \frac{f(ax + b)}{a} + c \quad (a \neq 0)$$

FUNDAMENTAL INTEGRALS

Fundamental Integrals of Functions

Standard integrals are as follows

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c \quad (n \neq -1,\ a \neq 0)$$

$$\int \frac{f'(x)}{f(x)}dx = \log_e |f(x)| + c$$

$$\int \frac{1}{x}dx = \log_e |x| + c; \quad (x \neq 0)$$

$$\int \frac{1}{ax + b}dx = \frac{1}{a}\log_e |ax + b| + c; \quad (a \neq 0)$$

$\therefore$ If $\int f(x)dx = F(x)$

Then $\int f(ax + b)dx = \frac{1}{a}F(ax + b) + c$

Fundamental Integrals of Trigonometric Functions

$$\int \sin x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \sin x + c$$

$$\int \sec^2 x\, dx = \tan x + c$$

$$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$$

$$\int \sec x \tan x\, dx = \sec x + c$$

$$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$$

Fundamental Integrals of Exponential Functions

$$\int e^x\, dx = e^x + c$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c\ ;\ (a \neq 0)$$

$$\int a^x dx = \frac{a^x}{\log_e a} + c\ ;\ a > 0\ \&\ a \neq 1$$

$$\int a^{px+q} dx = \frac{a^{px+q}}{p \log_e a} + c\ ;\ (p \neq 0)$$

Some more Fundamental Integrals

$$\int \cot x\, dx = \log_e |\sin x| + c = -\log_e |\operatorname{cosec} x| + c$$

$$\int \tan x\, dx = -\log_e |\cos x| + c = \log_e |\sec x| + c$$

$$\int \operatorname{cosec} x\, dx = \log_e |\operatorname{cosec} x - \cot x| + c = \log_e \tan\left(\frac{x}{2}\right) + c$$

$$\int \sec x\, dx = \log_e |\sec x + \tan x| + c = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c = -\cos^{-1}\left(\frac{x}{a}\right) + c' \qquad [|x| < a,\ a > 0]$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + c' \qquad (a \neq 0)$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c = -\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + c' \qquad [x > a > 0]$$

Method of Expansion or Transformation

1. **Expansion Method :** We expand the integrand into the linear combination of simpler functions and integrate each term using the fundamental formulae.
2. **Transformation Method :** We change the integrand into sum of standard integrands by simplification using algebraic or trigonometrical simplification or transformations.

Some useful Trigonometric Identities for Transformation

$$\sin^2 mx = \frac{1-\cos 2mx}{2}$$

$$\cos^2 mx = \frac{1+\cos 2mx}{2}$$

$$\sin mx = 2\sin\frac{mx}{2}\cos\frac{mx}{2}$$

$$\sin^3 mx = \frac{3\sin mx - \sin 3mx}{4}$$

$$\cos^3 mx = \frac{3\cos mx + \cos 3mx}{4}$$

$\tan^2 mx = \sec^2 mx - 1$

$\cot^2 mx = \text{cosec}^2 mx - 1$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Method of Substitution : (Change of independent variable)

Integration can be made easier with the help of substitution. Here x is substituted by $\phi(t)$, where $\phi(t)$ is a continuously differentiable function and we have

$$\int f(x)dx = \int f\{\phi(t)\}.\phi'(t)dt \text{ . where } x = \phi(t)$$

There is no general rule for finding a proper substitution and the best method in this matter is experience. However, the following suggestions will prove useful.

(i) If the integrand is of the form $f'(ax+b)$, then we put

$ax + b = t$ so that $dx = \frac{1}{a}dt$.

Thus, $\int f'(ax+b)dx = \int f'(t)\frac{dt}{a}$

$$= \frac{1}{a}\int f'(t)dt$$

$$= \frac{f(t)}{a} + c = \frac{f(ax+b)}{a} + c$$, where c is a constant of integration.

(ii) When the integrand is of the form $x^{n-1} f'(x^n)$, we put

$x^n = t$ so that $nx^{n-1}dx = dt$

Thus, $\int x^{n-1} f'(x^n) dx = \int f'(t)\frac{dt}{n}$

$= \frac{1}{n}\int f'(t) dt$

$= \frac{1}{n} f(t) + c = \frac{1}{n} f(x^n) + c$, where c is a constant of integration.

(iii) When the integrand is of the form $[f(x)]^n \cdot f'(x)$, we put

$f(x) = t \Rightarrow f'(x)\,dx = dt$

Thus, $\int [f(x)]^n f'(x) dx = \int t^n dt = \frac{t^{n+1}}{n+1} + c$

$= \frac{[f(x)]^{n+1}}{n+1} + c$, where c is a constant of integration.

(iv) When the integrand is of the form $\frac{f'(x)}{f(x)}$, we put

$f(x) = t \Rightarrow f'(x)dx = dt$

Thus $\int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t} = \log_e |t| + c$, where c is a constant of integration.

$= \log_e |f(x)| + c$

Some Standard Substitution

	Form of function	Substitution
(i)	$\sqrt{a^2 - x^2}$ or $\frac{1}{\sqrt{a^2 - x^2}}$	$x = a \sin \theta$ or $a \cos \theta$
(ii)	$\frac{1}{a^2 + x^2}$ or $\frac{1}{\sqrt{a^2 + x^2}}$	$x = a \tan \theta$ or $a \cot \theta$
(iii)	$\frac{1}{\sqrt{x^2 - a^2}}$ or $\sqrt{x^2 - a^2}$	$x = a \sec \theta$ or $a \operatorname{cosec} \theta$
(iv)	$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
(v)	$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a \sin^2 \theta$ or $a \cos^2 \theta$

(vi) $\sqrt{\dfrac{x}{a+x}}$ or $\sqrt{\dfrac{a+x}{x}}$ $\quad x = a\tan^2\theta$ or $a\cot^2\theta$

(vii) $\sqrt{\dfrac{x-a}{b-x}}$ or $\sqrt{(x-a)(b-x)}$ $\quad x = a\cos^2\theta + b\sin^2\theta$ (Imp.)

(viii) $\sqrt{\dfrac{x-a}{x-b}}$ or $\sqrt{(x-a)(x-b)}$ $\quad x = a\sec^2\theta - b\tan^2\theta$ (Imp.)

(ix) $\dfrac{1}{\sqrt{(x-a)(x+b)}}$ $\quad x - a = t^2$ or $x + b = t^2$

(x) $\sqrt{\dfrac{x-a}{x+a}}$ or $\sqrt{\dfrac{x+a}{x-a}}$ $\quad$ then $x = a\sec\theta$ or $a\sec 2\theta$

Integration by Parts

If u and v are the differentiable function of x, then

$$\int uv\,dx = u\int v dx - \int\left[\frac{du}{dx}\int v dx\right]dx$$

i.e., The integral of the product of two functions = (first function) × (integral of the second function) - integral of {(differentiation of first function) × (integral of second function)}.

Note : *While using integration by parts, choose u and v such that*

(i) $\int v dx$ *is simple and*

(ii) $\int\left[\dfrac{du}{dx}\cdot\int v dx\right]dx$ *is simple to integrate*

(iii) This is generally obtained, by keeping the order of uv as per the order of the letters in ***ILATE****, where* ***I****-Inverse function,* ***L****-Logarithmic function,* ***A****-Algebraic function;* ***T****-Trigonometric function and* ***E****-Exponential function.*

(iv) $\int e^x[f(x)+f'(x)]dx = e^x f(x)+c$

(v) $\int [xf'(x)+f(x)]dx = xf(x)+c$

(vi) In $\int f(x)x^n dx$ *take* x^n *as first function*

(vii) In $\int (\log x)^n dx$ *take 1 as second function*

(viii) If required, integration by parts can be used repeatedly.

(ix) When both functions are trigonometric take that function as second whose integral is simpler.

(x) If both functions are algebraic take that function as first whose derivative is simpler.

(xi) If the integral consists of an inverse trigonometric function of an algebraic expression in x, use suitable trigonometric substitution to simplify and then integrate.

Method of Multiple Integration by Parts

Sometimes, we have to use method of integration by parts successively many times. In such case a rapid way is to use method of multiple integration by parts.

$$\int f(x)g(x)dx = f(x).g_1(x) - f^1(x)g_2(x) + f^2(x)g_3(x) + \ldots.. + (-1)^{n-1}f^{n-1}(x)g_n(x) + (-1)^n \int f^n(x)g_n(x)dx$$

where $f^r(x)$ means r^{th} derivative of $f(x)$ and $g_k(x)$ means k^{th} integration of $g(x)$

It is possible only when $f(x)$ is a polynomial function in x.

INTEGRATION OF RATIONAL FUNCTION

Method of Partial Fractions for Rational Functions

Integrals of the type $\int \frac{p(x)}{g(x)}dx$ can be integrated by resolving the integrand into partial fractions. We proceed as follows:

(i) Check degree of $p(x)$ and $g(x)$.

(ii) If degree of $p(x)$ > degree of $g(x)$, then divide $p(x)$ by $g(x)$ till its degree is less, *i.e.* put in the form

$$\frac{p(x)}{g(x)} = r(x) + \frac{f(x)}{g(x)}$$

where degree of $f(x)$ < degree of $g(x)$

Case 1: When the denominator contains non-repeated linear factors. That is

$g(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$

In such a case write $\frac{f(x)}{g(x)}$ as:

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \ldots + \frac{A_n}{x-\alpha_n}$$

where $A_1, A_2, \ldots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

Case 2: When the denominator contains repeated as well as non-repeated linear factors. That is

$g(x) = (x - \alpha_1)^2 (x - \alpha_3) \ldots (x - \alpha_n)$.

In such a case express $\frac{f(x)}{g(x)}$ as :

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{(x-\alpha_1)^2} + \frac{A_3}{x-\alpha_3} + \ldots + \frac{A_n}{(x-\alpha_n)}$$

where $A_1, A_2, \ldots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

Case 3: When the denominator contains a non repeated quadratic factor which cannot be factorised further

$g(x) = (ax^2 + bx + c)(x - \alpha_3)(x - \alpha_4) \ldots (x - \alpha_n)$

In such a case express $\frac{f(x)}{g(x)}$ as :

$$\frac{f(x)}{g(x)} = \frac{A_1 x + A_2}{ax^2 + bx + c} + \frac{A_3}{x - \alpha_3} + \ldots + \frac{A_n}{x - \alpha_n}$$

where $A_1, A_2, \ldots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

Case 4: When the denominator contains a repeated quadratic factor which cannot be factorised further.

That is $g(x) = (ax^2 + bx + c)^2 (x - \alpha_5)(x - \alpha_6) \ldots (x - \alpha_n)$

In such a case write $\frac{f(x)}{g(x)}$ as :

$$\frac{f(x)}{g(x)} = \frac{A_1 x + A_2}{ax^2 + bx + c} + \frac{A_3 x + A_4}{(ax^2 + bx + c)^2} + \frac{A_5}{x - \alpha_5} + \ldots + \frac{A_n}{(x - \alpha_n)}$$

where $A_1, A_2, \ldots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

> ***Note:*** *Corresponding to repeated linear factor $(x - a)^r$ in the denominator, a sum of r partial fractions of the type* $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_r}{(x-a)^r}$ *is taken.*

Case 5: If the integrand contains only even powers of x

1. Put $x^2 = z$ in the integrand
2. Resolve the resulting rational express in z into partial fractions
3. Put $z = x^2$ again in the partial fractions and then integrate both sides.

Integration of rational function containing only even power:

(i) $\int \frac{x^2 + 1}{x^4 + 1} dx$

(ii) $\int \frac{x^2 + 1}{x^4 + x^2 + 1} dx$

(iii) $\int \frac{dx}{x^4 + 1}$

To find integral of such function, first we divide numerator and denominator by x^2 then express numerator as $d\left(x \pm \frac{1}{x}\right)$ and denominator as a function of $\left(x \pm \frac{1}{x}\right)$

When denominator can not be factorised :

(i) $\int \frac{dx}{ax^2 + bx + c}$ in this case $ax^2 + bx + c = a^2\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$

(ii) $\int \frac{px + q}{ax^2 + bx + c} dx$ in this case put $px + q = A$ [differentiation of $ax^2 + bx + c$] + B

$\Rightarrow \quad px + q = A(2ax + b) + B$

Some Important Formulae

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$; $a \neq 0$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$; $a \neq 0$

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$; $a \neq 0$

4. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$; $a \neq 0$

5. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$; $a \neq 0$

6. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$; $a \neq 0$

7. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$; $a \neq 0$

8. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|\,x+\sqrt{x^2+a^2}\,| + C$; $a \neq 0$

9. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|\,x+\sqrt{x^2-a^2}\right| + C$; $a \neq 0$

Integral of type $\int R\left\{x,[(ax+b)/(cx+d)]^{1/n}\right\}dx$ **(where R is a rational function of its arguments):**

Reduce to the integral of rational fraction by the substitution $(ax + b)/(cx + d) = t^n$

Integral of the form $\int \frac{dx}{X\sqrt{Y}}$ **(where X and Y are linear or quadratic expression in x)**:

Substitution table is given by :

X	Y	Substitution
Linear	Linear	$z^2 = Y$
Quadratic	Linear	$z^2 = Y$
Linear	Quadratic	$z = 1/X$
Quadratic	Quadratic	$z^2 = Y/X$ or $z = \frac{1}{X}$

Integral of the type $\int R(x, \sqrt{(ax^2+bx+c)}\, dx$ (where R is a rational function of x and $\sqrt{(ax^2+bx+c)}$)

Reduced to an integral of rational fraction by the Euler substitutions:

$$\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}(a>0), \sqrt{ax^2+bx+c} = tx \pm \sqrt{c}(c>0) \text{ or}$$

$\sqrt{(ax^2+bx+c)} = t(x-\alpha)(4ac-b^2<0)$ where α is the root of the quadratic $ax^2+bx+c = 0$

Integrals of the Form

1. $\int \frac{x^m}{(a+bx)^p} dx$

2. $\int \frac{dx}{x^m (a+bx)^p}$

3. $\int x^m(a+bx^n)^p\, dx$

Integral	Substitution
$\int \frac{x^m}{(a+bx)^p} dx$, m is a + ve integer	Put $a + bx = z$
$\int \frac{dx}{x^m (a+bx)^p}$, where either ($m$ and p are positive integers) or (m and p are fractions, but $m+p$ = integer >1)	Put $a + bx = zx$
$\int x^m(a+bx^n)^p\, dx$, where m, n, p are rationals,	
(a) p is a + ve integer	Apply Binomial theorem to $(a + bx^n)^p$.
(b) p is a – ve integer	Put $x = z^k$ where k = common denominator of m and n.
(c) $\frac{m+1}{n}$ is an integer	Put $a + bx^n = z^k$, where k = denominator of p.
(d) $\frac{m+1}{n} + p$ is an integer	Put $a + bx^n = x^n z^k$

Integration of irrational function

(A) If integrand are in the form of $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ and $\int \sqrt{ax^2+bx+c}\, dx$, then we integrate it by expressing $ax^2 + bx + c = (x + \alpha)^2 + \beta^2$, also for integrals of the form

(B) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$, and $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ first we express $px + q$ in form $px + q = A$ {diff $(ax^2 + bx + c)$} + B and then reduce as usual standard form

Integration of Trigonometric Functions

(a) $\int (\sin^m x.\cos^n x)dx,$ where $m, n, \in N$.

(i) If one of them is odd (m or n) then substitute for term of even power

(ii) If both are odd, substitute either of them

(iii) If both are even, use trigometric identities..

(iv) If m and n are rational and $(m + n)$ is a negative even integer then substitute $\cot x = p$ or $\tan x = p$.

(b) Integral of the form $\int \frac{p\sin x + q\cos x}{a\sin x + b\cos x}, \int \frac{p\sin x}{a\sin x + b\cos x}$ and $\int \frac{p\cos x}{a\sin x + b\cos x}$ for integration of these type question we first substitute numerator as follows.

Numerator = A (denominator) + B (differentiation of denominator w.r.t. x)

(c) Integral of the form $\int \frac{p\sin x + q\cos x + r}{a\sin x + b\cos x + c}.dx,$ then substitute $p\sin x + q\cos x + r$ as follows

$p\sin x + q\cos x + r = A$ (denominator) + B (differentiation of denominator w.r.t. x) + C

(d) Integral of the form $\int \frac{dx}{a\sin^2 x + b\cos^2 x}, \int \frac{dx}{a + b\cos^2 x}$ and $\int \frac{dx}{a + b\sin^2 x}$.

These type of question divide both the numerator and denominator by $\cos^2 x$ and put $\tan x = t$

Some Special Integrals

$\int e^x \{f(x) + f'(x)\}\,dx = e^x f(x) + c$

Proof :

$$I = \int e^x f(x)dx + \int e^x f'(x)\,dx$$

$$= f(x)e^x - \int f'(x)e^x dx + \int e^x f'(x)\,dx = e^x f(x) + c$$

Algebraic Twins

$$\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx.$$

$$\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx.$$

Trigonometric Twins

$$\int \sqrt{\tan x}\,dx = \frac{1}{2}\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx + \frac{1}{2}\int\left(\sqrt{\tan x}-\sqrt{\cot x}\right)dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sin x+\cos x}{\sqrt{2\sin x.\cos x}}dx + \frac{1}{\sqrt{2}}\int \frac{\sin x-\cos x}{\sqrt{2\sin x.\cos x}}dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos x+\sin x}{\sqrt{1-(\sin x-\cos x)^2}}dx - \frac{1}{\sqrt{2}}\int \frac{\cos x-\sin x}{\sqrt{(\sin x+\cos x)^2-1}}dx$$

$$\int \frac{dx}{\cos x+\operatorname{cosec} x} = \int \frac{2\sin x}{2+\sin 2x}dx$$

$$= \int \frac{\sin x+\cos x}{3-(1-\sin 2x)}dx + \int \frac{\sin x-\cos x}{1+(1+\sin 2x)}dx$$

$$= \int \frac{\sin x+\cos x}{3-(\sin x-\cos x)^2}dx + \int \frac{\sin x-\cos x}{1+(\sin x+\cos x)^2}dx$$

DEFINITE INTEGRAL

Here we shall define integration as a process of summation or definite integral as the limit of a sum. Then we discuss some properties of definite integral. The concept of definite integral is then used to find the area enclosed by certain curves.

Fundamental theorem of calculus

Let $f(x)$ be a continuous real valued function defined on $[a, b]$ such that $\int f(x)dx = F(x)+C$. Then $\int_a^b f(x)dx = F(b)-F(a)$, called definite integral of $f(x)$ in $[a, b]$.

Remark :

1. If $f(x)$ is discontinuous at $x = a$ and continuous at $x = b$

   then $\int_a^b f(x)dx = F(b) - \lim\limits_{x\to a+} F(x)$

2. If $f(x)$ is discontinuous at $x = b$ and continuous at $x = a$ then

   $\int_a^b f(x)dx = \lim\limits_{x\to b-} F(x) - F(a)$

3. If $f(x)$ is discontinuous at $x = a$ and $x = b$ then

   $\int_a^b f(x)dx = \lim\limits_{\in\to 0}\int_{a+\in}^{b-\in} f(x)dx$ or $\lim\limits_{x\to b-} F(x) - \lim\limits_{x\to a+} F(x)$

4. If $f(x)$ is discontinuous at $x = c$ $(a < c < b)$ then

   $\int_a^b f(x)dx = \lim\limits_{\in\to 0}\int_a^{c-\in} f(x)dx + \lim\limits_{\in\to 0}\int_{c+\in}^{b} f(x)dx$

Note that even if $f(x)$ is not defined at $x = a$ or $x = b$ or at both, $\int_a^b f(x)dx$ can be evaluated. a and b are called lower and upper limits of integration respectively. If we make change in variable (i.e. substitution) then limit of integration should be changed accordingly.

Definite Integral as the Limit of a Sum

(i) Express the given series in the form of $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$

(ii) The limit when $n \to \infty$ is its sum $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \frac{1}{n} . f\left(\frac{r}{n}\right)$ which is equal to $\int_0^1 f(x)dx$

Replace r/n by x, $1/n$ by dx and $\lim\limits_{n\to\infty} \sum$ by the sign of integration $\int$

(iii) The lower and upper limits of integration will be the value of r/n for the first and last term (or the limits of these values respectively).

Lower limit is 0 and

Upper limit is $\lim\limits_{n\to\infty} \frac{n-1}{n} = 1$

Some Important Formulae

1. $\sum\limits_{r=1}^{n} r = \frac{n(n+1)}{2}$.

2. $\sum\limits_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

3. $\sum\limits_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.

4. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta) = \frac{\sin(n\beta/2)}{\sin(\beta/2)} \sin\left[\frac{1^{\text{st}}\text{angle} + \text{last angle}}{2}\right]$.

5. $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta) = \frac{\sin(n\beta/2)}{\sin(\beta/2)} \cos\left[\frac{1^{\text{st}}\text{angle} + \text{last angle}}{2}\right]$.

6. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\ldots \text{ to } \infty = \log_e 2$.

7. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$.

8. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$.

9. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$.

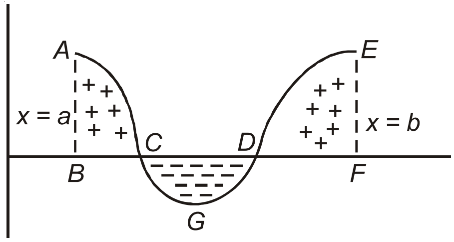
10. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{24}$.

11. $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

12. $\cos h\theta = \frac{e^{\theta} + e^{-\theta}}{2}$ and $\sin h\theta = \frac{e^{\theta} - e^{-\theta}}{2}$.

Geometrical Interpretation of Definite Integral

$\int_a^b f(x)\,dx$ represents algebraic sum of the areas of the figures bounded by the graph of the function $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. The areas above x-axis contribute to the sum as positive and below it contribute to the sum as negative.



$$\int_a^b f(x)\,dx = \text{Area } ABC - \text{Area } CGD + \text{Area } DEF$$

Basic Properties of Definite Integral

Property 1 $\int_a^b f(x)dx = \int_a^b f(z)dz$

Property 2 $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Property 3 $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots\ldots + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^{b} f(x)dx$

Where $a < c_1 < c_2 < \ldots < c_{n-1} < c_n < b$

Property 4 $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Property 5 $\int_{-a}^a f(x)dx = \begin{cases} 0 & \text{if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{if } f(-x) = f(x) \end{cases}$

Property 6 $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Proof : $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$

In second integral of RHS substitute $x = 2a - y$, $dx = -\ dy$

Then $\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_a^0 f(2a-y)dy = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Property 7 In particular, $\int_0^{2a} f(x)dx = \left\{\begin{matrix} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \end{matrix}\right\}$.

Property 8	In Particular	$\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx;\ n \in I$; T is the fundamental period of f $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx;\ m, n \in I$; T is the fundamental period of f

Evaluation of Definite Integral by change of variable

When the variable (x) in a definite integral is changed with (t) then, substitution affects at three places

(1) Integrand is changed

(2) $d(x)$ is changed with $d\,f(t)$

(3) Upper and lower limits are changed

Some more properties of Definite Integral

Property 9 If $f(x) \geq 0,\ \forall\ x \in [a, b]$, then

$$\int_a^b f(x)\,dx \geq 0$$

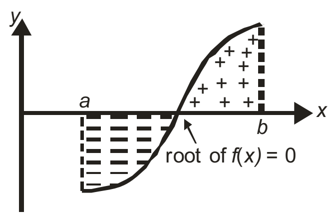
Property 10 If $f(x)$ is an odd function of x, then $\int_a^x f(t)\,dt$ is an even function of x.

Property 11 If $f(x)$ is an even function of x, then $\int_0^x f(t)\,dt$ is an odd function of x.

Property 12 (Differentiation under the integral sign) (Leibnitz Rule):

$$\frac{d}{dx}\left\{\int_{\phi(x)}^{\psi(x)} f(t)dt\right\} = f(\psi(x))\,\psi'(x) - f(\phi(x))\,\phi'(x)$$

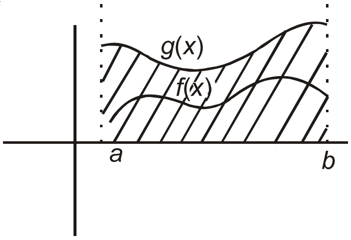
Property 13 If $f(x)$ is continuous in $[a, b]$ and $\int_a^b f(x)dx = 0$, then equation $f(x) = 0$ has at least one root in $[a, b]$



Property 14 If $f(x) < g(x)\ \forall x \in (a, b)$ then

$$\int_a^b f(x)dx < \int_a^b g(x)dx$$

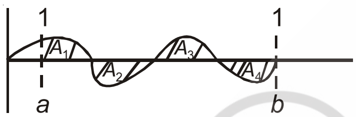
Geometrical Proof:



It is clear from the figure that area under $y = f(x)$ is less than area under $y = g(x)$.

Property 15	$\left\|\int_a^b f(x)dx\right\| \le \int_a^b \|f(x)\|\,dx$ Equality sign holds where $f(x)$ has the same sign on $[a, b]$

Proof: Let the graph of $y = (x)$ is as shown



then LHS $= |A_1 - A_2 + A_3 - A_4|$

& RHS $= A_1 + A_2 + A_3 + A_4$

Hence LHS $\le$ RHS

Property 16	(Estimation of an Integral) If m is the least value and M is the greatest value of function f(x) in the interval $[a, b]$ then, $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Property 17	(Mean Value Theorem) If function f(x) is continuous in the interval $[a, b]$ then $\exists\ a<c<b$ such that $\int_a^b f(x)dx = f(c)(b-a)\left[\begin{array}{c} f(c)=\frac{1}{(b-a)}\int_a^b f(x)dx \\ \text{is called mean value of } f(x) \text{ in } [a,b]\end{array}\right]$

Property 18	(Generalized Mean-Value Theorem) If $f(x)$ and $\phi(x)$ are continuous on $[a, b]$ and $\phi(x)$ has same sign throughout $[a, b]$ then $\exists\ a<c<b$ such that $\int_a^b f(x)\varphi(x)dx = f(c)\int_a^b \varphi(x)dx.$

Property 19	If $f(x)$ is continuous in $[a, b]$ then integral function $\varphi(x) = \int_a^x f(t)dt \quad (x \in [a,b])$ is differentiable in (a, b) with $\varphi'(x) = f(x)$.

Property 20	$\left\|\int_a^b f(x)g(x)dx\right\| \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}$ (Here, $f^2(x) \equiv (f(x))^2$)

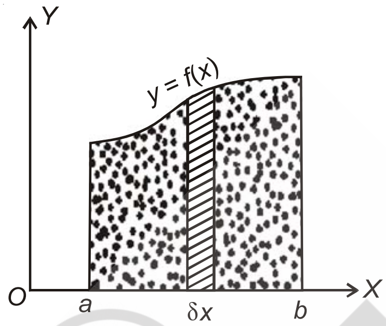
Leibnitz theorem

If $F(x) = \int_{g(x)}^{h(x)} f(t)dt$, then $\dfrac{dF(x)}{dx} = h'(x)f(h(x)) - g'(x)f(g(x))$

Computing Areas bounded by Simple Curves

Computing areas in Rectangular coordinates :

1. If $y = f(x)$ be a continuous function in $[a , b]$. Then area of plane figure bounded by $y = f(x)$, $x = a$, $x = b$ and x-axis $(b > a)$ is given by

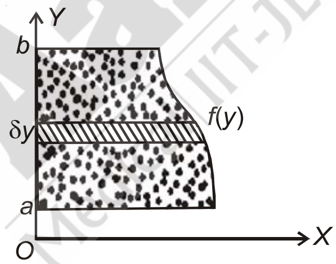


$$A = \int_a^b ydx = \int_a^b f(x)dx$$

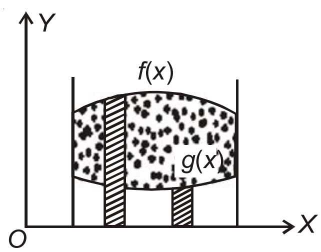
(Here $f(x)$ is above the x-axis therefore $\int_a^b f(x)dx$ is +ve, otherwise we have to consider $\left|\int_a^b f(x)dx\right|$

2. Area bounded by the curve $x = f(y)$, y-axis, $y = a$, $y = b$ $(b > a)$ is given by

$$A = \int_a^b xdy = \int_a^b f(y)dy$$



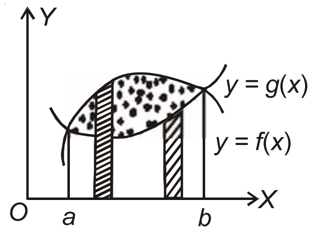
3. Let $y = f(x)$ and $y = g(x)$ be two curves such that graph of $f(x)$ lies above the graph of $g(x)$ and both are above x-axis, then area bounded by $y = f(x)$, $y = g(x)$, $x = a$, $x = b$ $(b > a)$ is given by



$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

Special case of the above case:

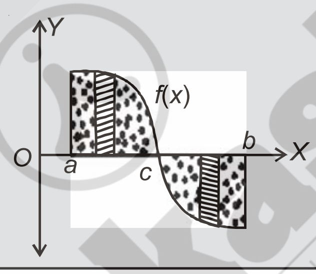
$x = a$ and $x = b$ are abscissas of points of intersection of given curves (that is left boundary $x = a$ and right boundary $x = b$ degenerate into point of intersection of $y = f(x)$, $y = g(x)$.



Remarks :

1. If whole region (whose area is required) is below x-axis then area will come out negative (we consider only numerical value). In such cases above formulae are valid after taking modulus of integral.
2. If some part of the bounded region is below x-axis (see figure) then above formulae may not work. First we divide the regions above x-axis and below x-axis and compute area separately.

$$A = \left| \int_a^c f(x)dx \right| + \left| \int_c^b f(x)dx \right|$$



Note : *Your original formulae will not give you the correct answer.*

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