MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

61. The mean and variance of the marks obtained by the students in a test are 10 and 4 respectively. Later, the marks of one of the students is increased from 8 to 12. If the new mean of the marks is 10.2, then their new variance is equal to :

(1) 4.08

(2) 3.92

(3) 3.96

(4) 4.04

Answer (3)

Sol. $\vec{x} = 10$ & $\sigma^2 = 4$, No. of students = N (let)

$$\therefore \quad \frac{\sum x_i}{N} = 10 \;\; \& \;\; \frac{\sum x_i^2}{N} - (10)^2 = 4$$

Now if one of x_i is changed from 8 to 12 we have

$$\text{New mean } \frac{\sum x_i + 4}{N} = 10 + \frac{4}{N} = 10.2$$

$\Rightarrow \quad N = 20$

$$\text{and } \sigma_{\text{new}}^2 = \frac{\sum x_i^2 - (8)^2 + (12)^2}{20} - (10 \cdot 2)^2$$

$$= \frac{\sum x_i^2}{20} + \frac{144 - 64}{20} - (10 \cdot 2)^2$$

$= 104 + 4 - (10 \cdot 2)^2$

$= 108 - 104.04 = 3.96$

62. The statement $(p \wedge (\sim q)) \Rightarrow (p \Rightarrow (\sim q))$ is

(1) a contradiction

(2) equivalent to $p \vee q$

(3) equivalent to $(\sim p) \vee (\sim q)$

(4) a tautology

Answer (4)

Sol. Making truth table (Let $(p \wedge \sim q) \Rightarrow (p \Rightarrow \sim q) = E$)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$p \Rightarrow \sim q$	E
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\therefore$ E is a tautology

63. The points of intersection of the line $ax + by = 0$, $(a \neq b)$ and the circle $x^2 + y^2 - 2x = 0$ are $A(\alpha, 0)$ and $B(1, \beta)$. The image of the circle with AB as a diameter in the line $x + y + 2 = 0$ is:

(1) $x^2 + y^2 + 5x + 5y + 12 = 0$

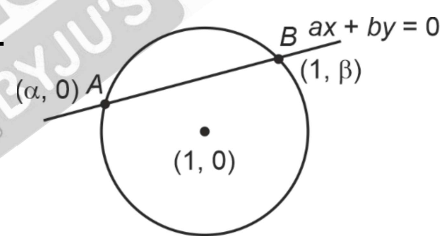
(2) $x^2 + y^2 + 3x + 5y + 8 = 0$

(3) $x^2 + y^2 - 5x - 5y + 12 = 0$

(4) $x^2 + y^2 + 3x + 3y + 4 = 0$

Answer (1)

Sol.



As A and B satisfy both line and circle we have

$\alpha = 0 \Rightarrow A(0, 0)$ and $\beta = 1$ i.e. $B(1, 1)$

Centre of circle as AB diameter is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and

radius $= \frac{1}{\sqrt{2}}$

$\therefore$ For image of $\left(\frac{1}{2}; \frac{1}{2}\right)$ in $x + y + z$ we get

$$\frac{x - \frac{1}{2}}{1} = \frac{y - \frac{1}{2}}{1} = \frac{-2(3)}{2}$$

$\Rightarrow$ Image $\left(-\frac{5}{2}, -\frac{5}{2}\right)$

∴ Equation of required circle

$$\left(x+\frac{5}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=\frac{1}{2}$$

$$\Rightarrow\ x^2+y^2+5x+5y+\frac{50}{4}-\frac{1}{2}=0$$

$\Rightarrow\ x^2 + y^2 + 5x + 5y + 12 = 0$

64. Consider the lines L_1 and L_2 given by

$$L_1:\frac{x-1}{2}=\frac{y-3}{1}=\frac{z-2}{2}$$

$$L_2:\frac{x-2}{1}=\frac{y-2}{2}=\frac{z-3}{3}.$$

A line L_3 having direction ratios, 1, –1, –2, intersects L_1 and L_2 at the points P and Q respectively. Then the length of line segment PQ is

(1) $3\sqrt{2}$ (2) $2\sqrt{6}$

(3) $4\sqrt{3}$ (4) 4

Answer (2)

Sol. Let,

$P \equiv (2\lambda + 1, \lambda + 3, 2\lambda + 2)$ and $Q(\mu + 2, 2\mu + 2, 3\mu + 3)$

d.r's of $PQ \equiv \langle 2\lambda - \mu - 1, \lambda - 2\mu + 1, 2\lambda - 3\mu - 1\rangle$

$$\therefore\ \frac{2\lambda-\mu-1}{1}=\frac{\lambda-2\mu-1}{-1}=\frac{2\lambda-3\mu-1}{-2}$$

∴ $-2\lambda + \mu + 1 = \lambda - 2\mu + 1$ and $-2\lambda + 4\mu - 2 = -2\lambda + 3\mu + 1$

$\Rightarrow\ 3\lambda - 3\mu = 0$ and $\mu = 3$

∴ $\lambda = \pm 3$ and $\mu = 3$

∴ $P \equiv (7, 6, 8)$ and $Q(5, 8, 12)$

$$\therefore\ |PQ|=\sqrt{2^2+2^2+4^2}=\sqrt{24}=2\sqrt{6}$$

65. Let $f : (0, 1) \to R$ be a function defined by $f(x)=\dfrac{1}{1-e^{-x}}$, and $g(x) = (f(-x) - f(x))$. Consider two statements

(I) g is an increasing function in (0, 1)

(II) g is one-one in (0, 1)

Then,

(1) Only (I) is true

(2) Both (I) and (II) are true

(3) Only (II) is true

(4) Neither (I) nor (II) is true

Answer (2)

Sol. $g(x) = f(-x) - f(x)$

$$=\frac{1}{1-e^{x}}-\frac{1}{1-e^{-x}}$$

$$=\frac{1}{1-e^{x}}-\frac{e^{x}}{e^{x}-1}$$

$$=\frac{1+e^{x}}{1-e^{x}}$$

$$g'(x)=\frac{(1-e^{x})e^{x}-(1+e^{x})(-e^{x})}{(1-e^{x})^2}$$

$$=\frac{e^{x}-2e^{x}+e^{x}+2e^{x}}{(1-e^{x})^2}>0$$

So both statements are correct

66. Let S_1 and S_2 be respectively the sets of $a\in\mathbb{R}-\{0\}$ for which the system of linear equations

$ax + 2ay - 3az = 1$

$(2a + 1)\,x + (2a + 3)\,y + (a + 1)\,z = 2$

$(3a + 5)\,x + (a + 5)\,y + (a + 2)\,z = 3$

has unique solution and infinitely many solutions. Then

(1) $S_1=\Phi$ and $S_2=\mathbb{R}-\{0\}$

(2) $S_1=\mathbb{R}-\{0\}$ and $S_2=\Phi$

(3) S_1 is an infinite set and $n(S_2) = 2$

(4) $n(S_1) = 2$ and S_2 is an infinite set

Answer (2)

Sol. Given system of equations

$ax + 2ay - 3az = 1$

$(2a + 1)x + (2a + 3)y + (a + 1)\,z = 2$

$(3a + 5)x + (a + 5)y + (a + 2)z = 3$

$$\text{Let } A=\begin{vmatrix} a & 2a & -3a \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$$

$$=a\begin{vmatrix} 1 & 0 & 0 \\ 2a+1 & 1-2a & 7a+4 \\ 3a+5 & -5a-5 & 10a+17 \end{vmatrix}$$

$= a(15a^2 + 31a + 37)$

Now $A = 0$

$\Rightarrow\ \boxed{a=0}$

So, $S_1 = R - \{0\}$ and at $a = 0$

System has infinite solution but $a\in R-\{0\}$

$\therefore\ S_2=\Phi$

67. The minimum value of the function $f(x)\int_0^2 e^{|x-t|}\,dt$ is

(1) $e\,(e-1)$

(2) $2e-1$

(3) $2\,(e-1)$

(4) 2

Answer (3)

Sol. $f(x)=\int_0^2 e^{|x-t|}dt$

For $x > 2$

$f(x)=\int_0^2 e^{x-t}dt=e^x\left(1-e^{-2}\right)$

For $x < 0$

$f(x)=\int_0^2 e^{t-x}dt=e^{-x}\left(e^2-1\right)$

For $x\in[0,2]$

$f(x)=\int_0^x e^{x-t}dt\in\int_x^2 e^{t-x}dt$

$=e^{2-x}+e^x-2$

For $x > 2$

$f(x)\big|_{\min}=e^2-1$

For x < 0

$f(x)\big|_{\min}=e^2-1$

For $x\in[0,2]$

$f(x)\big|_{\min}=2(e-1)$

68. Let $y(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})$. Then $y' - y''$ at $x = -1$ is equal to :

(1) 496 (2) 944

(3) 976 (4) 464

Answer (1)

Sol. $y=\dfrac{1-x^{32}}{1-x}=1+x+x^2+x^3+\ldots+x^{31}$

$y' = 1 + 2x + 3x^2 + \ldots + 31x^{30}$

$y'(-1) = 1 - 2 + 3 - 4 + \ldots + 31 = 16$

$y''(x) = 2 + 6x + 12x^2 + \ldots + 31.30\, x^{29}$

$y''(-1) = 2 - 6 + 12 \ldots 31.30 = 480$

$y''(-1) - y'(-1) = -496$

69. Let $x = 2$ be a local minima of the function $f(x) = 2x^4 - 18x^2 + 8x + 12$, $x\in(-4,4)$. If M is local maximum value of the function f in (-4, 4), then M =

(1) $12\sqrt{6}-\dfrac{33}{2}$ (2) $12\sqrt{6}-\dfrac{31}{2}$

(3) $18\sqrt{6}-\dfrac{31}{2}$ (4) $18\sqrt{6}-\dfrac{33}{2}$

Answer (1)

Sol. $f'(x) = 8x^3 - 36x + 8$

$= 4(2x^3 - 9x + 2)$

$= 4(x - 2)(2x^2 + 4x - 1)$

$=4(x-2)\left(x-\dfrac{-2+\sqrt{6}}{2}\right)\left(x-\dfrac{-2\sqrt{6}}{2}\right)$

Local maxima occurs at $x=\dfrac{-2+\sqrt{6}}{2}=x_0$

$f(x_0)=12\sqrt{6}-\dfrac{33}{2}$

70. The value of

$$\lim_{n\to\infty}\frac{1+2-3+4+5-6+\ldots+(3n-2)+(3n-1)-3n}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$$

is :

(1) $3\left(\sqrt{2}+1\right)$ (2) $\dfrac{3}{2}\left(\sqrt{2}+1\right)$

(3) $\dfrac{\sqrt{2}+1}{2}$ (4) $\dfrac{3}{2\sqrt{2}}$

Answer (2)

Sol. $I=\lim\limits_{n\to\infty}\dfrac{(1+2+3+\ldots+3n)-2(3+6+9+..+3n)}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$

$$=\lim_{n\to\infty}\frac{\dfrac{3n(3n+1)}{2}-6\dfrac{n(n+1)}{2}}{\left(\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}\right)}$$

$$=\lim_{n\to\infty}\frac{3n(n-1)\left[\sqrt{2n^4+4n+3}+\sqrt{n^4+5n+4}\right]}{2\left[\left(2n^4+4n-3\right)-\left(n^4+5n+4\right)\right]}$$

$$=\lim_{n\to\infty}\frac{3\cdot1\left(1-\dfrac{1}{n}\right)\left[\sqrt{2+\dfrac{4}{n^3}+\dfrac{3}{n^4}}+\sqrt{1+\dfrac{5}{n^3}+\dfrac{4}{n^4}}\right]}{2\left[1-\dfrac{1}{n^3}-\dfrac{7}{n^4}\right]}$$

$$=\frac{3\left(\sqrt{2}+1\right)}{2}$$

71. The distance of the point $\left(6, -2\sqrt{2}\right)$ from the common tangent $y = mx + c$, $m > 0$, of the curves $x = 2y^2$ and $x = 1 + y^2$ is

(1) $\frac{14}{3}$ (2) $\frac{1}{3}$

(3) $5\sqrt{3}$ (4) 5

Answer (4)

Sol. $y^2 = \frac{x}{2} \Rightarrow$ tangent $y = mx + \frac{1}{8m}$

$y^2 = x - 1 \Rightarrow$ tangent $y = m(x-1) + \frac{1}{4m}$

For common tangent $\frac{1}{8m} = -m + \frac{1}{4m}$

$\Rightarrow$ $1 = -8m^2 + 2$

$\because$ $m > 0 \Rightarrow m = \frac{1}{2\sqrt{2}}$

$\Rightarrow$ Common tangent is $y = \frac{x}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$\Rightarrow x - 2\sqrt{2}y + 1 = 0$

Distance of point $(6, -2\sqrt{2})$ from common tangent = 5

72. Let $x, y, z > 1$ and $A = \begin{bmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 2 & \log_y z \\ \log_z x & \log_z y & 3 \end{bmatrix}$.

Then $\left|\text{adj}(\text{adj } A^2)\right|$ is equal to

(1) 2^4 (2) 6^4

(3) 2^8 (4) 4^8

Answer (3)

Sol. $|A| = \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & 2\log y & \log z \\ \log x & \log y & 3\log z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 2$

$\Rightarrow \left|\text{adj}(\text{adj } A^2)\right| = \left|\text{adj}(A^2)\right|^2 = \left(\left|A^2\right|^2\right)^2 = |A|^8 = 2^8$

73. Let $f(x) = \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx$. If $f(3) = \frac{1}{2}(\log_e 5 - \log_e 6)$, then $f(4)$ is equal to

(1) $\log_e 17 - \log_e 18$ (2) $\log_e 19 - \log_e 20$

(3) $\frac{1}{2}(\log_e 17 - \log_e 19)$ (4) $\frac{1}{2}(\log_e 19 - \log_e 17)$

Answer (3)

Sol. $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)}dx$

Put $x^2 = t \Rightarrow 2xdx = dt$

$f(x) = \int \frac{dt}{(t+1)(t+3)} = \int \frac{dt}{(t+2)^2 - 1}$

$= \frac{1}{2}\log_e\left|\frac{t+1}{t+3}\right| + C$

$f(x) = \frac{1}{2}\log_e\left(\frac{x^2+1}{x^2+3}\right) + C \Rightarrow$

$f(3) = \frac{1}{2}\log_e\left(\frac{10}{12}\right) + C$

$\because$ $f(3) + \frac{1}{2}(\log_e 5 - \log_e 6) \Rightarrow C = 0$

$f(x) = \frac{1}{2}\log_e\left(\frac{x^2+1}{x^2+3}\right) \Rightarrow$

$f(4) = \frac{1}{2}(\log_e 17 - \log_e 19)$

74. The distance of the point P (4. 6, –2) from the line passing through the point (–3, 2, 3) and parallel to a line with direction ratios 3, 3, –1 is equal to

(1) 3

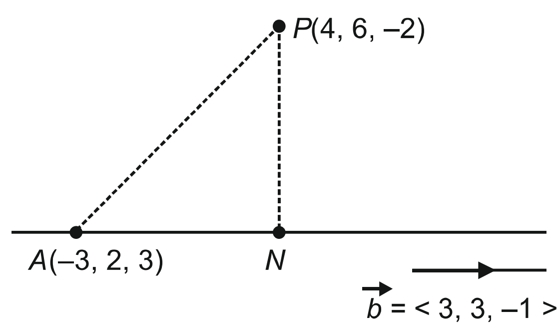
(2) $2\sqrt{3}$

(3) $\sqrt{6}$

(4) $\sqrt{14}$

Answer (4)

Sol.



$\overrightarrow{AP} = 7\hat{i} + 4\hat{j} - 5\hat{k} \Rightarrow \left|\overrightarrow{AP}\right| = \sqrt{49+16+25} = \sqrt{90}$ AN

= projection of $\overrightarrow{AP}$ on $\vec{b} = \overrightarrow{AP}\cdot\vec{b} = \frac{21+12+5}{\sqrt{19}} = \frac{38}{\sqrt{19}}$

$(PN)^2 = (AP)^2 - (AN)^2 = 90 - 76 = 14 \Rightarrow PN = \sqrt{14}$

75. Let M be the maximum value of the product of two positive integers when their sum is 66. Let the sample space $S=\left\{x\in\mathbb{Z}: x(66-x)\geq\frac{5}{9}M\right\}$ and the event $A = \{x \in S : x \text{ is a multiple of } 3\}$. Then $P(A)$ is equal to

(1) $\frac{7}{22}$ (2) $\frac{1}{3}$

(3) $\frac{1}{5}$ (4) $\frac{15}{44}$

Answer (2)

Sol. $x + y = 66$

$\frac{x+y}{2}\geq\sqrt{xy}$

$\Rightarrow\ 33\geq\sqrt{xy}$

$\Rightarrow\ xy\leq 1089$

$\therefore\ M = 1089$

$S:\ x(66-x)\geq\frac{5}{9}\cdot 1089$

$66x - x^2 \geq 605$

$\Rightarrow\ x^2 - 66x + 605 \leq 0$

$\Rightarrow\ (x-61)(x-5)\leq 0$

$x \in [5, 61]$

$A = \{6, 9, 12, \ldots\ldots 60\}$

$x(A) = 19$

$x(S) = 57$

$\therefore\ P(A)=\frac{1}{3}$

76. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three non zero vectors such that $\vec{b}\cdot\vec{c}=0$ and $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{\vec{b}-\vec{c}}{2}$. If $\vec{d}$ be a vector such that $\vec{b}\cdot\vec{d}=\vec{a}\cdot\vec{b}$, then $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)$ is equal to

(1) $\frac{1}{4}$ (2) $\frac{1}{2}$

(3) $-\frac{1}{4}$ (4) $\frac{3}{4}$

Answer (1)

Sol. $\vec{b}\left(\vec{a}\cdot\vec{c}\right)-\vec{c}\left(\vec{a}\cdot\vec{b}\right)=\frac{\vec{b}-\vec{c}}{2}$

$\vec{a}\cdot\vec{c}=\frac{1}{2},\quad \vec{a}\cdot\vec{b}=\frac{1}{2}$

$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=\left(\vec{b}\cdot\vec{d}\right)\left(\vec{a}\cdot\vec{c}\right)-\left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)$

$=\left(\vec{a}\cdot\vec{b}\right)\left(\vec{a}\cdot\vec{c}\right)$

$=\frac{1}{4}$

77. If a_r is the coefficient of x^{10-r} in the Binomial expansion of $(1 + x)^{10}$, then $\sum_{r-1}^{10} r^3\left(\frac{a_r}{a_{r-1}}\right)^2$ is equal to

(1) 1210 (2) 5445

(3) 3025 (4) 4895

Answer (1)

Sol. $T_r = {}^{10}C_r x^r$

Coefficient of $x^{10-r} = {}^{10}C_{10-r} = {}^{10}C_r$

$\sum_{r=1}^{10} r^3\left(\frac{{}^{10}C_r}{{}^{10}C_{r-1}}\right)^2$

$=\sum_{r=1}^{10} r^3\left(\frac{11-r}{r}\right)^2 \Rightarrow \sum r(11-r)^2$

$\Rightarrow\ \sum r\left(121+r^2-22r\right)$

$\Rightarrow\ \sum 121r+\sum r^3-22\sum r^2$

$\Rightarrow\ 121\times\frac{10\times 11}{2}+\left(\frac{10\times 11}{2}\right)^2-22\times\left(\frac{10\times 11\times 21}{6}\right)$

$= 6655 + 3025 - 8470$

$= 1210$

78. Let $y = y(x)$ be the solution curve of the differential equation

$\frac{dy}{dx}=\frac{y}{x}\left(1+xy^2\left(1+\log_e x\right)\right), x>0, y(1)=3$. Then

$\frac{y^2(x)}{9}$ is equal to

(1) $\frac{x^2}{7-3x^3\left(2+\log_e x^2\right)}$

(2) $\frac{x^2}{2x^3\left(2+\log_e x^3\right)-3}$

(3) $\frac{x^2}{5-2x^3\left(2+\log_e x^3\right)}$

(4) $\frac{x^2}{3x^3\left(1+\log_e x^2\right)-2}$

Answer (3)

Sol. $\frac{dy}{dx}=\frac{y}{x}\left(1+xy^2\left(1+\log_e x\right)\right),\quad y(1)=3$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx}-\frac{1}{x}\cdot\frac{1}{y^2}=(1+\ln x)$

$-\frac{1}{y^2}=t\Rightarrow\frac{2}{y^3}\frac{dy}{dx}=\frac{dt}{dx}$

$\Rightarrow \frac{1}{2}\frac{dt}{dx}+\frac{t}{x}=1+\ln x$

$\Rightarrow \frac{dt}{dx}+\frac{2t}{x}=2(1+\ln x)$

IF = x^2

$t\cdot x^2=\int(1+\ln x)x^2dx$

$\Rightarrow -\frac{1}{y^2}\cdot x^2=2\left[\frac{x^3}{3}(1+\ln x)-\frac{x^3}{9}\right]+c$

$y(1)=3$

$\Rightarrow c=-\frac{5}{9}$

$\therefore \frac{x^2}{y^2}=-2\left(\frac{x^3}{3}(1+\ln x)-\frac{x^3}{9}\right]+\frac{5}{9}$

$\Rightarrow \frac{y^2}{9}=\frac{x^2}{5-2x^3\left(2+\ln x^3\right)}$

79. Let $z_1=2+3i$ and $z_2=3+4i$. The set

$S=\left\{z\in\mathbb{C}:|z-z_1|^2-|z-z_2|^2=|z_1-z_2|^2\right\}$

represents a

(1) straight line with the sum of its intercepts on the coordinate axes equals – 18

(2) hyperbola with eccentricity 2

(3) straight line with the sum of its intercepts on the coordinate axes equals 14

(4) hyperbola with the length of the transverse axis 7

Answer (3)

Sol. $|z-z_1|^2-|z-z_2|^2=|z_1-z_2|^2$

$\Rightarrow (x-2)^2+(y-3)^2-(x-3)^2-(y-4)^2=1+1$

$\Rightarrow -4x+4+9-6y-9+6x-16+8y=2$

$\Rightarrow 2x+2y=14$

$\Rightarrow x+y=7$

80. The vector $\vec{a}=-\hat{i}+2\hat{j}+\hat{k}$ is rotated through a right angle, passing through the y-axis in its way and the resulting vector is $\vec{b}$. Then the projection of $3\vec{a}+\sqrt{2}\vec{b}$ on $\vec{c}=5\hat{i}+4\hat{j}+3\hat{k}$ is

(1) $\sqrt{6}$ (2) $2\sqrt{3}$

(3) 1 (4) $3\sqrt{2}$

Answer (4)

Sol. Let $\vec{b}=\mu\vec{a}+\lambda\hat{j}$

Now $\vec{b}\cdot\vec{a}=0$

$\Rightarrow (\mu\vec{a}+\lambda\hat{j})\cdot\vec{a}=0$

$\Rightarrow \mu|\vec{a}|^2+2\lambda=0\Rightarrow 6\mu+2\lambda=0\ \ldots$(i)

$\Rightarrow \vec{b}=\lambda(\vec{a}-3\hat{j})=\lambda(-\hat{i}-\hat{j}+\hat{k})$

$\Rightarrow |\vec{b}|=|\vec{a}|\Rightarrow\lambda=\pm\sqrt{2}$

$\therefore \vec{b}=-\sqrt{2}(-\hat{i}-\hat{j}+\hat{k})$

$\therefore 3\vec{a}+\sqrt{2}\vec{b}=3(-\hat{i}+2\hat{j}+\hat{k})-2(-\hat{i}-\hat{j}+\hat{k})$

$=-\hat{i}+8\hat{j}+\hat{k}$

$\therefore$ projection $3\sqrt{2}$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. Let the equation of the plane passing through the line $x-2y-z-5=0=x+y+3z-5$ and parallel to the line $x+y+2z-7=0=2x+3y+z-2$ be $ax+by+cz=65$. Then the distance of the point (a, b, c) from the plane $2x+2y-z+16=0$ is ________

Answer (09)

Sol. Let the equation of the plane is

$(x-2y-z-5)+\lambda(x+y+3z-5)=0\quad\ldots$(i)

$\because$ it's parallel to the line

$x+y+2z-7=0=2x+3y+z-2$

So, vector along the line $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

$= -5\hat{i} + 3\hat{j} + \hat{k}$

$\because$ Plane is parallel to line

$\therefore$ $-5(1 + \lambda) + 3(-2 + \lambda) + 1(-1 + 3\lambda) = 0$

$\boxed{\lambda = 12}$

So, by (i)

$13x + 10y + 35z = 65$

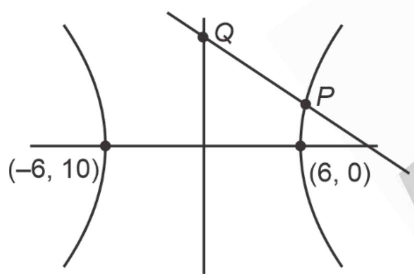
$\therefore$ $a = 13$, $b = 10$, $c = 35$

and $d = \dfrac{26 + 20 - 35 + 16}{\sqrt{9}} = 9$

82. The vertices of a hyperbola H are $(\pm 6, 0)$ and its eccentricity is $\dfrac{\sqrt{5}}{2}$. Let N be the normal to H at point in the first quadrant and parallel to the line $\sqrt{2}x + y = 2\sqrt{2}$. If d is the length of the line segment of N between H and the y-axis then d^2 is equal to ____________.

Answer (216)

Sol.



$a = 6$, $e = \dfrac{\sqrt{5}}{2}$

$\therefore$ $\dfrac{5}{4} = 1 + \dfrac{b^2}{36} \Rightarrow b^2 = 36 \times \dfrac{1}{4} = 9$

$\therefore$ $H : \dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$

$P(6\sec\theta, 3\tan\theta)$

Slope of tangent at $P = \dfrac{6\sec\theta}{4 \times 3\tan\theta}$

So, $\dfrac{1}{2\sin\theta} \times -\sqrt{2} = -1 \Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}}$

$\boxed{Q = 45^\circ}$ (for first quad)

$\therefore$ $P \equiv \left(6\sqrt{2}, 3\right)$ and $N : \sqrt{2}x + y = 15$

$\therefore$ $Q(0, 15)$ Now, $PQ^2 = 72 + 144 = 216$

83. Let

$s = \left\{\alpha : \log_2\left(9^{2a-4} + 13\right) - \log_2\left(\dfrac{5}{2} \cdot 3^{2\alpha-4} + 1\right) = 2\right\}.$

Then the maximum value of β for which the equation $x^2 - 2\left(\sum_{a \in s} \alpha\right)^2 x + \sum_{a \in s} (\alpha + 1)^2 \beta = 0$ has real roots, is ________.

Answer (25)

Sol. $S = \left\{\alpha : \log_2\left(9^{2\alpha-4} + 13\right) - \log_2\left(\dfrac{5}{2} \cdot 3^{2\alpha-4} + 1\right) = 2\right\}$

So,

$\dfrac{9^{2\alpha-4} + 13}{\dfrac{5}{2} 3^{2\alpha-4} + 1} = 4 \Rightarrow 9^{2\alpha-4} + 13 = 103^{2\alpha-4} + 4$

Let $3^{2\alpha-4} = t$ then $t^2 - 10t + 9 = 0$

$(t - 9)(t - 1) = 0$

$\therefore$ $3^{2\alpha-4} = 3^2$ or $3^{2\alpha-4} = 3^\circ$

$\therefore$ $\alpha = 3, 2$

Now equation

$x^2 - 50x + 25\beta = 0$

$D \geq 0 \Rightarrow (50)^2 - 4 \times 25\beta \geq 0$

$\beta \leq 25$

$\therefore$ Max. $\beta = 25$

84. For some $a, b, c \in N$, let $f(x) = ax - 3$ and $g(x) = x^b + c$, $x \in R$. If $(fog)^{-1}(x) = \left(\dfrac{x-7}{2}\right)^{\frac{1}{3}}$, then $(fog)(ac) + (gof)(b)$ is equal to ____________.

Answer (2039)

Sol. $f(x) = ax - 3$

$g(x) = x^b + c$

$(fog)^{-1} = \left(\dfrac{x-7}{2}\right)^{\frac{1}{3}}$

$(fog)^{-1}(x) = \left(\dfrac{x + 3 - ca}{a}\right)^{\frac{1}{b}} = \left(\dfrac{x-7}{2}\right)^{\frac{1}{3}}$

$\Rightarrow$ $a = 2$, $b = 3$, $c = 5$

$fog(ac) + gof(b)$

$\because$ $f(x) = 2x - 3$

$g(x) = x^3 + 5$

$fog(10) + gof(3)$

$= 2007 + 32$

$= 2039$

85. Let x and y be distinct integers where $1 \le x \le 25$ and $1 \le y \le 25$. Then, the number of ways of choosing x and y, such that $x + y$ is divisible by 5, is ________.

Answer (120)

Sol.

Type	Numbers
$5k$	5, 10, 15, 20, 25
$5k + 1$	1, 6, 11, 16, 21
$5k + 2$	2, 7, 12, 17, 22
$5k + 3$	3, 8, 13, 18, 23
$5k + 4$	4, 9, 14, 19, 24

To select x and y.

Case I : 1 of $(5k + 1)$ and 1 of $(5k + 4) = 5 \times 5 = 25$

Case II : 1 of $(5k + 2)$ and 1 of $(5k + 3) = 5 \times 5 = 25$

Case III : Both of type $5k$ (both cannot be same) $= 5 \times 4 = 20$

Total = 120

86. The constant term in the expansion of

$$\left(2x + \frac{1}{x^7} + 3x^2\right)^5$$ is ________.

Answer (1080)

Sol. Constant term in the expansion of

$$\left(2x + \frac{1}{x^7} + 3x^2\right)^5$$

$$\frac{1}{x^{35}}(2x^8 + 1 + 3x^9)^5$$

$$\frac{1}{x^{35}}\left(1 + x^8(3x + 2)\right)^5$$

Term independent of x = coefficient of x^{35} in

$${}^5C_4\left(x^8(3x + 2)\right)^4$$

$= {}^5C_4$ coefficient of x^3 in $(2 + 3x)^4$

$= {}^5C_4 \times {}^4C_3(2)^1(3)^3$

$= 5 \times 4 \times 2 \times 27$

$= 1080$

87. Let $S = \{1, 2, 3, 5, 7, 10, 11\}$. The number of non-empty subsets of S that have the sum of all elements a multiple of 3, is

Answer (43)

Sol. Out of the given numbers one is $(3k)$ type and 3 of $(3k + 1)$ type and remaining 3 are $(3k + 2)$ type

Number of subsets of 1 element = 1

(1 of $3k$ type)

Number of subsets of 2 elements

1 of $(3k + 1)$ type + 1 of $(3k + 2)$ type = 9

Number of subsets of 3 elements

1 of $3k$ type + 1 of $(3k + 1)$ type + 1 of $(3k + 2)$ type = 9

3 of $(3k + 1)$ type = 1

3 of $(3k + 2)$ type = 1

Number of subsets of 4 elements

1 of $3k$ type + 3 of $(3k + 1)$ type = 1

1 of $3k$ type + 3 of $(3k + 2)$ type = 1

2 of $(3k + 1)$ type + 2 of $(3k + 2)$ type = 9

Number of subsets of 5 elements

1 of $3k$ + 2 of $(3k + 1)$ type + 2 of $(3k + 2)$ type = 9

Number of subsets of 6 elements

3 of $(3k + 1)$ type + 3 of $(3k + 2)$ type = 1

The set itself = 1

Total = 43

88. If the sum of all the solutions of

$$\tan^{-1}\left(\frac{2x}{1 - x^2}\right) + \cot^{-1}\left(\frac{1 - x^2}{2x}\right) = \frac{\pi}{3}, -1 < x < 1, x \neq 0,$$

is $\alpha - \frac{4}{\sqrt{3}}$, then α is equal to ________.

Answer (02)

Sol. Case-I

$-1 < x < 0$

$$\tan^{-1}\left(\frac{2x}{1 - x^2}\right) + \pi + \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \frac{\pi}{3}$$

$$\tan^{-1}\frac{2x}{1 - x^2} = \frac{-\pi}{3}$$

$$2\tan^{-1}x = \frac{-\pi}{3}$$

$$\tan^{-1}x = \frac{-\pi}{6}$$

$$x = \frac{-1}{\sqrt{3}}$$

Case-II

$0 < x < 1$

$$\tan^{-1}\frac{2x}{1 - x^2} + \tan^{-1}\frac{2x}{1 - x^2} = \frac{\pi}{3}$$

$$\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{6}$$

$$2\tan^{-1}x=\frac{\pi}{6}$$

$$\tan^{-1}x=\frac{\pi}{12}$$

$$x=2-\sqrt{3}$$

$$\text{Sum }=\frac{-1}{\sqrt{3}}+2-\sqrt{3}=2-\frac{4}{\sqrt{3}}$$

$\Rightarrow\ \alpha = 2$

89. Let A_1, A_2, A_3 be the three A.P. with the same common difference d and having their first terms as A, $A + 1$, $A + 2$, respectively. Let a, b, c be the 7th, 9th, 17th terms of A_1, A_2, A_3, respectively such that

$$\begin{vmatrix} a & 7 & 1 \\ 2b & 17 & 1 \\ c & 17 & 1 \end{vmatrix}+70=0.$$

If $a = 29$, then the sum of first 20 terms of an AP whose first term is $c - a - b$ and common difference is $\frac{d}{12}$, is equal to ________.

Answer (495)

Sol. $a = A + 6d$

$b = A + 8d + 1$

$c = A + 16d + 2$

$$\begin{vmatrix} a & 7 & 1 \\ 26 & 17 & 1 \\ c & 17 & 1 \end{vmatrix}=-70$$

$$\Rightarrow \begin{vmatrix} A+6d & 7 & 1 \\ 2A+16d+2 & 17 & 1 \\ A+16d+2 & 17 & 1 \end{vmatrix}=-70$$

$R_3 \to R_3 - R_2,\quad R_2 \to R_2 - R_1$

$$\Rightarrow \begin{vmatrix} A+6d & 7 & 1 \\ A+10d+2 & 10 & 0 \\ -A & 0 & 0 \end{vmatrix}=-70$$

$\Rightarrow\ A = -7$

$a = A + 6d = 29 \Rightarrow d = 6$

$b = -7 + 48 + 1 = 42$

$c = -7 + 96 + 2 = 91$

$c - a - b = 91 - 29 - 42 = 20$

$$\text{Sum }=\frac{20}{2}\left[2\times 20+19\times\frac{6}{12}\right]=10\left[40+\frac{19}{2}\right]=495$$

90. In the area enclosed by the parabolas $P_1 : 2y = 5x^2$ and $P_2 : x^2 - y + 6 = 0$ is equal to the area enclosed by P_1 and $y = ax$, $a > 0$, then a^3 is equal to ________.

Answer (600)

Sol. $x^2+6=\frac{5}{2}x^2 \Rightarrow x=\pm 2$

Area between P_1 and P_2 [Say A_1]

$$=\int_{-2}^{2}(x^2+6)-\frac{5}{2}x^2dx$$

$$=2\int_0^2\left(6-\frac{3}{2}x^2\right)dx=2\left[6x-\frac{x^3}{2}\right]_0^2=16$$

$$ax=\frac{5}{2}x^2 \Rightarrow x=0,\ \frac{2a}{5}$$

Area between P_1 and $y = ax$ [Say A_2]

$$=\int_0^{\frac{2\alpha}{5}} ax-\frac{5}{2}x^2dx$$

$$=\left.\frac{ax^2}{2}-\frac{5}{6}x^3\right]_0^{\frac{2a}{5}} : \frac{2a^3}{75}$$

$$A_1 = A_2 \Rightarrow \frac{2a^3}{75}=16$$

$a^3 = 600$

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