MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

61. The set of all values of a for which $\lim_{x\to a}([x-5]-[2x+2])=0$, where $[\propto]$ denotes the greatest integer less than or equal to $\propto$ is equal to

(1) [–7.5, –6.5) (2) (–7.5, –6.5]

(3) [–7.5, –6.5] (4) (–7.5, –6.5)

Answer (4)

Sol. $\lim_{x\to a}([x-5]-[2x+2])=0$

$\Rightarrow [x - 5] = [2x + 2]$

$\Rightarrow [x] -5 = [2x] + 2$

$\Rightarrow [x] = [2x] + 7$ …(i)

if $x \in Z$ we have

$x = -7$

also $2x \in Z$ if x is of form $z \pm \frac{1}{2}$

Hence, if $x \in (-7.5, -7)$ eq. (1) become

$-8 = -15 + 7 \Rightarrow 7 = 7$

Similarly, if $x \in (-7, -6.5)$ in eq. (1)

$-7 = -14 + 7 \Rightarrow 7 = 7$

At $x = -6.5$ in eq. (1)

$-7 = -13 + 7 \Rightarrow -14 \neq -13$ not possible

At $x = -7.5$ in eq. (1)

$-8 = -15+7 \Rightarrow 8 = 8$

But $x \to a$ $a \neq -6.5$ or -7.5

$\therefore a \in (-7.5, -6.5)$

62. Let p and q be two statements. Then ~ $(p \wedge (p \Rightarrow \sim q))$ is equivalent to

(1) $(\sim p) \vee q$ (2) $p \vee ((\sim p) \wedge q)$

(3) $p \vee (p \wedge q)$ (4) $p \vee (p \wedge (\sim q))$

Answer (1)

Sol. Making truth table $(E \equiv \sim (p \wedge (p \Rightarrow \sim q)))$

p	q	$\sim p$	$\sim q$	$p \vee q$	$p \wedge q$	$p \Rightarrow \sim q$	$p \wedge (p \Rightarrow \sim q)$	E
T	T	F	F	T	T	F	F	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	F	T	T	F	F	T	F	T

&

$\sim p \vee q$	$p \vee (\sim p \wedge q)$	$p \vee (p \wedge q)$	$p \vee (p \wedge \sim q)$
T	T	T	T
F	T	T	T
T	T	F	F
T	F	F	F

$\therefore \sim (p \wedge (p \Rightarrow \sim q))$ is equivalent to $\sim p \vee q$

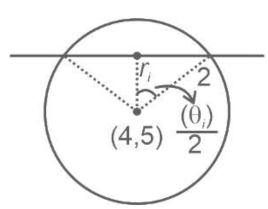
63. The locus of the mid points of the chords of the circle C_1 : $(x - 4)^2 + (y - 5)^2 = 4$ which subtend an angle θ_i at the centre of the circle C_1, is a circle of radius r_i. If $\theta_1 = \frac{\pi}{3}$, $\theta_3 = \frac{2\pi}{3}$ and $r_1^2 = r_2^2 + r_3^2$, then θ_2 is equal to

(1) $\frac{3\pi}{4}$ (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

Answer (4)

Sol.



$\therefore \cos\left(\frac{\theta_1}{2}\right) = \frac{r_i}{2} \Rightarrow r_i = 2\cos\left(\frac{\theta_i}{2}\right)$

Given $r_1^2 = r_2^2 + r_3^3$

$$\Rightarrow \left(\cos\left(\frac{\theta_1}{2}\right)\right)^2 = \left(\cos\left(\frac{\theta_2}{2}\right)\right)^2 + \left(\cos\left(\frac{\theta_3}{2}\right)\right)^2$$

$$\Rightarrow \frac{3}{4} = \cos^2\left(\frac{\theta_2}{2}\right) + \frac{1}{4}$$

$$\Rightarrow \cos^2\left(\frac{\theta_2}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \frac{\theta_2}{2} = \frac{\pi}{4}$$

$$\Rightarrow \theta_2 = \frac{\pi}{2}$$

64. If $f(x) = \frac{2^{2x}}{2^{2x}+2}, x \in \mathbb{R},$ then $f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \ldots + f\left(\frac{2022}{2023}\right)$ is equal to

(1) 1010 (2) 2011

(3) 1011 (4) 2010

Answer (3)

Sol. $f(x) = \frac{2^{2x}}{2^{2x}+2}$, and $f(1-x) = \frac{2^{2(1-x)}}{2^{2(1-x)}+2}$

$\therefore$ $f(x) + f(1-x) = 1$

$$\sum_{K=1}^{2022} f\left(\frac{K}{2022}\right) = \underbrace{f\left(\frac{1}{2022}\right) + f\left(\frac{2022}{2023}\right)}_{1}$$

$$+ \underbrace{f\left(\frac{2}{2022}\right) + f\left(\frac{2021}{2022}\right)}_{1} + \ldots.. \to 1011 \text{ Pairs}$$

= 1011

65. If the system of equations

$x + 2y + 3z = 3$

$4x + 3y - 4z = 4$

$8x + 4y - \lambda z = 9 + \mu$

has infinitely many solutions, then the ordered pair (λ, μ) is equal to :

(1) $\left(-\frac{72}{5}, \frac{21}{5}\right)$ (2) $\left(\frac{72}{5}, -\frac{21}{5}\right)$

(3) $\left(\frac{72}{5}, \frac{21}{5}\right)$ (4) $\left(-\frac{72}{5}, -\frac{21}{5}\right)$

Answer (2)

Sol. $D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & -4 \\ 8 & 4 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = \frac{72}{5}$

$$D_z = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & 4 \\ 8 & 4 & 9+\mu \end{vmatrix} = 0$$

$$\Rightarrow \quad \mu = \frac{-21}{5}$$

66. Let the plane containing the line of intersection of the planes $P1: x + (\lambda+4)y + z = 1$ and $P2 : 2x + y + z = 2$ pass through the points (0, 1, 0) and (1, 0, 1). Then the distance of the point $(2\lambda, \lambda, -\lambda)$ from the plane $P2$ is

(1) $4\sqrt{6}$ (2) $3\sqrt{6}$

(3) $5\sqrt{6}$ (4) $2\sqrt{6}$

Answer (2)

Sol. Equation of plane :

$$(x + (\lambda+4)y + z - 1) + k(2x + y + z - 2) = 0$$

Passes through (0, 1, 0) and (1, 0, 1)

$\Rightarrow \quad \lambda + 4 - 1 + k(-1) = 0$

$\boxed{\lambda - k = -3}$ ...(i)

& $(1+0+0) + k(1) = 0$

$k = -1 \Rightarrow \lambda = -4$

P_2: $2x + y + z = 2$, Point$(-8, -4, 4)$

$$\text{Distance} = \left|\frac{-16-4+4-2}{\sqrt{6}}\right| = 3\sqrt{6} \text{ units}$$

67. If $\left({}^{30}C_1\right)^2 + 2\left({}^{30}C_2\right)^2 + 3\left({}^{30}C_3\right)^2 + \ldots + 30\left({}^{30}C_{30}\right)^2 = \frac{\alpha 60!}{(30!)^2}$ then α is equal to :

(1) 60

(2) 30

(3) 15

(4) 10

Answer (3)

Sol. $\sum_{r=1}^{30} r\cdot\left({}^{30}C_r\right)^2 = \sum_{r=1}^{30} 30\cdot{}^{29}C_{r-1}\cdot{}^{30}C_r$

$= \sum_{r=1}^{30} 30\cdot{}^{29}C_{r-1}\cdot{}^{30}C_{30-r}$

$= 30\cdot{}^{59}C_{30}$

$= 30\cdot\frac{59!}{30!\cdot 29!}\cdot\frac{30}{30}$

$= \frac{15\cdot 60!}{(30!)^2}$

68. If the foot of the perpendicular drawn from (1, 9, 7) to the line passing through the point (3, 2, 1) and parallel to the planes $x + 2y + z = 0$ and $3y - z = 3$ is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to

(1) –1 (2) 1

(3) 3 (4) 5

Answer (4)

Sol. Direction of line

$$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{vmatrix}$$

$= \hat{i}(-5) - \hat{j}(-1) + \hat{k}(3)$

$= -5\hat{i} + \hat{j} + 3\hat{k}$

Equation of line

$$\frac{x-3}{-5} = \frac{y-2}{1} = \frac{z-1}{3}$$

Let foot of perpendicular be = $(-5k + 3, k + 2, 3k + 1)$

$\Rightarrow\ (-5k+2)(-5) + (k-7)(1) + (3k-6)(3) = 0$

Or $25k - 10 + k - 7 + 9k - 18 = 0$

Or $k = 1$

$\alpha + \beta + \gamma = -k + 6 = 5$

69. Let A be a 3 × 3 matrix such that $|\text{adj}(\text{adj}(\text{adj } A))| = 12^4$. Then $|A^{-1}\ \text{adj } A|$ is equal to

(1) 12 (2) $2\sqrt{3}$

(3) $\sqrt{6}$ (4) 1

Answer (2)

Sol. $|A|^{(n-1)^3} = 12^4$

$|A|^8 = 12^4$

$|A| = \sqrt{12}$

$|A^{-1}\ \text{adj}A| = |A^{-1}|\cdot|A|^2$

$= |A|$

70. The value of $\left(\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$ is

(1) $\frac{1}{2}\left(\sqrt{3}+i\right)$ (2) $-\frac{1}{2}\left(1-i\sqrt{3}\right)$

(3) $\frac{1}{2}\left(1-i\sqrt{3}\right)$ (4) $-\frac{1}{2}\left(\sqrt{3}-i\right)$

Answer (4)

Sol. $z = \left(\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$

$1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9} = 1+\cos\frac{5\pi}{18}+i\sin\frac{5\pi}{18}$

$= 1+2\cos^2\frac{5\pi}{36}-1+2i\sin\frac{5\pi}{36}\cos\frac{5\pi}{36}$

$= 2\cos\frac{5\pi}{36}\left(\cos\frac{5\pi}{36}+i\sin\frac{5\pi}{36}\right) = 2\cos\frac{5\pi}{36}e^{i\frac{5\pi}{36}}$

$\Rightarrow\ z = \left(\frac{2\cos\left(\frac{5\pi}{36}\right)e^{i\frac{5\pi}{36}}}{2\cos\left(\frac{5\pi}{36}\right)e^{-i\frac{5\pi}{36}}}\right)^3 = e^{i\frac{5\pi}{6}}$

$z = -\frac{\sqrt{3}}{2}+\frac{1}{2}i = \frac{1}{2}\left(i-\sqrt{3}\right) = -\frac{1}{2}\left(\sqrt{3}-i\right)$

71. The number of square matrices of order 5 with entries from the set {0, 1}, such that the sum of all the elements in each row is 1 and the sum of all the elements in each column is also 1, is

(1) 120 (2) 225

(3) 150 (4) 125

Answer (1)

Sol. $\begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix}$

$\because$ In every row and every column there would be exactly one 1 and four zeroes.

Number of matrices = ${}^5C_1 \cdot {}^4C_1 \cdot {}^3C_1 \cdot {}^2C_1 \cdot {}^1C_1$ = 120

Option (1) is correct.

72. $\int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}dx$ is equal to

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$

(3) $\frac{\pi}{3}$ (4) 2π

Answer (4)

Sol. $I = \int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}dx = \left[48 \cdot \sin^{-1}\left(\frac{2x}{3}\right) \cdot \frac{1}{2}\right]_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}}$

$= 24\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$

$= 24\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = 24 \cdot \frac{\pi}{12} = 2\pi$

Option (4) is correct.

73. The number of real solutions of the equation $3\left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) + 5 = 0$, is

(1) 3 (2) 0

(3) 2 (4) 4

Answer (2)

Sol. $3\left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) + 5 = 0$

$3\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - 2\left(x + \frac{1}{x}\right) + 5 = 0$

Put $x + \frac{1}{x} = t \Rightarrow t \in (-\infty, -2] \cup [2, \infty)$

$3t^2 - 2t - 1 = 0$

$3t^2 - 3t + t - 1 = 0$

$\Rightarrow 3t(t-1) + 1(t-1) = 0 \Rightarrow t = 1, = -\frac{1}{3}$

$\Rightarrow t = 1, -\frac{1}{3}$

$\because t \in (-\infty, -2] \cup [2, \infty)$

No real value of $t \Rightarrow$ no real value of x.

Option (2) is correct.

74. Let $\vec{\alpha} = 4\hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{\beta} = \hat{i} + 2\hat{j} - 4\hat{k}$. Let $\vec{\beta}_1$ be parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ be perpendicular to $\vec{\alpha}$. If $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, then the value of $5\vec{\beta}_2 \cdot (\hat{i} + \hat{j} + \hat{k})$ is

(1) 7 (2) 9

(3) 6 (4) 11

Answer (1)

Sol. $\vec{\beta}_1 = \lambda(4\hat{i} + 3\hat{j} + 5\hat{k}) = \lambda\vec{\alpha}, \vec{\beta}_2 \cdot \vec{\alpha} = 0$

$\Rightarrow \vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2 = \lambda\vec{\alpha} + \vec{\beta}_2$

$\vec{\beta} \cdot \vec{\alpha} = \lambda |\vec{\alpha}|^2 + 0 \Rightarrow \lambda = \frac{-10}{50} = -\frac{1}{5}$

$\vec{\beta} = -\frac{1}{5}\vec{\alpha} + \vec{\beta}_2$

$5\vec{\beta}_2 = \vec{\alpha} + 5\vec{\beta} = (9\hat{i} + 13\hat{j} - 15\hat{k})$

$5\vec{\beta}_2 \cdot (\hat{i} + \hat{j} + \hat{k}) = 9 + 13 - 15 = 7$

Option (1) is correct.

75. Let $f(x)$ be a function such that $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{N}$. If $f(1) = 3$ and $\sum_{k=1}^{n} f(k) = 3279$, then the value of n is

(1) 8 (2) 9

(3) 6 (4) 7

Answer (4)

Sol. $f(x + y) = f(x) \cdot f(y)$

$\Rightarrow f(x) = a^x$

$\Rightarrow f(1) = 3$

$\Rightarrow f(x) = 3^x$

$\sum_{k=1}^{n} f(x)$

$\Rightarrow \quad \frac{3(3^n - 1)}{2} = 3279$

$\Rightarrow \quad 3^n = 2187$

$\Rightarrow \quad n = 7$

76. The number of integers, greater than 7000 that can be formed, using the digits 3, 5, 6, 7, 8 without repetition, is

(1) 48 (2) 168

(3) 220 (4) 120

Answer (2)

Sol. $\underline{7}$ _ _ _ _

Number of integers = 4 × 3 × 2 = 24

$\underline{8}$ _ _ _ _

4 × 3 × 2 = 24

$\therefore$ 24 + 24 = 48

_ _ _ _ _

5 digit integers = 5! = 120

$\therefore$ 120 + 48 = 168

77. If $f(x) = x^3 - x^2 f'(1) + xf''(2) - f'''(3)$, $x \in R$, then

(1) $2f(0) - f(1) + f(3) = f(2)$

(2) $f(3) - f(2) = f(1)$

(3) $3f(1) + f(2) = f(3)$

(4) $f(1) + f(2) + f(3) = f(0)$

Answer (1)

Sol. $f(x) = x^3 - x^2 f'(1) + xf''(2) - f'''(3)$ ...(i)

$f'(x) = 3x^2 - 2xf'(1) + f''(2)$ ...(ii)

$f''(x) = 6x - 2f'(1)$ ...(iii)

$f'''(x) = 6$

$\Rightarrow \quad f'''(3) = 6$

from (iii) : $f''(2) = 12 - 2f'(1)$ ...(iv)

from (ii) : $f'(1) = 3 - 2f'(1) + f''(2)$

$\Rightarrow \quad f''(2) = 3f'(1) - 3$ ...(v)

from (iv) and (v)

$f''(2) = 6, f'(1) = 3$

$f(x) = x^3 - 3x^2 + 6x - 6$

$f(0) = -6, f(1) = -2, f(2) = 2, f(3) = 12$

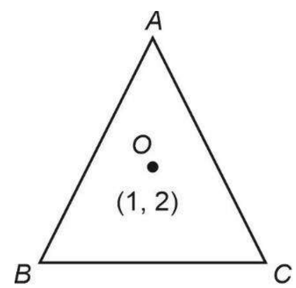
78. The equations of the sides AB and AC of a triangle ABC are $(\lambda + 1)x + \lambda y = 4$ and $\lambda x + (1 - \lambda)y + \lambda = 0$ respectively. Its vertex A is on the y-axis and its orthocentre is (1, 2). The length of the tangent from the point C to the part of the parabola $y^2 = 6x$ in the first quadrant is :

(1) $2\sqrt{2}$ (2) $\sqrt{6}$

(3) 2 (4) 4

Answer (1)

Sol. Let point A be $\left(0, \frac{4}{\lambda}\right)$ or $\left(0, \frac{-\lambda}{1-\lambda}\right)$



$\frac{4}{\lambda} = \frac{-\lambda}{1-\lambda}$

$\Rightarrow \quad \lambda = 2$

$\therefore \quad A(0, 2)$

Now,

$AB : 3x + 2y = 4$

$AC : 2x - y = -2$

$M_{AB} = -\frac{3}{2} \Rightarrow M_{OC} = \frac{2}{3}$

Let $C(h, 2h + 2)$

$M_{OC} = \frac{2h}{h-1}$

$\therefore \quad \frac{2}{3} = \frac{2h}{h-1} \Rightarrow h = -\frac{1}{2}$

$\therefore \quad C\left(\frac{-1}{2}, 1\right)$

Now, $ty = x + \frac{3}{2}t^2$ is tangent passing through $C\left(-\frac{1}{2}, 1\right)$

$\Rightarrow \quad t = 1 \Rightarrow P_{OC}$ is $\left(\frac{3}{2}, 3\right)$

$\therefore \quad$ Length = $2\sqrt{2}$

79. Let $y = y(x)$ be the solution of the differential equation $(x^2 - 3y^2)dx + 3xy\, dy = 0$, $y(1) = 1$. Then $6y^2(e)$ is equal to

(1) $3e^2$ (2) $\frac{3}{2}e^2$

(3) e^2 (4) $2e^2$

Answer (4)

Sol. $\left(x^2 - 3y^2\right)dx + 3xy\,dy = 0,\ y(1) = 1$

$$\Rightarrow\ 3xy\frac{dy}{dx} - 3y^2 = -x^2$$

$$\Rightarrow\ y\frac{dy}{dx} - \frac{y^2}{x} = \frac{-x}{3}$$

$$\Rightarrow\ 2y\frac{dy}{dx} - \frac{2y^2}{x} = \frac{-2x}{3}$$

Let $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} - \frac{2t}{x} = \frac{-2x}{3}$$

$$IF = \frac{1}{x^2}$$

$$\therefore\ \frac{y^2}{x^2} = \frac{-2}{3}\ln|x| + c$$

$x = 1,\ y = 1\ \Rightarrow c = 1$

$$\Rightarrow\ y^2 = \frac{-2}{3}x^2\ln|x| + x^2$$

Now, $x = e$

$$y^2 = \frac{-2e^2}{3} + e^2 = \frac{e^2}{3}$$

$\therefore\ 6y^2 = 2e^2$

80. Let the six numbers a_1, a_2, a_3, a_4, a_5, a_6 be in A.P. and $a_1 + a_3 = 10$. If the mean of these six numbers is $\frac{19}{2}$ and their variance is σ^2, then $8\sigma^2$ is equal to:

(1) 200 (2) 105

(3) 220 (4) 210

Answer (4)

Sol. Given : a, a_2, a_3, a_4, a_5, a_6 ______ A.P.

and $a_1 + a_3 = 10$

Let the common difference be d

then, $2a_1 + 2d = 10$

$\therefore\ a_1 + d = 5$ ...(i)

and $\frac{a_1 + a_2 + ... + a_6}{6} = \frac{19}{2}$

$\Rightarrow\ 2a_1 + 5d = 19$ ...(ii)

By (i) and (ii)

$\boxed{d = 3}$ and $a_1 = 2$

$\therefore\ a_2 = 5,\ a_3 = 8,\ a_4 = 11,\ a_5 = 14,\ a_6 = 17$

$$\therefore\ \text{Variance} = \frac{233}{2} - \frac{361}{4} = \frac{105}{4} = \sigma^2$$

$\therefore\ 8\sigma^2 = 210$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. Let f be a differentiable function defined on $\left[0, \frac{\pi}{2}\right]$ such that $f(x) > 0$ and

$$f(x) + \int_0^x f(t)\sqrt{1 - (\log_e f(t))^2}\,dt = e,\quad \forall x \in \left[0, \frac{\pi}{2}\right].$$

Then $\left(6\log_e f\left(\frac{\pi}{6}\right)\right)^2$ is equal to ____________.

Answer (27)

Sol. $f(x) + \int_0^x f(t)\sqrt{1 - (\log_e f(t))^2}\,dt = e$ ...(1)

So, $f(0) = e$

Now differentiate w.r.t. to x

$$f'(x) + f(x)\sqrt{1 - (\log_e f(x))^2} = 0$$

$$\frac{f'(x)}{f(x)\sqrt{1 - (\log_e f(x))^2}} = -1$$

Let $\log_e f(x) = t$

$$\therefore\ \int\frac{dt}{\sqrt{1 - t^2}} = -x + c$$

$\Rightarrow\ \sin^{-1}t = -x + c$

Now $f(0) = e \Rightarrow \boxed{t = 1}$ So, $\boxed{c = \frac{\pi}{2}}$

$$\therefore\ t = \sin\left(\frac{\pi}{2} - x\right) = \cos x \quad \left(\because x \in \left[0, \frac{\pi}{2}\right]\right)$$

$$\therefore\ \left(6\log_e f\left(\frac{\pi}{6}\right)\right)^2 = 27$$

82. If $\frac{1^3+2^3+3^3+\ldots \text{ up to } n \text{ terms}}{1.3+2.5+3.7+\ldots \text{ up to } n \text{ terms}}=\frac{9}{5}$, then the value of n is

Answer (05)

Sol. Given $\frac{1^3+2^3+3^3+\ldots \text{up to } n \text{ terms}}{1.3+2.5+3.7+\ldots \text{up to } n \text{ terms}}=\frac{9}{5}$ …(1)

Now

Let $S = 1.3 + 2.5 + 3.7 + \ldots$

$T_n = n.(2n + 1)$

$\therefore \quad S=\frac{2n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$

$\Rightarrow \quad \frac{\left(\frac{n(n+1)}{2}\right)^2}{n(n+1)\left[\frac{2n+1}{3}+\frac{1}{2}\right]}=\frac{9}{5}$

$\Rightarrow \quad 5n^2 - 19n - 30 = 0$

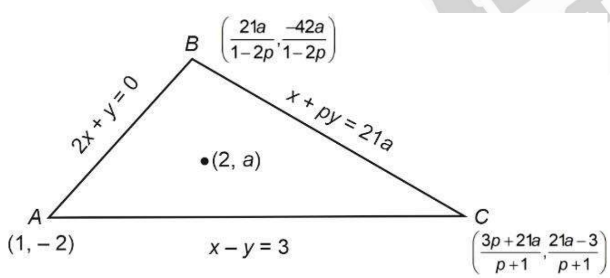
$\Rightarrow \quad (5n + 6)(n - 5) = 0$

$\therefore \quad \boxed{n=5}$

83. The equations of the sides AB, BC and CA of a triangle ABC are : $2x + y = 0$, $x + py = 21a$, $(a \neq 0)$ and $x - y = 3$ respectively. Let $P(2, a)$ be the centroid of ΔABC. Then $(BC)^2$ is equal to

Answer (122)

Sol.



$\because \quad \frac{21a}{1-2p}+1+\frac{3p+21a}{p+1}=6$

$\therefore \quad 4p^2 - 21ap + 8p + 42a - 5 = 0$ …(1)

And $\frac{-42a}{1-2p}-2+\frac{21a-3}{p+1}=3a$

$\therefore \quad 4p^2 - 81ap + 6ap^2 - 24a + 8p - 5 = 0$ …(2)

From equation (1) –equation (2) we get;

$60ap + 66a - 6ap^2 = 0$

$\because \quad a \neq 0 \Rightarrow p^2 - 10p - 11 = 0$

$p = -1 \text{ or } 11 \Rightarrow p = 11.$

When $p = 11$ then $a = 3$

Coordinate of $B = (-3, 6)$

And coordinate of $C = (8, 5)$

$\therefore \quad BC^2 = 122$

84. The minimum number of elements that must be added to the relation $R = \{(a, b), (b, c), (b, d)\}$ on the set $\{a, b, c, d\}$ so that it is an equivalence relation, is _____.

Answer (13)

Sol. $R = \{(a, b)(b, c)(b, d)\}$

$S : \{a, b, c, d\}$

Adding (a, a), (b, b), (c, c), (d, d) make reflexive.

Adding (b, a), (c, b), (d, b) make Symmetric

And adding (a, d), (a, c) to make transitive

Further (d, a) & (c, a) to be added to make Symmetricity.

Further (c, d) & (d, c) also be added.

So total 13 elements to be added to make equivalence.

85. Let $S = \{\theta \in [0, 2\pi) : \tan(\pi\cos\theta) + \tan(\pi \sin\theta) = 0\}$. Then $\sum_{\theta \in S} \sin^2\left(\theta+\frac{\pi}{4}\right)$ is equal to __________.

Answer (02)

Sol. $S=\{\theta \in [0, \pi) : \tan(\pi\cos\theta)+\tan(\pi\sin\theta)=0\}$

$\tan(\pi\cos\theta)+\tan(\pi\sin\theta)=0$

$\tan(\pi\cos\theta)=\tan(-\pi\sin\theta)$

$\pi\cos\theta = n\pi - \pi\sin\theta \qquad n \in I$

$\sin\theta+\cos\theta = n$

$\therefore \quad \sin\theta+\cos\theta=\{-1, 0, 1\}$

$\therefore \quad \theta = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}, \pi, \frac{3\pi}{2}$

Now $\sum_{\theta \in S} \sin^2\left(\theta+\frac{\pi}{4}\right)$

$=\sin^2\left(\frac{\pi}{4}\right)+\sin^2\left(\frac{\pi}{2}+\frac{\pi}{4}\right)+\sin^2\left(\frac{3\pi}{4}+\frac{\pi}{4}\right)$

$+\sin^2\left(\frac{7\pi}{4}+\frac{\pi}{4}\right)+\sin^2\left(\pi+\frac{\pi}{4}\right)+\sin^2\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)$

$=\frac{1}{2}+\frac{1}{2}+0+0+\frac{1}{2}+\frac{1}{2}$

$= 2$

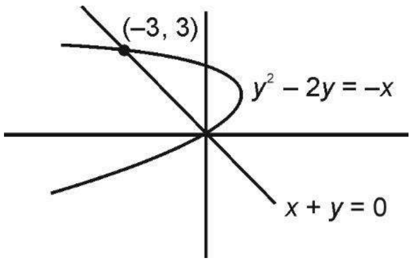
86. If the area of the region bounded by the curves $y^2 - 2y = -x$, $x + y = 0$ is A, then 8 A is equal to ________.

Answer (36)

Sol. Area enclosed by

$y^2 - 2y = -x$

$x + y = 0$



$$\text{Area} = \int_0^3 (2y - y^2) - (-y)dy$$

$$= \int_0^3 (3y - y^2)dy$$

$$= \left.\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$$

$$= \frac{27}{2} - 9$$

$$= \frac{27-18}{2} = \frac{9}{2} = A$$

$$8A = \frac{9}{2} \times 8 = 36 \text{ sq. units}$$

87. If the shortest between the lines

$$\frac{x+\sqrt{6}}{2} = \frac{y-\sqrt{6}}{3} = \frac{z-\sqrt{6}}{4} \text{ and}$$

$$\frac{x-\lambda}{3} = \frac{y-2\sqrt{6}}{4} = \frac{z+2\sqrt{6}}{5}$$

is 6, then the square of sum of all possible values of λ is

Answer (384)

Sol. Shortest distance between

$$\frac{x+\sqrt{6}}{2} = \frac{y-\sqrt{6}}{3} = \frac{z-\sqrt{6}}{4} \text{ and}$$

$$\frac{x-\lambda}{3} = \frac{y-2\sqrt{6}}{4} = \frac{z+2\sqrt{6}}{5} \text{ is 6}$$

$$\vec{b_1} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{b_2} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

$$\vec{a_2} - \vec{a_1} = \left(\lambda + \sqrt{6}\right)\hat{i} + \sqrt{6}\hat{j} - 3\sqrt{6}\,\hat{k}$$

$$d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot \vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|}\right| = 6$$

$$\left|\frac{-\lambda - \sqrt{6} + 2\sqrt{6} + 3\sqrt{6}}{\sqrt{6}}\right| = 6$$

$$|-\lambda + 4\sqrt{6}| = 6\sqrt{6}$$

$$-\lambda + 4\sqrt{6} = \pm 6\sqrt{6}$$

$$-\lambda + 4\sqrt{6} = 6\sqrt{6} \;\Rightarrow\; \lambda_1 = -2\sqrt{6} \quad\Big|\quad -\lambda + 4\sqrt{6} = -6\sqrt{6} \;\Rightarrow\; \lambda_2 = 10\sqrt{6}$$

$$(\lambda_1 + \lambda_2)^2 = \left(8\sqrt{6}\right)^2$$

$$= 384$$

88. Let the sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^n$, $x \neq 0.\ n \in \mathbb{N}$, be 376. Then the coefficient of x^4 is ________.

Answer (405)

Sol. $S = 1 - 3n + \frac{9n(n-1)}{2} = 376$

$3n^2 - 5n - 250 = 0$

$n = 10, \frac{-25}{3}$ (Rejected)

$$T_{r+1} = {}^nC_r \cdot x^{n-r}\left(\frac{-3}{x^2}\right)^r$$

$$= {}^nC_r \quad x^{n-3r}(-3)^r$$

$$= {}^{10}C_r \quad x^{10-3r}(-3)^r$$

Here $r = 2$

Required coefficient = ${}^{10}C_2\ (-3)^2$

= 45 × 9

= 405

89. Three urns A, B and C contain 4 red, 6 black; 5 red, 5 black, and λ red, 4 black balls respectively. One of the urns is selected at random and a ball is drawn. If the ball drawn is red and the probability that it is drawn from urn C is 0.4 then the square of the length of the side of the largest equilateral triangle, inscribed in the parabola $y^2 = \lambda x$ with one vertex at the vertex of the parabola, is

Answer (432)

Sol. E_1 : Ball is drawn from urn A $(4R + 6B)$

E_2 : Ball is drawn from urn B $(5R + 5B)$

E_3 : Ball is drawn from urn C $(\lambda R + 4B)$

$A \to$ Ball drawn is red.

Required probability $= P\left(\frac{E_3}{A}\right)$

$$= \frac{\frac{1}{3}\times\frac{\lambda}{\lambda+4}}{\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}+\frac{1}{3}\times\frac{\lambda}{\lambda+4}} = \frac{2}{5}$$

$\Rightarrow \quad = \frac{10\lambda}{19\lambda+36} = \frac{2}{5}$

$\Rightarrow \quad \lambda = 6$

Parabola: $y^2 = 6x = 4ax$

Let length of side $= l$

Point $\left(\frac{\sqrt{3}}{2}l, \frac{l}{2}\right)$ lies on parabola

$$\frac{l^2}{4} = 4a\left(\frac{\sqrt{3}}{2}l\right)$$

$\Rightarrow \quad l = 8a\sqrt{3}$

$l = 12\sqrt{3}$

$l^2 = 432$

90. Let $\vec{a} = \hat{i} + 2\hat{j} + \lambda\hat{k}$, $\vec{b} = 3\hat{i} - 5\hat{j} - \lambda\hat{k}$, $\vec{a}\cdot\vec{c} = 7$, $2\vec{b}\cdot\vec{c} + 43 = 0$, $\vec{a}\times\vec{c} = \vec{b}\times\vec{c}$. Then $|\vec{a}\cdot\vec{b}|$ is equal to

Answer (08)

Sol. $\vec{a} = \hat{i} + 2\hat{j} + \lambda\hat{k}$

$\vec{b} = 3\hat{i} - 5\hat{j} - \lambda\hat{k}$

$\vec{a}\times\vec{c} = \vec{b}\times\vec{c}$

$\Rightarrow \quad (\vec{a}-\vec{b})\times\vec{c} = 0$

$\Rightarrow \quad \vec{c} \parallel \vec{a}-\vec{b}$

$\Rightarrow \quad \vec{c} = \propto (\vec{a}-\vec{b})$

$\therefore \quad \vec{c} = \alpha(-2\hat{i} + 7\hat{j} + 2\lambda\hat{k})$

$\vec{a}\cdot\vec{c} = \alpha\left(12 + 2\lambda^2\right) = 7 \quad \ldots(i)$

$\vec{b}\cdot\vec{c} = \alpha\left(-41 - 2\lambda^2\right) = \frac{-43}{2} \quad \ldots(ii)$

(i) and (ii)

$\Rightarrow \quad \frac{12+2\lambda^2}{41+2\lambda^2} = \frac{14}{43}$

$\Rightarrow \quad \lambda^2 = 1$

$|\vec{a}\cdot\vec{b}| = |3 - 10 - \lambda^2| = 8$

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