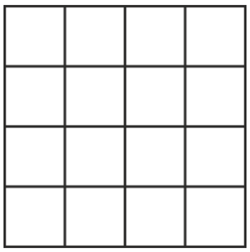
MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer:

1. If a square is divided in 4 × 4 squares.



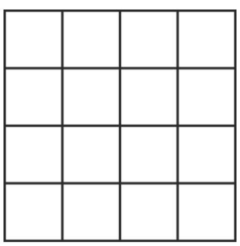
If two squares are chosen randomly then the probability that the squares doesn't share common side is

(1) $\frac{3}{5}$ (2) $\frac{4}{5}$

(3) $\frac{3}{20}$ (4) $\frac{7}{10}$

Answer (2)

Sol.



Total = ${}^{16}C_2$

Required ways = Total – (adjacent squares)

= ${}^{16}C_2$ – [3 pairs in vertical and horizontal for each row and column]

= ${}^{16}C_2$ – [3 × 4 + 3 × 4]

Probability = $\frac{{}^{16}C_2 - 24}{{}^{16}C_2} = \frac{120-24}{120} = \frac{96}{120} = \frac{4}{5}$

2. If $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$, then the value of definite integration $\int_0^{2I} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ is

(1) $\frac{\pi}{16}$ (2) $\frac{\pi^2}{16}$

(3) $\frac{\pi}{8}$ (4) $\frac{\pi^2}{8}$

Answer (2)

Sol. $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$

$\int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$

$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} (1) dx = \frac{\pi}{2}$

$\Rightarrow \quad I = \frac{\pi}{4}$

$\Rightarrow \quad 2I = \frac{\pi}{2}$

$\Rightarrow \quad I_1 = \int_0^{2I} \frac{x \sin x \cos x}{(\sin^4 + \cos^4 x)} dx = \int_0^{2I} \frac{x \sin x \cos x}{\sin^4 + \cos^4 x} dx$

$\Rightarrow \quad I_1 = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - 2\right) \cos \sin x dx}{\cos^4 x + \sin^4 x}$

Adding,

$2I_1 = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} \cos x \sin x dx}{(\sin^4 x + \cos^4 x)}$

$(\sin^4 x + \cos^4 x) = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$$= \left(1 - \frac{\sin^2 2x}{2}\right) = \frac{2 - \sin^2 x}{2}$$

$$= \frac{1 - \cos^2 2x}{2}$$

$$2I = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\frac{1 + \cos^2 2x}{2}}\,dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_{-1}^{1} \frac{dt}{(1 + t^2)}, \quad t = \cos 2x$$

$$\Rightarrow \quad I = \frac{\pi}{8}\cdot\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi^2}{16}$$

3. Consider the terms 8, 21, 34, 47, …, 320. The variance of the given data set is

(1) 8788 (2) 8614

(3) 720 (4) 9402

Answer (1)

Sol. μ = Mean = $\frac{8 + 21 + ...320}{25} = \frac{25}{2}\frac{(8 + 320)}{25} = 164$

Variance = $\frac{8^2 + 21^2 + ...320^2}{25} - (164)^2 = 8788$

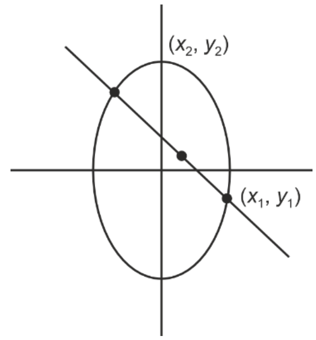
4. Let $M\left(\frac{1}{2}, 1\right)$ be the mid-point of a chord to the Ellipse $\frac{x^2}{2} + \frac{y^2}{4} = 1$, then the length of chord is

(1) $\frac{2}{3}\sqrt{5}$ (2) $\frac{\sqrt{5}}{3}$

(3) $2\sqrt{\frac{5}{3}}$ (4) $\frac{\sqrt{5}}{2}$

Answer (3)

Sol. $\frac{x^2}{2} + \frac{y^2}{4} = 1$



The equation of chord bisected at $\left(\frac{1}{2}, 1\right)$ will be $T = S_1$.

$$\Rightarrow \frac{x}{2}\left(\frac{1}{2}\right) + \frac{y}{4}(1) - 1 = \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{1^2}{4} - 1$$

$$\Rightarrow \frac{x}{4} + \frac{y}{4} = \frac{1}{8} + \frac{1}{4}$$

$$\Rightarrow x + y = \frac{3}{2} \Rightarrow x_1 + y_1 = \frac{3}{2}$$

$$x_2 + y_2 = \frac{3}{2}$$

$$\Rightarrow (x_1 - x_2) + (y_1 - y_2) = 0$$

$$\Rightarrow (x_1 - x_2)^2 = (y_1 - y_2)^2$$

$2x^2 + y^2 = 4$

$$2x^2 + \left(\frac{3}{2} - x\right)^2 = 4$$

$$3x^2 - 3x + \frac{9}{4} - 4 = 0$$

$$\Rightarrow 3x^2 - 3x - \frac{7}{4} = 0$$

$$x_1 + x_2 = 1$$

$$x_1 . x_2 = \frac{-7}{12}$$

$$(x_1 - x_2)^2 = (x_1 + x_2) - 4x_1x_2$$

$$= 1 + \frac{28}{72} = \frac{10}{3}$$

$\Rightarrow$ Length of chord = $\sqrt{\frac{10}{3} + \frac{10}{3}} = \sqrt{\frac{20}{3}} = 2\sqrt{\frac{5}{3}}$

5. If the square of the shortest distance between the lines $\frac{x-2}{1}=\frac{y-1}{2}=\frac{z+3}{-3}$ and $\frac{x+1}{2}=\frac{y+3}{4}=\frac{z+5}{-5}$ is $\frac{m}{n}$ (where m, n are coprime number) then $m+n$ equals to

(1) 6 (2) 9
(3) 21 (4) 14

Answer (2)

Sol. $\frac{x-2}{1}=\frac{y-1}{2}=\frac{z+3}{-3}$

$\frac{x+1}{2}=\frac{y+3}{4}=\frac{z+5}{-5}$

$a_1 = (2, 1, -3)$

$a_2 = (-1, -3, -5)$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix}$$

$2\hat{i}-\hat{j}$

$$(SD)^2=\left|\frac{(a_2-a_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|^2$$

$$=\left|\frac{(3\hat{i}+4\hat{j}+2k)\cdot(2\hat{i}-\hat{j})}{\sqrt{4+1}}\right|^2$$

$$=\left(\frac{2}{\sqrt{5}}\right)^2=\frac{4}{5}$$

$m+n=9$

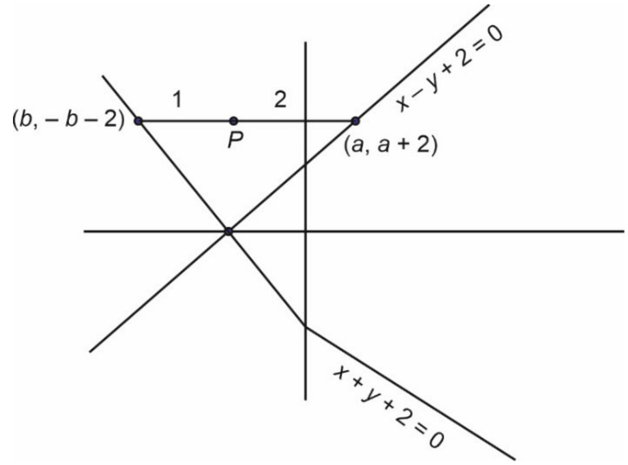
6. A rod of length 8 units having two end points always lie on $x-y+2=0$ and $x+y+2=0$. A point P divide this line in ratio 2 : 1. Then locus of P is

(1) $9x^2+9y^2+36x-28=0$
(2) $8x^2+8y^2+36x+27=0$
(3) $9x^2+9y^2-36x+28=0$
(4) $8x^2+8y^2-36x-27=0$

Answer (1)

Sol. $x-y+z=0$

$x+y+z=0$



$PQ = 8$

$\Rightarrow\ (a-b)^2+(a+2+b+2)^2=64$

$(a-b)^2+(a+b+4)=64$

$2a^2+2b^2-2ab+16+2(ab+4b+4a)=64$

$\Rightarrow\ 2a^2+2b^2+8a+8b=48$

$\Rightarrow\ a^2+b^2+4a+4b-24=0=(a+2)^2+(b+2)^2=32$

$$h=\frac{2b+1(a)}{3},K=\frac{2(-2-b)+1(a+2)}{3}$$

$3h=2b+a,\ 3K=a-2-2b\rightarrow$ Solve for a and b.

$$a=\frac{3h+3K+2}{2},\ b=\frac{3h-3K-2}{4}$$

$$(a+2)=\left(\frac{3h+3K+6}{2}\right),\ b+2=\left(\frac{3h-3K+6}{2}\right)$$

$$\Rightarrow\left(\frac{3h+3K+6}{2}\right)^2+\left(\frac{3h-3K+6}{2}\right)^2=32$$

$9(x+y+z)^2+9(x-y+z)^2=128$

$18[x^2+y^2+4x+4]=128\Rightarrow x^2+y^2+4x-\frac{28}{9}=0$

7. Evaluate

$$\lim_{x\to\infty}\left(\frac{2x^7+3x-5}{3x^7+5x-2}\right)\frac{(3x-1)^{\frac{x}{2}}}{\left(\sqrt{3x+2}\right)^x}$$

(1) $\frac{2}{3\sqrt{e}}$ (2) $\frac{3}{\sqrt{e}}$
(3) $\frac{2}{5\sqrt{e}}$ (4) $\frac{2}{\sqrt{e}}$

Answer (1)

Sol. $\lim\limits_{x\to\infty} \underbrace{\left[\frac{2x^7+3x-5}{3x^7+5x-2}\right]}_{M} \underbrace{\left(\frac{(3x-1)^{\frac{x}{2}}}{\left(\sqrt{3x+2}\right)^x}\right)}_{N}$

$\lim\limits_{x\to\infty} M = \lim\limits_{x\to\infty}\left[\frac{2+\frac{3}{x^6}-\frac{5}{x^7}}{3+\frac{5}{x^6}-\frac{2}{x^7}}\right] = \frac{2}{3}$

$\lim\limits_{x\to\infty} N = \lim\limits_{x\to\infty}\left(\frac{3x-1}{3x+2}\right)^{\frac{x}{2}} = (1^\infty \text{ form})$

$= e^L$

$L = \lim\limits_{x\to\infty}\left(\frac{3x-1}{3x+2}-1\right)\frac{x}{2}$

$= \lim\limits_{x\to\infty} -\frac{3x}{2(3x+2)} = \lim\limits_{x\to\infty} -\frac{3}{2\left(3+\frac{2}{x}\right)} = -\frac{3}{6} = -\frac{1}{2}$

$\therefore \quad \lim\limits_{x\to\infty} N = e^{-\frac{1}{2}}$

$\therefore \quad \lim\limits_{x\to\infty} MN = \frac{2}{3}e^{-\frac{1}{2}}$

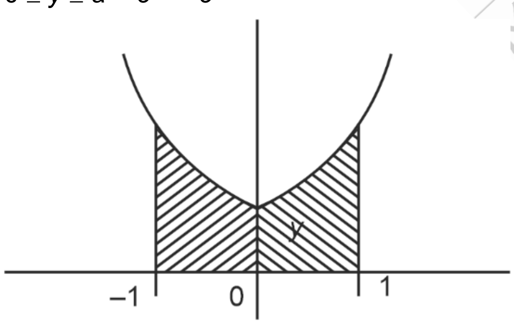
8. Let S be the region consisting of points (x, y) such that $-1 \le x \le 1$ and $0 \le y \le a + e^{|x|} - e^{-|x|}$. If area of bounded region is $\frac{2(e^2+8e+1)}{e}$ square units, then a is equal to

(1) 8 (2) 10

(3) 7 (4) 5

Answer (2)

Sol. $0 \le y \le a + e^{|x|} - e^{-|x|}$



$\text{Area} = 2\int_0^1 \left(a+e^{|x|}-e^{-|x|}\right)dx$

$= 2\int_0^1 \left(a+e^{x}-e^{-x}\right)dx$

$= 2\left(ax+e^x+e^{-x}\right|_0^1$

$= 2\,(a + e + e^{-1}) - 2\,(e^0 + e^{-0})$

$= 2\left(a+e+\frac{1}{e}\right)-2(2)$

$= 2e+\frac{2}{e}+2a-4 = 2\left(e+\frac{1}{e}+(a-2)\right)$

$= \frac{2(e^2+(a-2)(e+1)}{e}$

Comparing, $a - 2 = 8 \Rightarrow a = 10$

9. If z is a complex number such that $|z| = 1$ and $\left|\frac{z}{\bar{z}}+\frac{\bar{z}}{z}\right| = 1$, then the number of complex number z is

(1) 8 (2) 4

(3) 2 (4) Zero

Answer (1)

Sol. $\left|\frac{z}{\bar{z}}+\frac{\bar{z}}{z}\right| = 1$

$\Rightarrow \left|z^2+(\bar{z})\right| = 1$

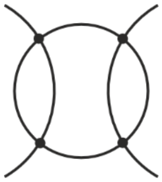
$Z = x+iy$

$\Rightarrow \left|Z^2+(\bar{Z})^2\right| = \left|2(x^2-y^2)\right|$

$\Rightarrow x^2-y^2 = \pm\frac{1}{2}$

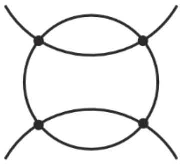
and $x^2 + y^2 = 1$

$x^2-y^2 = \frac{1}{2}$ and $x^2-y^2 = 1$



4 sol.

$$x^2 - y^2 = \frac{-1}{2}, x^2 + y^2 = 1$$



4 sol.

$\Rightarrow$ Total 8 sol.

10. If $A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $A\begin{bmatrix}4\\1\\3\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $A\begin{bmatrix}2\\1\\2\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$. Then a_{23} equals to

(1) 3
(2) –1
(3) 2
(4) –2

Answer (2)

Sol. Let $A=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$

$$\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\Rightarrow \quad b = 0 \quad e = 0 \quad h = 1$

$$\begin{bmatrix}a & 0 & c\\ d & 0 & f\\ g & 1 & i\end{bmatrix}\begin{bmatrix}4\\1\\3\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$4a + 3c = 0$

$4d + 3f = 1$ ...(1)

$4g + p + 3i = 0$

$$\begin{bmatrix}a & 0 & c\\ d & 0 & f\\ g & 1 & i\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$2a + 2c = 1$

$2d + 2f = 0$ ...(2)

$2g + 1 + 2f = 0$

Solving (1) and (2)

$d = 1 \quad f = -1$

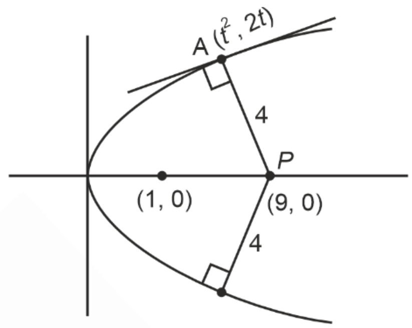
$\Rightarrow \quad a_{23} = -1$

11. Let $(a, 0)$ be a point such that its shortest distance from the parabola $y^2 = 4x$ is 4. Then the equation of circle passing through $(a, 0)$ and focus of the parabola having centre on the axis of parabola is

(1) $x^2 + y^2 - 6x + 8 = 0$
(2) $x^2 + y^2 + 6x + 5 = 0$
(3) $x^2 + y^2 - 6x + 5 = 0$
(4) $x^2 + y^2 - 9x + 5 = 0$

Answer (3)

Sol.



The shortest distance lie along the normal.

Slope of normal at A is $(-t)$

$\Rightarrow \quad (-t) = \dfrac{2t - 0}{(t^2 - a)}$

$\Rightarrow \quad t = 0,\ t^2 - a = -2$

$t^2 = (a - 2)$

$(t^2 - a)^2 + (2t)^2 = 16$

$\Rightarrow \ 4 + 4t^2 = 16 \Rightarrow t^2 = 3,\ a = 5$

Equation of circle passing through (1, 0) and (5, 0) and having centre on the axis will be diametric form of circle.

$(x - 1)(x - 5) + (y - 0)(y - 0) = 0$

$x^2 + y^2 - 6x + 5 = 0$

12. If 10th and 12th terms of an arithmetic progression are roots of equation $3x^2 - px + q = 0$ and common difference of the arithmetic progression is $\frac{3}{2}$. Also, the sum of first 11 terms of the arithmetic progression is 88, then $q - 2p$ is

(1) 625
(2) 474
(3) 729
(4) 476

Answer (2)

Sol. $\Rightarrow \quad S_{11} = \frac{11}{2}(2a + (11-1)d) = 88$

$\Rightarrow \quad (a + 5d) = 8$

$a = 8 - 5\left(\frac{3}{2}\right) = \frac{1}{2}$

$T_{10} = a + 9d = \frac{1}{2} + 9\left(\frac{3}{2}\right) = \frac{1+27}{2} = 14$

$T_{11} = a + 10d = \frac{1}{2} + 10\left(\frac{3}{2}\right) = \frac{31}{2}$

$\Rightarrow$ Sum of roots $= 14 + \frac{31}{2} = \frac{P}{3}$

Product of roots $= (14)\frac{(31)}{2} = \frac{q}{3}$

$\frac{p}{3} = \frac{59}{2}, \quad q = (7.31)\cdot 3$

$P = \frac{59 \times 3}{2}, \quad q = 651$

$\Rightarrow \quad q - 2p = 651 - 59 \times 3 = 474$

13.

14.

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. There are 5 boys and 4 girls. The sum of number of ways in which they can be seated such that all boys sit together and number of ways such that no boys sit together is equal to

Answer (17280)

Sol. All boys sit together

$\boxed{B_1B_2B_3B_4B_5}$ $G_1G_2G_3G_4$

= 5! 5!

no boys sit together

$\uparrow G_1 \uparrow G_2 \uparrow G_3 \uparrow G_4 \uparrow$

$4! \cdot 5!$

$\therefore$ sum $= 5! \cdot 5! + 5! \cdot 4!$

$= 5!\,(4! + 5!)$

$= 5! \cdot 4!\,(1 + 5) = 6 \cdot 4!\,5! = 17280$

22. Let $f(x) = 6 + 16\cos\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right)\cos x\ \sin 3x$ $\cos 6x$ if range of $f(x)$ is $[\alpha, \beta]$ then distance of (α, β) from $3x + 4y + 12 = 0$

Answer (11)

Sol. $f(x) = 6 + 16 \times \frac{1}{4}\cos 3x \sin 3x \cos 6x$

$= 6 + 2\sin 6x \cos 6x$

$= 6 + \sin 12x$

$R_{f(x)} \in [5, 7]$

Now distance $= \left|\frac{15 + 28 + 12}{\sqrt{3^2 + 4^2}}\right| = \frac{55}{5} = 11$

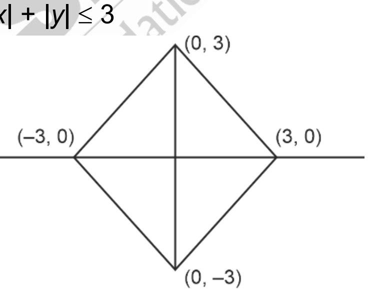
23. $A = \{(x, y) \mid |x + y| \geq 3\}$;

$B = \{(x, y) \mid |x| + |y| \leq 3\}$

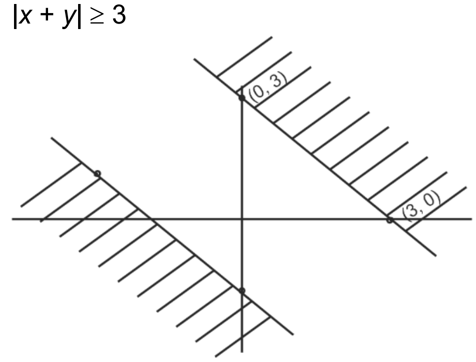
Let $C = A \cap B$. Find the sum of $(x, y) \forall\ x, y \in C$.

Answer (0)

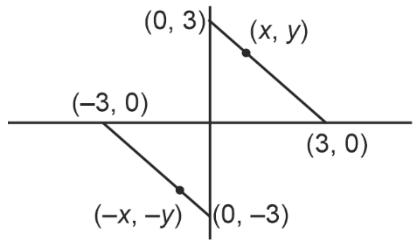
Sol. $|x| + |y| \leq 3$



$|x + y| \geq 3$



$A \geq B =$



$\therefore$ Sum = 0

24. If system of linear equations

$x + y + z = 6$

$x + 2y + 5z = 9$

$x + 5y + \lambda z = \mu$

has no solutions. Then value of λ equals to

Answer (17)

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 5 & \lambda \end{vmatrix} = 0$

$\lambda - 17 = 0$

$\Rightarrow \lambda = 17$

For no solution $\Delta = 0$ at least one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 9 & 2 & 5 \\ \mu & 5 & 17 \end{vmatrix} \neq 0$$

$\Rightarrow 3\mu - 54 \neq 0$

$\mu \neq 18$

$\Rightarrow$ for $\mu \neq 18$ and $\lambda = 17$ given system of equations have no solution.

25. Let $\int x^3 \sin x dx = g(x) + C$, where $g(0) = 0$. If $8\left(g\left(\frac{\pi}{2}\right) + g'\left(\frac{\pi}{2}\right)\right) = \alpha\pi^3 + \beta\pi^2 + \gamma$, where α, β and γ are integers, then the value of $\alpha + \beta + \gamma$ is

Answer (55)

Sol. $I = \int x^3 \sin x dx = x^3(-\cos x) - \int 3x^2(-\cos x)dx$

$= -x^3 \cos x + 3\left[x^2(\sin x) - \int 2x \sin x \, dx\right]$

$= -x^3 \cos x + 3x^2 \sin x - 6\left[x(-\cos x) - \int(-\cos x)dx\right]$

$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$

$\because g(0) = 0 \Rightarrow g(x) = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$

$= g\left(\frac{\pi}{2}\right) = \frac{3\pi^2}{4} - 6, \quad g'\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^3 \sin\left(\frac{\pi}{2}\right) = \frac{\pi^3}{8}$

$\Rightarrow 8\left[g\left(\frac{\pi}{2}\right) + g'\left(\frac{\pi}{2}\right)\right] = 6\pi^2 - 48 + \pi^3 = \pi^3 + 6\pi^2 - 48$

$\Rightarrow \alpha = 1, \ \beta = 6, \ \gamma = -48$

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