MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

61. Let the shortest distance between the line $L: \frac{x-5}{-2} = \frac{y-\lambda}{1} = \frac{z+\lambda}{1}$, $\lambda \ge 0$ and $L_1 : x + 1 = y - 1 = 4 - z$ be $2\sqrt{6}$. If (α, β, γ) lies on L, then which of the following is **NOT** possible?

(1) $\alpha + 2\gamma = 24$ (2) $2\alpha - \gamma = 9$

(3) $2\alpha + \gamma = 7$ (4) $\alpha - 2\gamma = 19$

Answer (1)

Sol. $\frac{x-5}{-2} = \frac{y-\lambda}{0} = \frac{z+\lambda}{1}$, $\lambda \ge 0$

$\frac{x+1}{1} = \frac{y-1}{1} = \frac{z-4}{-1}$

$\vec{a}_1 = 5\hat{i} + \lambda\hat{j} - \lambda\hat{k}, \vec{a}_2 = -\hat{i} + \hat{j} + 4\hat{k}$

$\vec{a}_1 - \vec{a}_2 = 6\hat{i} + (\lambda - 1)\hat{j} - (\lambda + 4)\hat{k}$

$\vec{b}_1 = -2\hat{i} + \hat{k}, \vec{b}_2 = \hat{i} + \hat{j} - \hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -\hat{i} - \hat{j} - 2\hat{k}$

$(\vec{a}_1 - \vec{a}_2) \cdot \vec{b}_1 \times \vec{b}_2 = -6 + 1 - \lambda + 2\lambda + 8 = \lambda + 3$

and $|\vec{b}_1 \times \vec{b}_2| = \sqrt{6}$

$\because \frac{|\lambda + 3|}{\sqrt{6}} = 2\sqrt{6}$

$\therefore \lambda = 9, \because \lambda \ge 0$

$\therefore L: \frac{x-5}{-2} = \frac{y-9}{0} = \frac{z+9}{1} = k$

$\therefore \alpha = -2k + 5, \beta = 9, \gamma = k - 9$

Here k is real then

$\alpha + 2\gamma = -13 \ne 24$.

But all other are in terms of k hence possible.

Correct option is (1).

62. Let $\alpha \in (0, 1)$ and $\beta = \log_e (1 - \alpha)$.

Let $P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \frac{x^n}{n}$, $x \in (0,1)$.

Then the integral $\int_0^\alpha \frac{t^{50}}{1-t} dt$ is equal to

(1) $-(\beta + P_{50}(\alpha))$

(2) $\beta + P_{50}(\alpha)$

(3) $P_{50}(\alpha) - \beta$

(4) $\beta + P_{50}(\alpha)$

Answer (1)

Sol. $\int_0^\alpha \frac{t^{50}}{1-t} dt = -\int_0^\alpha \left(\frac{1-t^{50}}{1-t} - \frac{1}{1-t}\right) dt$

$= -\left(\int_0^\alpha 1 + t + t^2 + \ldots.. + t^{49}\right) dt + \ln|1-t|\Big|_0^\alpha$

$= -\left(\alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \ldots.. + \frac{\alpha^{50}}{50}\right) + \ln(1-\alpha)$

$= -\beta - P_{50}(\alpha)$

63. (S1) $(p \Rightarrow q) \vee (p \wedge (\sim q))$ is a tautology

(S2) $((\sim p) \Rightarrow (\sim q) \wedge ((\sim p) \vee q)$ is a contradiction.

Then

(1) both (S1) and (S2) are wrong

(2) both (S1) and (S2) are correct

(3) only (S1) is correct

(4) only (S2) is correct

Answer (3)

Sol. S1

p	q	$\sim q$	$p \to q$	$p \wedge (\sim q)$	$(p \to q) \vee p \wedge (\sim q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	T	F	T
F	F	T	T	F	T

$\therefore$ S1 is correct

S2

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$	$\sim p \vee q$	(S2)
T	T	F	F	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

$\therefore$ S2 is incorrect

Option (3) is correct.

64. A bag contains 6 balls. Two balls are drawn from it at random and both are found to be black. The probability that the bag contains at least 5 black balls is

(1) $\frac{3}{7}$ (2) $\frac{5}{6}$

(3) $\frac{2}{7}$ (4) $\frac{5}{7}$

Answer (4)

Sol. Let $E_i \to$ Bag have at least *i* black balls

E $\to$ 2 balls are drawn & both black

$$\therefore \quad P\left(\frac{E_5 \text{ or } E_6}{E}\right) = \frac{P\left(\frac{E}{E_5}\right) + P\left(\frac{E}{E_6}\right)}{\sum_{i=1}^{6} P\left(\frac{E}{E_i}\right)}$$

$$= \frac{\frac{{}^5C_2}{{}^6C_2} + \frac{{}^6C_2}{{}^6C_2}}{0 + \frac{{}^2C_2}{{}^6C_2} + \frac{{}^3C_2}{{}^6C_2} + \frac{{}^4C_2}{{}^6C_2} + \frac{{}^5C_2}{{}^6C_2} + \frac{{}^6C_2}{{}^6C_2}}$$

$$= \frac{10+15}{1+3+6+10+15} = \frac{25}{35} = \frac{5}{7}$$

65. Let R be a relation on N × N defined by (a, b) R (c, d) if and only if $ad(b - c) = bc(a - d)$. Then R is

(1) symmetric and transitive but not reflexive

(2) reflexive and symmetric but not transitive

(3) symmetric but neither reflexive nor transitive

(4) transitive but neither reflexive nor symmetric

Answer (3)

Sol. (a, b) R $(c, d) \Rightarrow ad(b - c) = bc(a - d)$

For Reflexive

(a, b) R $(a, b) \Rightarrow ab(b - a) = ba(a - b)$

So not reflexive

For symmetric

(c, d) R $(a, b) \Rightarrow cb(d - a) = ad(c - b)$

OR $\quad ad(b - c) = bc(a - d)$

So symmetric

For transitive

(a, b) R $(c, d) \Rightarrow ad(b - c) = bc(a - d)$

(c, d) R $(e, f) \Rightarrow cf(d - e) = de(c - f)$

So $adcf(b - c)(d - e) = bcde(c - d)(c - f)$

$af(b - c)(d - e) = be(a - d)(c - f)$

$\Rightarrow$ Not transitive

66. Let $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, and $\vec{b}$ and $\vec{c}$ be two nonzero vectors such that $|\vec{a} + \vec{b} + \vec{c}| = |\vec{a} + \vec{b} - \vec{c}|$ and $\vec{b}.\vec{c} = 0$. Consider the following two statements.

(A) $|\vec{a} + \lambda\vec{c}| \geq |\vec{a}|$ for all $\lambda \in \mathbb{R}$

(B) $\vec{a}$ and $\vec{c}$ are always parallel.

Then

(1) Neither (A) nor (B) is correct

(2) Both (A) and (B) are correct

(3) Only (B) is correct

(4) Only (A) is correct

Answer (3)

Sol. $|\vec{a} + \vec{b} + \vec{c}| = |\vec{a} + \vec{b} - \vec{c}|$

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{c} - \vec{c}\cdot\vec{a})$$

$$\Rightarrow \quad \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = 0 \;\Rightarrow\; \vec{c}\cdot\vec{a} = 0$$

$$|\vec{a} + \lambda\vec{c}|^2 = |\vec{a}|^2 + \lambda^2|\vec{c}|^2 + 0 \geq |\vec{a}|^2$$

So A is correct

B is incorrect

67. A wire of lenth 20 m to be cut into two pieces. A piece of length l_1 is bent to make a square of area A_1 and the other piece of length l_2 is made into a circle of area A_2. If $2A_1 + 3A_2$ is minimum then $(\pi l_1) : l_2$ is equal to

(1) 1 : 6 (2) 3 : 1

(3) 6 : 1 (4) 4 : 1

Answer (3)

Sol. $l_1 = 20 - x$, $l_2 = x$

$$2A_1 + 3A_2 = 2\left(\frac{20-x}{4}\right)^2 + 3\pi\left(\frac{x}{2\pi}\right)^2$$

$$f(x) = \frac{(20-x)^2}{8} + \frac{3x^2}{4\pi}$$

$$f'(x_0) = \frac{1}{8}2(20-x)(-1) + \frac{3}{4\pi}\cdot 2x\Big|_{x_0} = 0$$

$$0 = -\frac{1}{4}(20-x_0) + \frac{6x_0}{4\pi}$$

$$\Rightarrow \frac{20-x_0}{4} = \frac{6x_0}{4\pi}$$

$$\pi(20-x_0) = 6x_0$$

$$20\pi = (6+\pi)x_0$$

$$x_0 = \frac{20\pi}{\pi+6}$$

$$\frac{\pi l_1}{l_2} = \pi\left(\frac{20-x_0}{x_0}\right) = \pi\left(\frac{\pi+6}{\pi} - 1\right)$$

$= 6$

68. Let

$$y = f(x) = \sin^3\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{\frac{3}{2}}\right)\right)$$

Then at x =1,

(1) $2y' + 3\pi^2 y = 0$ (2) $2y' + \sqrt{3}\pi^2 y = 0$

(3) $2y' + 3\pi^2 y = 0$ (4) $\sqrt{2}y' - 3\pi^2 y = 0$

Answer (1)

Sol. $f(x) = \sin^3\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{3/2}\right)\right)$

$$f'(x) = 3\sin^2\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{3/2}\right)\right)$$

$$\cos\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{3/2}\right)\right)$$

$$\frac{\pi}{3}\left(-\sin\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{3/2}\right)\right)$$

$$\frac{\pi}{3\sqrt{2}}\frac{3}{2}\left(-4x^3+5x^3+1\right)^{1/2}\left(-12x^2+10x\right)$$

$$f'(1) = \frac{3\pi^2}{16}$$

$$f(1) = \sin^3\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}2\sqrt{2}\right)\right)$$

$$= \sin^3\left(-\frac{\pi}{6}\right) = \frac{-1}{8}$$

$$\therefore \quad 2f'(1) + 3\pi^2 f(1) = 0$$

69. Let $y = f(x)$ represent a parabola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $y = -\frac{1}{2}$.

Then

$$S = \left\{x \in \mathbb{R} : \tan^{-1}\left(\sqrt{f(x)}\right) + \sin^{-1}\left(\sqrt{f(x)+1}\right) = \frac{\pi}{2}\right\}:$$

(1) Is an empty set

(2) Contains exactly one element

(3) Is an infinite set

(4) Contains exactly two elements

Answer (4)

Sol. Equation of parabola

$$k^2 + \left(h + \frac{1}{2}\right)^2 = \left|k + \frac{1}{2}\right|^2$$

$$k^2 + h^2 + h + \frac{1}{4} = k^2 + \frac{1}{4} + k$$

$$\boxed{y = x^2 + x}$$

$$\tan^{-1}\sqrt{x^2+x} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\tan^{-1}\sqrt{x^2+x} = \cos^{-1}\sqrt{x^2+x+1}$$

$$\sqrt{x^2+x+1} = \frac{1}{\sqrt{x^2+x+1}}$$

$x = 0, -1$

70. If the domain of the function $f(x) = \frac{[x]}{1+x^2}$, where [x] is greatest integer $\le x$, is $[2, 6)$, then its range is

(1) $\left(\frac{5}{37}, \frac{2}{5}\right]$

(2) $\left(\frac{5}{37}, \frac{2}{5}\right] - \left\{\frac{9}{29}, \frac{27}{109}, \frac{18}{89}, \frac{9}{53}\right\}$

(3) $\left(\frac{5}{26}, \frac{2}{5}\right]$

(4) $\left(\frac{5}{26}, \frac{2}{5}\right] - \left\{\frac{9}{29}, \frac{27}{109}, \frac{18}{89}, \frac{9}{53}\right\}$

Answer (1)

Sol. $f(x)=\dfrac{k}{1+x^2}$ is a decreasing function

where k > 0

$\therefore\quad x\in[2,3)\Rightarrow f(x)=\dfrac{2}{1+x^2}\in\left(\dfrac{2}{10},\dfrac{2}{5}\right]=R_1$

$x\in[3,4)\Rightarrow f(x)=\dfrac{3}{1+x^2}\in\left(\dfrac{3}{17},\dfrac{3}{10}\right]=R_2$

$x\in[4,5)\Rightarrow f(x)=\dfrac{4}{1+x^2}\in\left(\dfrac{4}{26},\dfrac{4}{17}\right]=R_3$

$x\in[5,6)\Rightarrow f(x)=\dfrac{5}{1+x^2}\in\left(\dfrac{5}{37},\dfrac{5}{26}\right]=R_4$

Range = $R_1\cup R_2\cup R_3\cup R_4$

$=\left(\dfrac{5}{37},\dfrac{2}{5}\right]$

71. The number of real roots of the equation $\sqrt{x^2-4x+3}+\sqrt{x^2-9}=\sqrt{4x^2-14x+6}$, is

(1) 1
(2) 0
(3) 2
(4) 3

Answer (1)

Sol. Common domain of functions is $(-\infty,\ -3]\cup[3,\ \infty)$

$\sqrt{x^2-4x+3}+\sqrt{x^2-9}=\sqrt{4x^2-14x+6}$

$\sqrt{x-3}\left(\sqrt{x-1}+\sqrt{x+3}\right)=\sqrt{x-3}\sqrt{4x-2}$

$\sqrt{x-3}=0\ \Rightarrow x=3$

Or $\sqrt{x-1}+\sqrt{x+3}=\sqrt{4x-2}$

On squaring,

$x-1+x+3+2\sqrt{(x-1)(x+3)}=4x-2$

$2\sqrt{x^2+2x-3}=2x-4$

$4\left(x^2+2x-3\right)=4x^2-16x+16$

$x=\dfrac{7}{6}\notin(-\infty,\ -3]\cup[3,\ \infty)$

$\therefore$ Only 1 solution

72. The value of $\displaystyle\int_{\pi/3}^{\pi/2}\frac{(2+3\sin x)}{\sin x(1+\cos x)}dx$ is equal to

(1) $\dfrac{10}{3}-\sqrt{3}-\log_e\sqrt{3}$
(2) $-2+3\sqrt{3}+\log_e\sqrt{3}$
(3) $\dfrac{7}{2}-\sqrt{3}-\log_e\sqrt{3}$
(4) $\dfrac{10}{3}-\sqrt{3}+\log_e\sqrt{3}$

Answer (4)

Sol. $\underbrace{\displaystyle\int_{\pi/3}^{\pi/2}\frac{2\sin x}{\sin^2 x(1+\cos x)}dx}_{I_1}+\underbrace{\displaystyle\int_{\pi/3}^{\pi/2}\frac{3}{1+\cos x}dx}_{I_2}$

$\cos x=t$

$\displaystyle\int_{1/2}^{0}\frac{-2dt}{(1-t^2)(1+t)}+\int_{\pi/3}^{\pi/2}\frac{3}{2}\sec^2\frac{x}{2}dx$

$\displaystyle 2\int_{0}^{1/2}\frac{dt}{\left(1-t^2\right)(1+t)}+3\tan\frac{x}{2}\Big|_{\pi/3}^{\pi/2}$

$=\ln\sqrt{3}-\sqrt{3}+\dfrac{10}{3}$

73. For the system of linear equations

$x+y+z=6$

$\alpha x+\beta y+7z=3$

$x+2y+3z=14,$

which of the following is **NOT** true?

(1) If $\alpha=\beta$ and $\alpha\neq 7$, then the system has a unique solution

(2) If $\alpha=\beta=7$, then the system has no solution

(3) There is a unique point $(\alpha,\ \beta)$ on the line $x+2y+18=0$ for which the system has infinitely many solutions

(4) For every point $(\alpha,\ \beta)\neq(7,\ 7)$ on the line $x-2y+7=0$, the system has infinitely many solutions

Answer (4)

Sol. $\Delta=\begin{vmatrix}1&1&1\\ \alpha&\beta&7\\ 1&2&3\end{vmatrix}$

$=1(3\beta-14)-1(3\alpha-7)+1(2\alpha-\beta)$

$=3\beta-14+7-3\alpha+2\alpha-\beta$

$=2\beta-\alpha-7$

So, for $\alpha=\beta\neq 7$, $\Delta\neq 0$ so unique solution

$\alpha=\beta=7$, equation (i) & (ii) represent 2 parallel planes so no solution.

If $\alpha-2\beta+7=0$, but $(\alpha,\beta)\neq(7,7)$, then no solution.

74. Let a differentiable function f satisfy

$f(x)+\int_3^x \frac{f(t)}{t}dt=\sqrt{x+1},\ x\geq 3$. Then $12f(8)$ is equal to

(1) 1 (2) 34

(3) 17 (4) 19

Answer (3)

Sol. Differentiating both sides we get

$$f'(x)+\frac{f(x)}{x}=\frac{1}{2\sqrt{x+1}}$$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=\frac{1}{2\sqrt{x+1}}$$

$\Rightarrow$ IF = x

$$\Rightarrow yx=\frac{1}{2}\int\frac{x}{\sqrt{x+1}}dx+c$$

$$\Rightarrow yx=\frac{1}{2}\left(\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}-2(x+1)^{\frac{1}{2}}\right)+c$$

$$xy=\frac{1}{3}(x+1)^{\frac{3}{2}}-(x+1)^{\frac{1}{2}}+c$$

$f(3) = 2$

So, $x = 3$, $y = 2$

$$\Rightarrow c=\frac{16}{3}$$

Now, $x = 8$

$$8f(8)=\frac{27}{3}-3+\frac{16}{3}=\frac{34}{3}$$

$$12f(8)=\frac{34}{3}\times\frac{12}{8}=17$$

Option (3) is correct.

75. Let $A=\begin{pmatrix}1&0&0\\0&4&-1\\0&12&-3\end{pmatrix}$. Then the sum of the diagonal elements of the matrix $(A+I)^{11}$ is equal to

(1) 2050 (2) 4097

(3) 6144 (4) 4094

Answer (2)

Sol. $A^2=\begin{bmatrix}1&0&0\\0&4&-1\\0&12&-3\end{bmatrix}\begin{bmatrix}1&0&0\\0&4&-1\\0&12&-3\end{bmatrix}$

$$=\begin{bmatrix}1&0&0\\0&4&-1\\0&12&-3\end{bmatrix}=A$$

Now,

$$(A+I)^{11}={}^{11}C_0A^{11}+{}^{11}C_1A^{10}+\ldots{}^{11}C_{11}I$$

$$=A\left({}^{11}C_0+{}^{11}C_1\ldots{}^{11}C_{10}\right)+I$$

$$=A(2^{11}-1)+I$$

Trace of

$$(A+I)^{11}=2^{11}+4(2^{11}-1)+1-3(2^{11}-1)+1$$

$$=2\times2^{11}-4+3+2$$

$$=2^{12}+1$$

$= 4097$

76. Let a circle C_1 be obtained on rolling the circle $x^2+y^2-4x-6y+11=0$ upwards 4 units on the tangent T to it at the point (3,2). Let C_2 be the image of C_1 in T. Let A and B be the centers of circles C_1 and C_2 respectively, and M and N be respectively the feet of perpendiculars drawn from A and B on the x-axis. Then the area of the trapezium AMNB is:

(1) $2(2+\sqrt{2})$ (2) $3+2\sqrt{2}$

(3) $4(1+\sqrt{2})$ (4) $2(1+\sqrt{2})$

Answer (3)

Sol. Given circle is $x^2+y^2-4x-6y+11=0$, centre (2, 3)

Tangent at (3, 2) is $x-y=1$

After rolling up by 4 units centre of C_1 is

$$A\equiv\left(2+\frac{4}{\sqrt{2}},3+\frac{4}{\sqrt{2}}\right)$$

$$\Rightarrow A=\left(2+2\sqrt{2},3+2\sqrt{2}\right)$$

B is the image of A in $x-y=1$

$$\frac{x-\left(2+2\sqrt{2}\right)}{1}=\frac{y-\left(3+2\sqrt{2}\right)}{-1}=\frac{-2(-2)}{2}=2$$

$$\Rightarrow x=4+2\sqrt{2},\ y=1+2\sqrt{2}$$

Area of $AMNB$

$$=\frac{1}{2}\left(4+4\sqrt{2}\right)\left(4+2\sqrt{2}-\left(2+2\sqrt{2}\right)\right)$$

$$=4\left(1+\sqrt{2}\right)$$

77. If $\sin^{-1}\frac{\alpha}{17}+\cos^{-1}\frac{4}{5}-\tan^{-1}\frac{77}{36}=0$, $0<\alpha<13$, then $\sin^{-1}(\sin\alpha)+\cos^{-1}(\cos\alpha)$ is equal to

(1) π (2) $16-5\pi$

(3) 16 (4) 0

Answer (1)

Sol. $\sin^{-1}\left(\frac{\alpha}{17}\right)=-\cos^{4}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{77}{36}\right)$

Let $\cos^{-1}\left(\frac{4}{5}\right)=p$ and $\tan^{-1}\left(\frac{77}{36}\right)=q$

$\Rightarrow \sin\left(\sin^{-1}\frac{\alpha}{17}\right)=\sin(q-p)$

$= \sin q\cdot\cos p-\cos q\cdot\sin p$

$\Rightarrow \frac{\alpha}{17}=\frac{77}{85}\cdot\frac{4}{5}-\frac{36}{85}\cdot\frac{3}{5}$

$\Rightarrow \alpha=\frac{200}{25}=8$

$\sin^{-1}\sin 8+\cos^{-1}\cos 8$

$\Rightarrow -8+3\pi+8-2\pi$

$=\pi$

78. If the sum and product of four positive consecutive terms of a G.P., are 126 and 1296, respectively, then the sum of common ratio of all such GPs is

(1) 14 (2) 7

(3) 3 (4) $\frac{9}{2}$

Answer (3)

Sol. Let the terms be $\frac{a}{r^3},\frac{a}{r},ar,ar^3$

$\frac{a}{r^3}\cdot\frac{a}{r}\cdot ar\cdot ar^3=1296$

$\Rightarrow a=6$

Now, $\frac{a}{r^3}+\frac{a}{r}+ar+ar^3=126$

$\Rightarrow \frac{1}{r^3}+\frac{1}{r}+r+r^3=21$

$\Rightarrow \left(r+\frac{1}{r}\right)\left(\left(r+\frac{1}{r}\right)^2-3\right)+\left(r+\frac{1}{r}\right)=21$

Let $r+\frac{1}{r}=t$

$t^3-3t+t=21$

$\Rightarrow t^3-2t-21=0$

$\Rightarrow (t-3)(t^2+3t+7)=0$

$\Rightarrow t=3$

$r+\frac{1}{r}=3$

$\Rightarrow r^2-3r+1=0$

$\Rightarrow r_1+r_2=3$

79. For all $z\in C$ on the curve $C_1:|z|=4$, let the locus of the point $z+\frac{1}{z}$ be the curve C_2. Then:

(1) the curve C_1 lies inside C_2

(2) the curve C_2 lies inside C_1

(3) the curves C_1 and C_2 intersect at 4 points

(4) the curves C_1 and C_2 intersect at 2 points

Answer (3)

Sol. Let $z=4e^{i\theta}$

$\Rightarrow z+\frac{1}{z}=4e^{i\theta}+\frac{1}{4}e^{-i\theta}$

$\Rightarrow x+iy=\frac{17}{4}\cos\theta+i\frac{15}{4}\sin\theta$

$\Rightarrow x=\frac{17}{4}\cos\theta,\quad y=\frac{15}{4}\sin\theta$

$\Rightarrow \frac{x^2}{\left(\frac{17}{4}\right)^2}+\frac{y^2}{\left(\frac{15}{4}\right)^2}=1$

Which is an ellipse whose $a>r$.

80. If the maximum distance of normal to the ellipse $\frac{x^2}{4}+\frac{y^2}{b^2}=1$, $b<2$, from the origin is 1, then the eccentricity of the ellipse is:

(1) $\frac{\sqrt{3}}{2}$

(2) $\frac{1}{2}$

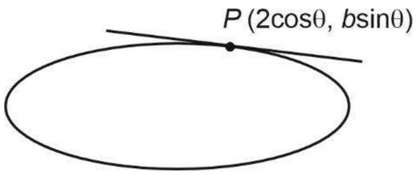
(3) $\frac{\sqrt{3}}{4}$

(4) $\frac{1}{\sqrt{2}}$

Answer (1)

Sol. $\frac{x^2}{4}+\frac{y^2}{b^2}=1,\quad b<2$

Equation of normal at P :



$2\sec\theta x - by\,\mathrm{cosec}\theta = 4 - b^2$ ...(i)

Distance from (0, 0)

$$d=\left|\frac{b^2-4}{\sqrt{4\sec^2\theta+b^2\mathrm{cosec}^2\theta}}\right|$$

$$d=\left|\frac{b^2-4}{\sqrt{4+b^2+4\tan^2\theta+b^2\cot^2\theta}}\right|$$

Now, $d_{max} = 1$

$$\therefore\quad \frac{4-b^2}{\sqrt{b^2+4+4b}}=1$$

$\Rightarrow\ 4 - b^2 = (b + 2) \Rightarrow b^2 + b - 2 = 0$

$\Rightarrow (b + 2)(b - 1) = 0$

$\Rightarrow b = 1$

$$\therefore\quad e=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$$

$\therefore$ option (1) is correct.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. Let θ be the angle between the planes $P_1:\vec{r}\cdot(\hat{i}+\hat{j}+2\hat{k})=9$ and $P_2:\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=15$. Let L be the line that meets P_2 at the point (4, –2, 5) and makes an angle θ with the normal of P_2. If α is the angle between L and P_2, then $(\tan^2\theta)(\cot^2\alpha)$ is equal to _______.

Answer (09)

Sol. $P_1:\vec{r}\cdot(\hat{i}+\hat{j}+2\hat{k})=9$

$P_2:\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=15$

then $\cos\theta=\frac{3}{\sqrt{6}\cdot\sqrt{6}}=\frac{1}{2}$

$\therefore\ \boxed{\theta=\frac{\pi}{3}}$ Now, $\alpha=\frac{\pi}{2}-\theta$

$\therefore\ \tan^2\theta\cdot\cot^2\alpha=\tan^4\theta$

$=(\sqrt{3})^4=9$

82. The remainder on dividing 5^{99} by 11 is _______.

Answer (09)

Sol. $5 \equiv 5 \pmod{11}$

$5^2 \equiv 3 \pmod{11}$

$5^4 \equiv -2 \pmod{11}$

$5^5 \equiv 1 \pmod{11}$

$5^{99} \equiv -2 \pmod{11}$

$\therefore$ remainder = 9

83. Number of 4-digit numbers that are less than or equal to 2800 and either divisible by 3 or 11, is equal to _______.

Answer (710)

Sol. Numbers which are divisible by 3 (4 digit) and less than or equal to 2800

$$=\frac{2799-1002}{3}+1=600$$

Numbers which are divisible by 11 (4 digit) and less than or equal to 2800

$$=\frac{2794-1001}{11}+1=164$$

Numbers which are divisible by 33 (4 digit) and less than or equal to 2800

$$=\frac{2772-1023}{33}+1=54$$

$\therefore$ Total no. = 710

84. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}|=\sqrt{14}, |\vec{b}|=\sqrt{6}$ and $|\vec{a}\times\vec{b}|=\sqrt{48}$. Then $(\vec{a}\cdot\vec{b})^2$ is equal to _______.

Answer (36)

Sol. $|\vec{a}| = \sqrt{14}, |\vec{b}| = \sqrt{6}$ and $|\vec{a} \times \vec{b}| = \sqrt{48}$

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

$$48 + (\vec{a} \cdot \vec{b})^2 = 6 \times 14$$

$$(\vec{a} \cdot \vec{b})^2 = 84 - 48$$

$$= 36$$

85. If the variance of the frequency distribution

x_i	2	3	4	5	6	7	8
Frequency f_i	3	6	16	α	9	5	6

is 3, then α is equal to ________.

Answer (05.00)

Sol. $$3 = \frac{3.2^2 + 6.3^2 + 16.4^2 + \alpha.5^2 + 9.6^2 + 5.7^2 + 6.8^2}{45 + \alpha} - \left(\frac{225 + 5\alpha}{45 + \alpha}\right)^2$$

$$3 = \frac{12 + 54 + 256 + 25\alpha + 324 + 245 + 384}{45 + \alpha} - 25$$

$28(45 + \alpha) = 1275 + 25\alpha$

OR $1260 + 28\alpha = 1275 + 25\alpha$

$\Rightarrow \alpha = 5$

86. Let $\alpha > 0$, be the smallest number such that the expansion of $\left(x^{\frac{2}{3}} + \frac{2}{x^3}\right)^{30}$ has a term $\beta x^{-\alpha}, \beta \in \mathbb{N}$.

Then α is equal to ________.

Answer (2.00)

Sol. $\because \left(x^{\frac{2}{3}} + \frac{2}{x^3}\right)^{30} = \sum_{r=0}^{30} {}^{30}C_r \left(x^{\frac{2}{3}}\right)^{30-r} \cdot \left(\frac{2}{x^3}\right)^r$

Here $\frac{60 - 2r}{3} - 3r \in$ integer.

$\because$ β is always a natural number.

$\therefore$ $r = 6$

Thus $\alpha = 2$

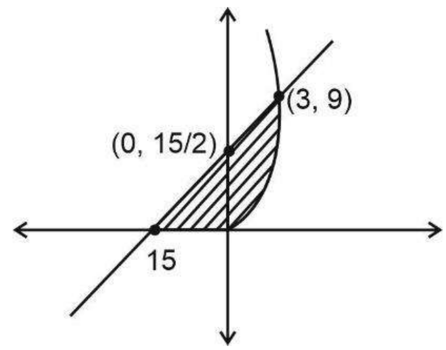
87. Let for $x \in \mathbb{R}$,

$$f(x) = \frac{x + |x|}{2} \quad \text{and} \quad g(x) = \begin{cases} x, & x < 0 \\ x^2, & x \ge 0 \end{cases}.$$

Then area bounded by the curve $y = (fog)(x)$ and the line $y = 0$, $2y - x = 15$ is equal to _______.

Answer (72)

Sol. $f_o g(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \ge 0 \end{cases}$



$$\text{Area} = \frac{1}{2} \times 15 \times \frac{15}{2} + \int_0^3 \left(\frac{x + 15}{2} - x^2\right) dx$$

$$\frac{225}{4} + \frac{99}{4} - 9$$

$$\frac{324}{4} - 9$$

$81 - 9$

$= 72$

88. Let 5 digit numbers be constructed using the digits 0, 2, 3, 4, 7, 9 with repetition allowed, and are arranged in ascending order with serial numbers. Then the serial number of the number 42923 is ______.

Answer (2997)

Sol.

2	_	_	_	_	$\rightarrow$	$6^4 = 1296$
3	_	_	_	_	$\rightarrow$	$6^4 = 1296$
4	0	_	_	_	$\rightarrow$	$6^3 = 216$
4	2	–	–	–	$\rightarrow$	$5 \times 6^2 = 180$
4	3	–	–	–		
4	4	–	–	–		
4	7	–	–	–		
4	2	9	0	–	$\rightarrow$	6
4	2	9	2	0	$\rightarrow$	1
4	2	9	2	2	$\rightarrow$	1
4	2	9	2	3	$\rightarrow$	1
						2997

89. Let $a_1, a_2, \ldots, a_n$ be in A.P. If $a_5 = 2a_7$ and $a_{11} = 18$, then

$$12\left(\frac{1}{\sqrt{a_{10}} + \sqrt{a_{11}}} + \frac{1}{\sqrt{a_{11}} + \sqrt{a_{12}}} + \ldots + \frac{1}{\sqrt{a_{17}} + \sqrt{a_{18}}}\right)$$

is equal to _______.

Answer (08)

Sol. $a_{11} = 18$

$a + 10d = 18$ ...(i)

$a_5 = 2a_7$

$a + 4d = 2(a + 6d)$

$a = -8d$ ...(ii)

(i) and (ii) $\Rightarrow a = -72, d = 9$.

On rationalising the denominator, given expression

$$= 12\left[\frac{\sqrt{a_{10}} - \sqrt{a_{11}}}{-d} + \frac{\sqrt{a_{11}} - \sqrt{a_{12}}}{-d} + \ldots + \frac{\sqrt{a_{17}} - \sqrt{a_{18}}}{-d}\right]$$

$$= 12\left[\frac{\sqrt{a_{10}} - \sqrt{a_{18}}}{-d}\right]$$

$$= 12\left[\frac{\sqrt{a_{11} - d} - \sqrt{a_{11} + 7d}}{-d}\right]$$

$$= 12\left[\frac{\sqrt{18 - 9} - \sqrt{18 + 63}}{-9}\right]$$

$$= 12 \times \frac{2}{3} = 8$$

90. Let the line $L : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{1}$ intersect the plane $2x + y + 3z = 16$ at the point P. Let the point Q be the foot of perpendicular from the point $R(1, -1, -3)$ on the line L. If α is the area of the triangle PQR then α^2 is equal to ______.

Answer (180)

Sol. $L : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{1} = r$ (say)

Let $P \equiv (2r_1 + 1, -r_1, r_1 + 3)$

P lies on $2x + y + 3z = 16$

$\therefore \quad 2(2r_1 + 1) + (-r_1 - 1) + 3(r_1 + 3) = 16$

$r_1 = 1$

$P \equiv (3, -2, 4)$

$R \equiv (1, -1, -3)$

Let $Q \equiv (2r_2 + 1, -r_2 - 1, r_2 + 3)$

DRs of $QR \equiv (2r_2 - r_2\ r_2 + 6)$

DRs of $L \equiv (2, -1, 1)$

$QR \perp L \Rightarrow 4r_2 + r_2 + r_2 + 6 = 0$

$r_2 = -1$

$Q \equiv (-1, 0, 2)$

$$\overrightarrow{QP} \times \overrightarrow{RP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & 2 \\ 2 & -1 & 7 \end{vmatrix} = -12\hat{i} - 24\hat{j} + 0\hat{k}$$

$$\alpha = [PQR] = \frac{1}{2}\left|\overrightarrow{QP} \times \overrightarrow{RP}\right| = \frac{1}{2} \times 12\sqrt{5}$$

$= 6\sqrt{5}$

$\alpha^2 = 180$

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