MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer:

1. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{k^3+6k^2+11k+5}{(k+3)!}$ is equal to

(1) $\dfrac{5}{3}$ (2) $\dfrac{8}{3}$

(3) 3 (4) $\dfrac{7}{3}$

Answer (1)

Sol. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{k^3+6x^2+11k+6-1}{(k+3)!}$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)(k+2)(k+3)-1}{(k+3)!}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k!}-\frac{1}{(k+3)!}$$

$$=\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\infty\right)-\left(\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots\infty\right)$$

$$=(e-1)-\left(e-1-\frac{1}{1!}-\frac{1}{2!}-\frac{1}{3!}\right)$$

$$=1+\frac{1}{2}+\frac{1}{6}=\frac{10}{6}=\frac{5}{3}$$

2. Sum of first three terms of an AP with integer common difference is 54 and sum of first twenty terms lies between 1600 to 1800, find a_{11}

(1) 108 (2) 90

(3) 111 (4) 115

Answer (2)

Sol. Let AP be a, $a + d$, $a + 2d$ ...

$3a + 3d = 54$

$a + d = 18$ ...(i)

$$1600<\frac{20}{2}[2a+19d]<1800$$

$160 < 2a + 19d < 180$

$160 < 18 \times 2 + 17d < 180$

$$\frac{124}{7}<d<\frac{144}{17}$$

$\because d \in \text{Integer} \Rightarrow d = 8$

$a + d = 18$

$\Rightarrow a = 10$

Now $a_{11} = a + 10d$

$= 10 + 10 \times 8$

$= 90$

3. Evaluate $I=80\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{(9\sin x+16\cos x)}dx$

(1) $\dfrac{80}{327}\left[\dfrac{25\pi}{2}+7\ln\left(\dfrac{9}{16}\right)\right]$

(2) $\dfrac{80}{337}\left[\dfrac{25\pi}{2}-7\ln\left(\dfrac{9}{16}\right)\right]$

(3) $\dfrac{40}{327}\left[\dfrac{25\pi}{2}+7\ln\left(\dfrac{9}{16}\right)\right]$

(4) $\dfrac{40}{327}\left[\dfrac{25\pi}{2}-7\ln\left(\dfrac{9}{16}\right)\right]$

Answer (2)

Sol. $\sin x \cos x = A[9\sin x + 16\cos x] + B[9\cos x - 16\sin x]$

$= \sin x[9A - 16B] + \cos x[16A + 9B]$

$\Rightarrow 9A - 16B = 16A + 9B = 1$

$\Rightarrow -7A = 25B \Rightarrow B=\dfrac{-7A}{25}$

$9A-16\left(\dfrac{-7A}{25}\right)=1 \Rightarrow 337A = 25,\ B=\dfrac{-7}{337}$

$$I=80\int_0^{\frac{\pi}{2}}\frac{\frac{25}{337}(9\sin x-16\cos x)\,\frac{-7}{337}[9\cos x-16\sin x]}{(9\sin x+16\cos x)}dx$$

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$$I = 80\int_0^{\frac{\pi}{2}} \frac{25}{337}\,dx - 80\int_0^{\frac{\pi}{2}} \frac{\frac{7}{337}d(9\sin x + 16\cos x)}{(9\sin x + 16\cos x)}$$

$$I = 80\left(\frac{25x}{337}\right)\Bigg|_0^{\frac{\pi}{2}} - \frac{80.7}{337}\ln(9\sin x + 16\cos x|_0^{\frac{\pi}{2}}$$

$$I = \frac{80.\,25}{337}\left(\frac{\pi}{2}\right) - \frac{80.(7)}{337}\ln\left(\frac{9}{16}\right)$$

4. If R be a relation defined on $(0, \pi/2)$ such that $xRy \Rightarrow \sec^2 x - \tan^2 y = 1$, then the relation R is

(1) Equivalence relation

(2) Reflexive and transitive only

(3) Symmetric and transitive only

(4) Neither reflexive nor transitive

Answer (1)

Sol. $xRy \Rightarrow \sec^2 x - \tan^2 y = 1$

- $xRx \Rightarrow \sec^2 x - \tan^2 x = 1$

$\Rightarrow$ R is reflexive

- $xRy \Rightarrow yRx$

$\Rightarrow$ $\sec^2 x - \tan^2 y = 1$

$\sec^2 y - \tan^2 x = (1 + \tan^2 y) - (\sec^2 x - 1)$

$= 2\sec^2 x + \tan^2 y$

$= 2 - (\sec^2 x - \tan^2 y) = 2 - 1 = 1$

$\Rightarrow$ R is symmetric

- $xRy \Rightarrow yRz$

$\Rightarrow$ $\sec^2 x - \tan^2 y = 1$

$\sec^2 y - \tan^2 z = 1$

Add $\Rightarrow \sec^2 x + \sec^2 y - \tan^2 y - \tan_z^2 = 2$

$\Rightarrow$ $\sec^2 x + (1) - \tan^2 z = 2$

$\Rightarrow$ $\sec^2 x - \tan^2 z = 1$

$\Rightarrow$ $x R z$

$\Rightarrow$ R is transitive.

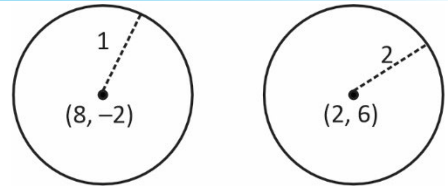
5. If z_1 lies on $|z - 8 + 2i| = 1$ and z_2 lies on $|z - 2 - 6i| = 2$, then $|z_1 - z_2|_{min}$ is

(1) 8 (2) 10

(3) 7 (4) 9

Answer (3)

Sol.



$$|Z_1 - Z_2|_{min} = \sqrt{(8-2)^2 + (2+6)^2} - 3$$

$$= \sqrt{36 + 64} - 3$$

$= 10 - 3 = 7$

6. If $\cos^{-1}x = \pi + \sin^{-1}x + \sin^{-1}(2x - 1)$, then find the sum of all values of 'x'.

(1) 1 (2) $\frac{1}{2}$

(3) 0 (4) $\frac{3}{2}$

Answer (3)

Sol. $\cos^{-1}x = \pi + \sin^{-1}x + \sin^{-1}(2x - 1)$

Now $-1 \le 2x - 1 \le 1$

$0 \le x \le 1$

$\Rightarrow \quad \pi + \sin^{-1}x + \sin^{1}(2x-1) \ge \frac{\pi}{2}$

and $\cos^{-1}x$ for $x \in [0, 1]$ always lies in $\left[0, \frac{\pi}{2}\right]$

$\Rightarrow$ LHS = RHS = $\frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} \Rightarrow \boxed{x = 0}$

Hence only $x = 0$ is the possible solution.

Sun of all solution = 0.

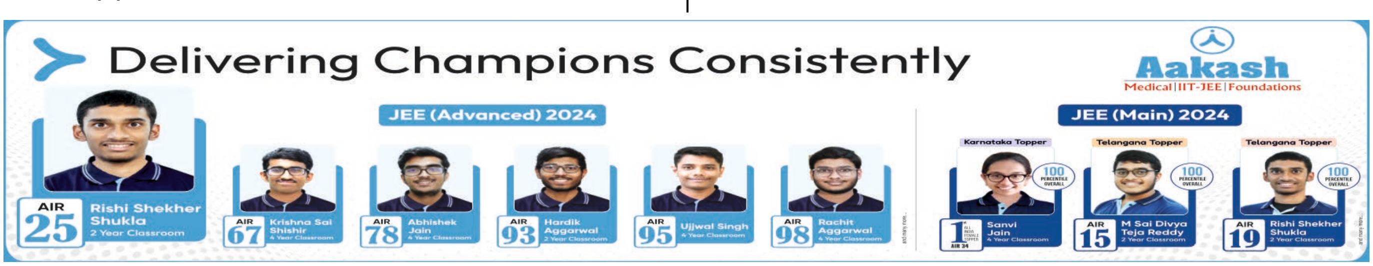
7. If $\begin{vmatrix} \sin^2 x & 1+\cos^2 x & \sin 4x \\ 1+\sin^2 x & \cos^2 x & \sin 4x \\ \sin^2 x & \cos^2 x & 1+\sin 4x \end{vmatrix} = L$

and $L_{min} = m$ and $L_{max} = M$, then $|M^4 - m^4|$ is

(1) 79 (2) 78

(3) 80 (4) 76

Answer (3)



Sol. $\begin{vmatrix} \sin^2 x & 1+\cos^2 x & \sin 4x \\ 1+\sin^2 x & \cos^2 x & \sin 4x \\ \sin^2 x & \cos^2 x & 1+\sin 4x \end{vmatrix} = -\sin(4x) - 2 = L$

$L_{min} = -3 = m \quad L_{max} = -1 = M$

$\therefore \quad m^4 - M^4 = 81 - 1 = 80$

8. If α, β are real numbers such that $\sec^2(\tan^{-1}\alpha) + \text{cosec}^2(\cot^{-1}\beta) = 36$ and $\alpha + \beta = 8$, where $\alpha > \beta$, then $(\alpha^3 + \beta^3)$ is equal to

(1) 146 (2) 152

(3) 148 (4) 150

Answer (2)

Sol. Let $A = \tan^{-1}(\alpha)$, $B = \cot^{-1}(\beta)$

$\Rightarrow \alpha = \tan A, \beta = \cot B$

$\Rightarrow \tan A + \cot B = 8$

$\sec^2(A) + \text{cosec}^2(B) = 36$

$\Rightarrow 1 + \tan^2 A + 1 + \cot^2 B = 36$

$\Rightarrow \tan^2 A + \cot^2 B = 34$

$\Rightarrow (\tan A + \cot B)^2 = 64 = 34 + 2 \tan A . \cot B$

$\Rightarrow \tan A . \cot B = 15$

$\Rightarrow x^2 - 8x + 25 = 0$ has roots $\tan A$, $\cot B$

$\Rightarrow \tan A = 5, \cot B = 3$

As $\alpha > \beta$

$\Rightarrow \alpha^3 > \beta^3 = (\tan A)^3 + (\cot B)^3 = 5^3 + 3^3 = 125 + 27 = 152$

9. How many 6 letter words can be formed using the word MATHS such that any letter can be used maximum two times.

(1) 6400 (2) 8100

(3) 10000 (4) 9824

Answer (2)

Sol. MATHS has only 5 letters, so in a 6-letter word at least one letter has to repeat.

Let's make cases:

(i) **Case-I:** Exactly one letter is repeated twice.

$^5C_1 \cdot \frac{6!}{2!}$ MM ATHS

(ii) **Case-II:** Exactly two letters are repeated twice.

$^5C_2 \cdot {}^3C_2 \frac{6!}{2!2!}$ MM AA THS

(iii) **Case-III:** Exactly 3 letters are repeated twice

$^5C_3 \frac{6!}{2!2!2!}$

$\therefore$ Required words = 8100

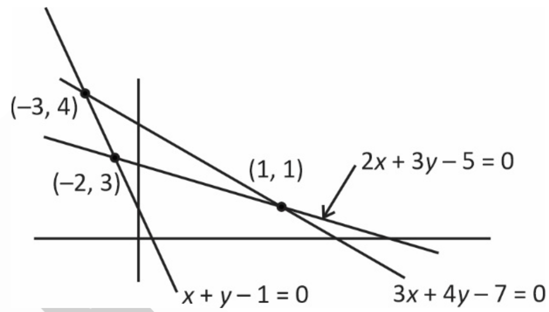
10. A triangle is formed by three lines $2x + 3y - 5 = 0$, $x + y - 1 = 0$, $3x + 4y - 7 = 0$. Let (h, k) be the image of the centroid of ΔABC in the line $2x + 4y - 7 = 0$ then $h^2 + k^2 + hk$ is

(1) $\frac{903}{225}$ (2) $\frac{223}{225}$

(3) $\frac{100}{23}$ (4) $\frac{10006}{225}$

Answer (1)

Sol.



Centroid $\left(\frac{-3+1-2}{3}, \frac{4+3+1}{3}\right)\left(\frac{-4}{3}, \frac{8}{3}\right)$

Image of $\left(\frac{-4}{3}, \frac{8}{3}\right)$ in $2x + 4y - 7 = 0$ is (h, k)

$$\frac{h+\frac{4}{3}}{2} = \frac{k-\frac{8}{3}}{4} = -2\left[\frac{\frac{-8}{3}+\frac{32}{3}-7}{4+16}\right]$$

$$\frac{h+\frac{4}{3}}{2} = \frac{k-\frac{8}{3}}{4} = \frac{-1}{10}$$

$$h = -\frac{1}{5} - \frac{4}{3} = -\frac{23}{15}$$

$$k = -\frac{2}{5} + \frac{8}{3} = -\frac{35}{15}$$

$$h^2 + k^2 + hk = \left(\frac{-23}{15}\right)^2 + \left(\frac{35}{15}\right)^2 - \frac{23}{15} \times \frac{34}{15}$$

$$= \frac{903}{225}$$

11. If two lines $L_1 : \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{2}$;

$L_2 : \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z}{1}$. Let the line L_3 passes through the point (α, β, γ) such that L_3 is perpendicular to L_1 to L_2 and L_3 intersects L_1. Then $|5\alpha - 11\beta - 8\gamma|$ is equal to

(1) 18

(2) 25

(3) 16

(4) 20

Answer (2)

Sol. Let the L_3 be

$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$, $(a\hat{i} + b\hat{b} + c\hat{k})$ is parallel to

$(\hat{i} - \hat{j} + 2\hat{k}) \times (-\hat{i} + 2\hat{j} + \hat{k})$

$(a, b, c) \equiv (5, 3, 1)$

$\Rightarrow \frac{x-\alpha}{5} = \frac{y-\beta}{3} = \frac{z-\gamma}{-1}$

$\Rightarrow$ Let the point of intersection be P.

$\Rightarrow 5\lambda + \alpha = P + 1,\ 3\lambda + \beta = P + 2,\ -\lambda + \gamma = 2P + 1$

$\Rightarrow \alpha = (P + 1 - 5\lambda),\ \beta = (-P + 2 - 3\lambda),\ \gamma = (2P + 1 + \lambda)$

$\Rightarrow |5\alpha - 11\beta - 8\gamma| = |-25| = 25$

12.

13.

14.

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. The minimum value of n for which the number of integer terms in the binomial expansion $\left(7^{\frac{1}{3}} + 11^{\frac{1}{12}}\right)^n$ is 183, is

Answer (2184)

Sol. $T_{k+1} = {}^nC_K \cdot \left(11^{\frac{1}{12}}\right)^k \cdot 7^{\frac{1}{3}(n-k)}$

$12 | k$ and $3 | (n-k) \Rightarrow 3 | n$

For integer terms.

$\Rightarrow$ Multiples of 12 for k would work.

$\Rightarrow$ $k = 0, 12, 24, \ldots$

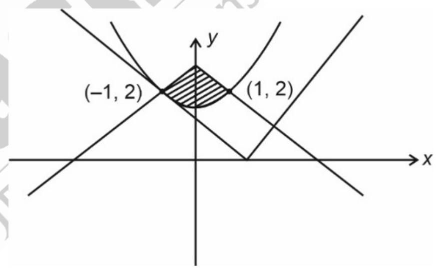
$\Rightarrow$ $k_{\max} = 12 \times 182 = 2184$

$\Rightarrow$ Minimum value of n will be 2184 as $3 | 2184$.

22. Area enclosed by $y \geq |x-1|$, $y + |x| \leq 3$, $x^2 \leq 2y - 3$ is A, then $6A$ is (in sq. units)

Answer (10)

Sol.



$$\text{Area} = 2\left[\int_0^1 (3-x) - \left(\frac{x^2+3}{2}\right) dx\right]$$

$$= 2\left[3x - \frac{x^2}{2} - \frac{1}{2}\left[\frac{x^3}{3} + 3x\right]\right]_0^1$$

$$= 2\left(3 - \frac{1}{2} - \frac{1}{2}\left[\frac{1}{3} + 3\right]\right)$$

$$= 2\left(\frac{5}{6} - \frac{1}{6} - \frac{3}{2}\right) = 2\left(\frac{5}{6}\right) = A$$

6 A = 10

23. Number of 7 digit numbers made with the digits 1, 2, 3 such that sum of the digits is 11 is equal to

Answer (161)

Sol. **Case-I :** 3 2 2 1 1 1 1

$$n_1 = \frac{7!}{4!2!} = 105$$

Case II: 2 2 2 2 1 1 1

$$\Rightarrow n_2 = \frac{7!}{4!3!} = 35$$

Case III : 3 3 1 1 1 1 1

$$\Rightarrow n_3 = \frac{7!}{5!2!} = 21$$

Total numbers $n_1 + n_2 + n_3$

= 105 + 35 + 21

= 161

24. The minimum value of p such that

$$\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\ldots+\left[\frac{p}{x}\right]\right)-x^2\left(\left[\frac{1}{x^2}\right]+\left[\frac{2}{x^2}\right]+\ldots+\left[\frac{9}{x^2}\right]\right)\geq 1,$$

is equal to (where [.] represents greatest integer function)

Answer (24)

Sol. Since $x^2\left[\frac{r^2}{x^2}\right] = x^2\left(\frac{r^2}{x^2} - \left\{\frac{r^2}{x^2}\right\}\right)$

$$= r^2 - x^2\left\{\frac{r^2}{x^2}\right\}$$

$$\lim_{x\to o^+} x^2\left[\frac{r^2}{x^2}\right] = \lim_{x\to o^+} r^2 - x^2\left\{\frac{r^2}{x^2}\right\} = r^2$$

Also,

$$\lim_{x\to o^+} x\left[\frac{k}{x}\right] = \lim_{x\to o^+} x\left(\frac{k}{x} - \left\{\frac{k}{x}\right\}\right) = \lim_{x\to o^+} k - x\left\{\frac{k}{x}\right\}$$

$$= k$$

$$\Rightarrow \lim_{x\to 0^+}\left(\sum_{k=1}^{p} x\left[\frac{k}{x}\right] - \sum_{k=1}^{9} x^2\left[\frac{k^2}{x^2}\right]\right)$$

$$= \sum_{k=1}^{p} \lim_{x\to 0^+} x\left[\frac{k}{x}\right] - \sum_{k=1}^{9} \lim_{x\to 0^+} x^2\left[\frac{k^2}{x^2}\right]$$

$$== \sum_{k=1}^{p} k - \sum_{k=1}^{9} k^2$$

$$= \frac{p(p+1)}{2} - \frac{(9)(10)(19)}{6} \geq 1$$

$$\Rightarrow \quad = \frac{p(p+1)}{2} - 285 \geq 1$$

$\Rightarrow$ $p(p + 1) \geq 2.286$

$\Rightarrow$ $p(p + 1) \geq 572$

Clearly p = 23 doesn't satisfy

$\Rightarrow$ Minimum value is $p = 24$, as $24^2 = 576 > 572$

25. Two parabolas having common focus at (4, 3) intersect at points A and B. Find the value of $(AB)^2$, given that directrices of these parabolas are along X-axis and Y-axis respectively.

Answer (192)

Sol. Equation of parabolas:

$(x - y)^2 + (y - 3)^2 = x^2$

$(x - y)^2 + (y - 3)^2 = y^2$

Let they intersect at (x_1, y_1) and (x_2, y_2)

$\therefore\ x_1^2 = y_1^2 \quad \Rightarrow x_1 = y_1 \quad (x_1 > 0, y_1 > 0)$

$\therefore\ (x_1 - 4)^2 + (x_1 - 3)^2 = x_1^2$

$\Rightarrow \quad x_1^2 - 14x_1 + 25 = 0$

$x_1 + x_2 = 14$, $x_1 \cdot x_2 = 25$

$(AB)^2 = \left(\sqrt{(x_1 - x_2) + (y_1 - y_2)}\right)^2$

$= 2(x_1 - x_2)^2$

$= 2((x_1 + x_2)^2 - 4x_1 x_2)$

$= 2(196 - 100)$

$= 192$

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