Chapter 9

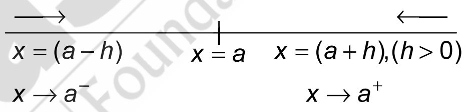
Limits, Continuity and Differentiability

Let y = f(x) be a real valued function which is defined in the neighbourhood of the point at x = a. A real number l is called limit of the function f when $x \to a$ i.e., $\lim_{x\to a} f(x) = l$. If given $\varepsilon > 0$, however small, there exists $\delta(\varepsilon) > 0$ such that $|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$.

The notion of continuity occurs in many application of calculus. We may intuitively think of continuous function as those functions whose graphs we can draw without lifting the pencil off the paper.

EXISTENCE OF LIMIT

$\lim_{x\to a} f(x) = l$ exists if



$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) =$ finite and definite

or $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a-h) =$ finite and definite

where $\lim_{x\to a^+} f(x)$ is called right hand limit and $\lim_{x\to a^-} f(x)$ is called left hand limit.

Conclusion :

$\lim_{x\to a} f(x)$ exist if its left hand limit (LHL) at *x* = *a*, and right hand limit (RHL) at *x* = *a* both exist and are equal and finite.

i.e., $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) =$ finite, otherwise limit does not exist.

Indeterminate Forms

A functional form whose value can't be fixed is called indeterminate form. For Example, $\frac{0}{0}$ is an indeterminate form as if, we take $\frac{0}{0} = k \Rightarrow 0 = k \times 0$. In this case we have infinitely many finite values of *k* for which *k* × 0 = 0. Thus *k* is not fixed. Hence $\left[\frac{0}{0}\right]$ form is indeterminate form.

Some indeterminate forms are $\frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$ etc.

Algebra of Limits

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist then

(i) $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$; $\left(\lim_{x\to a} g(x) \neq 0\right)$

(iii) $\lim_{x\to a}[k\ f(x)] = k \lim_{x\to a} f(x)$

(iv) $\lim_{x\to a}[f(x)]^{g(x)} = \left[\lim_{x\to a} f(x)\right]^{\lim_{x\to a} g(x)}$

(v) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$

(vi) $\lim_{x\to a} \log f(x) = \log\left[\lim_{x\to a} f(x)\right]$

(vii) $\lim_{x\to a} (f(x)\times g(x)) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$

(viii) $\lim_{x\to a} fog(x) = f\left[\lim_{x\to a} g(x)\right]$; (Provided f(x) is continuous at x = $\lim_{x\to a} g(x)$)

(ix) Sandwich theorem – If there is a function $h(x)$ such that $f(x) \le h(x) \le g(x)$ $\forall$ x and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

Then $\lim_{x\to a} h(x) = \lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x)$

L. Hospital's Rule

If $f(x)$ and $g(x)$ are functions of x such that $f(a) = 0$ and $g(a) = 0$, of $\lim_{x\to a}\frac{f(x)}{g(x)}$ reduces to $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then differentiation numerator and denominator until this form is removed.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} \text{ etc.}$$

Note : 1. *In applying this rule f(x) and g(x) are to be differentiated separately.*

2. *L. Hospital's Rule is true even if f(x) and g(x) $\to \infty$ when x $\to$ a*

i.e. $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

3. *Forms* $0 \times \infty$ *and* $\infty - \infty$, *reduced either to the form* $\frac{0}{0}$ or $\frac{\infty}{\infty}$ *for using L. Hospital's rule*

4. 0^0, 1^∞, ∞^0 *such types of form can be reduced to* $\frac{0}{0}$ or $\frac{\infty}{\infty}$ *by taking log of the given expression.*

Some Particular Limits

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\sin x} = \lim\limits_{x\to 0}\dfrac{\tan x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\tan x} = 1$

2. $\lim\limits_{x\to 0}\cos x = \lim\limits_{x\to 0}\sec x = 1$

3. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x} = \lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x} = 1$

4. $\lim\limits_{x\to\infty}\dfrac{\sin x}{x} = 0$

5. $\lim\limits_{x\to\infty}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$

6. $\lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right] = 0,$ where [.] represent step function

7. $\lim\limits_{x\to 0}\left[\dfrac{x}{\sin x}\right] = 1$

8. $\lim\limits_{x\to 0}\,[\cos x] = 0$ [$x \to 0$ then $\cos x \to 1$, not exactly equal to 1]

9. $\lim\limits_{x\to 0}\left[\dfrac{\tan x}{x}\right] = 1$, where [.] represents step functions.

> ***Note :*** *sin x < x < tan x for 0 < x <* $\dfrac{\pi}{2}$

10. $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x - a} = na^{n-1}$, if $a > 0$, $n \in$ R and if $a < 0$, $n \in$ Z.

11. $\lim\limits_{x\to a}\dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n}a^{m-n}$

12. $\lim\limits_{x\to 0}\dfrac{a^x - 1}{x} = \log a$; $a > 0$

13. $\lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{x} = 1$ and $\lim\limits_{x\to 0}\dfrac{\log_e(1-x)}{x} = -1$

14. $\lim\limits_{n\to\infty} x^n = \begin{cases} 0, & 0 < x < 1 \\ \infty, & x > 1 \end{cases}$

15. $\lim\limits_{x\to\infty}\left(1 + \dfrac{1}{x}\right)^x = e$

Some Important Expansions

1. $a^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + ... ;\ a > 0$

2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

3. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\quad (-1 < x \le 1)$

Note : *For getting expansion of a^{-x}, e^{-x}, $\log(1 - x)$ replace x by $-x$ in expansion of a^x, e^x and $\log(1 + x)$*

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

6. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...$

7. $\sin^{-1} x = x + \frac{1^2.x^3}{3!} + \frac{1^2.3^2.x^5}{5!} + \frac{1^2.3^2.5^2.x^7}{7!} + ...$

8. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

9. $(1+x)^{1/x} = e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - .....\right]$

EVALUATION OF LIMITS

By Factorization

In case of $\frac{0}{0}$ form we factorize numerator and denominator and then cancel the common factor. This process is continued till all the common factors are cancelled.

By Substitution

Evaluation of limit when $x \to \infty$:

In case of limits, when $x \to \infty$, put $x = \frac{1}{t}$ so that $t \to 0$ when $x \to \infty$ or if $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is in $\frac{\infty}{\infty}$ form where $f(x)$ and $g(x)$ are algebraic function, then divide both N^r and D^r by terms which have highest power of x, then apply the limit.

By Expansion

In this method use expansion formulae and solve

Using Some Standard Limits

In this method, we use some standard limits given below

If $\lim\limits_{x\to a} f(x)=0$ then

(i) $\lim\limits_{x\to a}\dfrac{\sin f(x)}{f(x)}=1$

(ii) $\lim\limits_{x\to a}\cos f(x)=1$

(iii) $\lim\limits_{x\to a}\dfrac{a^{f(x)}-1}{f(x)}=\log a$, $a > 0$

(iv) $\lim\limits_{x\to a}\dfrac{\log_e(1+f(x))}{f(x)}=1$

Evaluation of Exponential Limit

(a) Evaluation of limit of the form 1^∞:

(i) $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0,$ then

$$\lim_{x\to a}[1+f(x)]^{\frac{1}{g(x)}}=e^{\lim\limits_{x\to a}\frac{f(x)}{g(x)}}$$

(ii) If $\lim\limits_{x\to a} f(x)=1, \lim\limits_{x\to a} g(x)=\infty$

then $\lim\limits_{x\to a}[f(x)]^{g(x)}=e^{\lim\limits_{x\to a}[f(x)-1]\times g(x)}$

(b) Evaluation of the limit of the form 0^0 or ∞^0:

If $\lim\limits_{x\to a}[f(x)]^{g(x)}$ gives any one form (0^0 or ∞^0)

then $\lim\limits_{x\to a}[f(x)]^{g(x)}=e^{\lim\limits_{x\to a} g(x)\log f(x)}$

Note : $\lim\limits_{x\to a}(1+f(x))^{\frac{1}{f(x)}}=e\left(\text{where} \lim\limits_{x\to a} f(x)=0\right)$

(c) Evaluation of the limit of the form $[0\times\infty]$ or $[\infty-\infty]$ form

In this form we simplify the given expression by rationalising or by taking L.C.M. to convert this form to other form which can be solved easily.

Evaluation of limits using Newton-Leibnitz's Theorem

Let $k(x) = \int_{\phi(x)}^{\psi(x)} f(t)dt$

Newton-Leibnitz's theorem states that

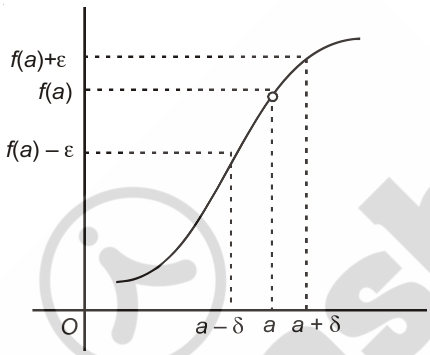
$k'(x) = f(\psi(x)).\psi'(x) - f(\phi(x)).\phi'(x)$

CONTINUITY & DIFFERENTIABILITY

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ in the domain of $f(x)$ if and only if $f(a^-) = f(a^+) = f(a)$ = finite number, *i.e.,* $\lim_{x \to a} f(x)$ exists finitely, $f(a)$ is a finite number and $\lim_{x \to a} f(x) = f(a)$.

More precisely, for given $\varepsilon > 0$, $\delta > 0$ such that $0 \le |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$



Continuity of a Function in Open Interval

A function is said to be continuous in an open interval (a, b), if it is continuous at each point of (a, b).

Continuity in Closed Interval

A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if

1. $f(x)$ is continuous from right at $x = a$, i.e.,

   $\lim_{h \to 0} f(a+h) = f(a)$

2. $f(x)$ is continuous from left at $x = b$, i.e.,

   $\lim_{h \to 0} f(b-h) = f(b)$

3. $f(x)$ is continuous at each point of the open interval (a, b)

Properties of Continuous Function

1. If f and g are continuous at $x = a$, then

   (a) $f + g$ is continuous at $x = a$

   (b) $f - g$ is continuous at $x = a$

   (c) fg is continuous at $x = a$

   (d) f/g is continuous at $x = a$, provided $g(a) \neq 0$

   (e) kf is continuous at $x = a$, where k is any real constant

   (f) $[f(x)]^{m/n}$ is continuous at $x = a$, provided $[f(x)]^{m/n}$ is defined on an interval containing a, and m, n are integers

2. If f is continuous at a and g is continuous at $f(a)$ then gof is continuous at a
3. If f is continuous at $x = a$ and g is discontinuous at $x = a$, then $f + g$ and $f - g$ are discontinuous at $x = a$, whereas fg may be continuous at $x = a$.
4. If f is continuous at $x = a$ and $f(a) \neq 0$, then there exists an open interval $(a - \delta, a + \delta)$ such that $\forall x \in (a-\delta, a+\delta), f(x)$ has the same sign as $f(a)$
5. If f is a continuous function defined on $[a, b]$ such that $f(a).f(b) < 0$, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).
6. If f is a continuous function defined on $[a, b]$ and k is any real number between $f(a)$ and $f(b)$, then there exists atleast one solution of the equation $f(x) = k$ in the open interval (a, b).
7. If a function f is continuous on a closed interval $[a, b]$, then it is bounded on $[a, b]$ i.e., there exists real number k such that $k \leq f(x)$ for all $x \in [a, b]$
8. Every polynomial is continuous at every point of the real line.
9. Every rational function is continuous at every point where its denominator is different from zero.
10. Logarithmic functions, Exponential functions, Trigonometric functions, Inverse circular function and Absolute value functions are continuous in their domain of definition.

DISCONTINUITY OF FUNCTIONS

A function $f(x)$, which is not continuous at a point $x = a$, is said to be discontinuous at that point.

The discontinuity may arise due to any of the following reasons

1. $\lim\limits_{h\to 0} f(a-h) \neq \lim\limits_{h\to 0} f(a+h)$, i.e., LHL and RHL exist, but are not equal
2. $\lim\limits_{h\to 0} f(a-h) = \lim\limits_{h\to 0} f(a+h) \neq f(a)$, LHL and RHL exist and are equal, but are different from $f(a)$.
3. $f(a)$ is not defined.
4. At least one of the limits $\lim\limits_{h\to 0} f(a-h)$ or $\lim\limits_{h\to 0} f(a+h)$ does not exist or at least one of these limits is ∞ or $-\infty$.

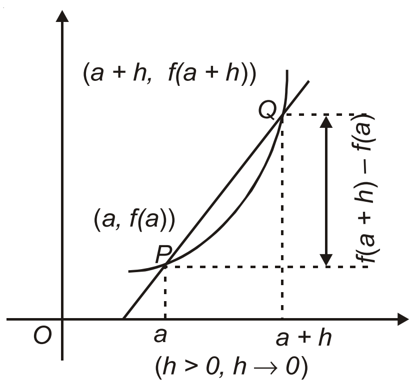
Types of Discontinuity:

1. Removable type : LHL = RHL = finite but not equal to $f(a)$
2. Discontinuity of 1st type : LHL $\neq$ RHL (both exist)
3. Discontinuity of 2nd type : At least one of LHL or RHL does not exist.

DIFFERENTIABILITY OF FUNCTIONS

Differentiability at a Point

A function $y = f(x)$ is said to be differentiable at a point a if at $x = a$ $f(x)$ is continuous and left hand derivative $f'(a^-)$ and right hand derivative $f'(a+)$ both exist finitely and are equal. Their common value is called derivative of $f(x)$ at $x = a$. Right hand derivative at $x = a$ is defined as:



$$f'(a^+) \equiv \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} (h > 0)$$

Left hand derivative at $x = a$:

$$f'(a^-) \equiv \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} (h > 0)$$

Thus $f(x)$ is differentiable at $x = a$ if $f'(a^-) = f'(a^+)$ (some fixed finite quantity).

Differentiability on an Interval

A function $f(x)$ is said to be differentiable on an open interval (a, b) if $f(x)$ is differentiable at every point of this interval (a, b).

It is differentiable on a closed interval [a, b] if it is differentiable on the open interval (a, b) and the limits

$$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

and $Lf'(b) = \lim_{h \to 0} \frac{f(b-h) - f(b)}{(b-h) - b}$ exist.

Properties of Differentiable Functions

1. Every polynomial function, exponential function and constant function are differentiable at each point of the real line.
2. Logarithmic functions and Trigonometric functions functions are differentiable in their domain of definition.
3. The sum, difference, product of two differentiable functions is differentiable.
4. The composition of differentiable functions is a differentiable function.
5. If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.
6. If $f(x)$ and $g(x)$ both are not differentiable at a point, then the sum function $f(x) + g(x)$ and the product function $f(x).g(x)$ can still be differentiable at that point.

Relation between Continuity and Differentiability

1. If a function $f(x)$ is differentiable at a point $x = a$ then it is continuous at $x = a$, converse is not true *i.e.*, $f(x)$ is only continuous at a point $x = a$, there is no guarantee that $f(x)$ is differentiable there.
2. If $f(x)$ is not differentiable at $x = a$ then it may or may not be continuous at $x = a$.
3. If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.
4. If left hand derivative and right hand derivative of $f(x)$ at $x = a$ are finite (they may or may not be equal) then $f(x)$ is continuous at $x = a$.

Differentiation of Basic Elementary Functions

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(\log_e x) = \frac{1}{x}\ (x > 0)$
9. $\frac{d}{dx}(a^x) = a^x(\ln a);\ (a > 0)$
10. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
11. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
12. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, -\infty < x < \infty$

13. $\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}, -\infty<x<\infty$

14. $\frac{d}{dx}(\text{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}, x\in R-[-1,1]$

15. $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}, x\in R-[-1,1]$

16. $\frac{d}{dx}$ (constant) = 0

17. $\frac{d}{dx}(x^n)=nx^{n-1}(n\in R)$

18. $\frac{d}{dx}\left(\frac{1}{x^n}\right)=\frac{-n}{x^{n+1}}; x\neq 0$

19. $\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e; x>0, a>0, a\neq 1$

20. $\frac{d}{dx}(|x|)=\frac{x}{|x|}$ or $\frac{|x|}{x}; x\neq 0$

Differentiation of Explicit Functions

1. Scalar product rule: $\frac{d}{dx}(cf(x))=cf'(x)$ (where c is a constant)
2. Sum and difference rule: $\frac{d}{dx}\{f(x)\pm g(x)\}=f'(x)\pm g'(x)$
3. Product rule: $\frac{d}{dx}\{f(x)g(x)\}=f'(x)g(x)+f(x)g'(x)$
4. Quotient rule: $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\}=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}(g(x)\neq 0)$

Differentiation of composite functions: (Chain Rule)

If function $h(x)$ is composition of two functions $f(x)$ and $g(x)$ as $h(x) = f(g(x))$. Now if $h(x) = f(t)$ and $t = g(x)$

then, $\frac{d}{dx}(h(x))=\frac{dh(x)}{dt}\times\frac{dt}{dx}=\frac{df(t)}{dt}\times\frac{dg(x)}{dx}=f'(t)g'(x)$

$\Rightarrow \frac{d}{dx}(h(x))=f'(g(x))g'(x)$.

This rule is called Chain Rule.

This rule can be extended to the compositions of any finite number of functions, for example, if $y = f(g(h(x)))$

Then, $\frac{dy}{dx}=f'(g(h(x)))g'(h(x))h'(x)$

Differentiation of Function represented Parametrically :

If x and y are represented as : $x = \phi(t)$ and $y = \psi(t)$ $(\alpha < t < \beta)$, where $\phi(t)$ and $\psi(t)$ are differentiable functions and $\phi'(t)\neq 0$, then y defined as a single valued function (continuous) of x is differentiable and its derivative is given by:

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\psi'(t)}{\phi'(t)}$$ (t is a parameter)

Remark : $\frac{d^2y}{dx^2}\neq\frac{\frac{d^2y}{dt^2}}{\frac{dx^2}{dt^2}}$

LOGARITHMIC DIFFERENTIATION

If the given functions are of the following types:

1. Consisting of product and quotient of a number of functions or
2. Consisting of function of x raised to a power which is also a function of x i.e., $\{f(x)\}^{g(x)}$ where $f(x)$ and $g(x)$ are function of x

   Then first we take logarithm of whole expression and then differentiate w.r.t. the suitable variable.

Remember : (Direct rule)

$$\boxed{\frac{dy}{dx} \equiv \text{Derivative of } f(x)^{g(x)} \text{ treating } f(x) \text{ constant} + \text{Derivative of } f(x)^{g(x)} \text{ treating } g(x) \text{ constant.}}$$

Differentiation by using Trigonometric Transformations

With the help of trigonometric transformations, the labour involved in computing derivation can be reduced. Note some standard trigonometric substitutions

1. $\sqrt{a^2 - x^2}$ $\Rightarrow$ put $x = a \sin\theta$ or $a\cos\theta$
2. $\sqrt{a^2 + x^2}$ $\Rightarrow$ put $x = a\tan\theta$ or $a\cot\theta$
3. $\sqrt{x^2 - a^2}$ $\Rightarrow$ put $x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
4. $\sqrt{(a-x)/(a+x)}$ or $\sqrt{\dfrac{a+x}{a-x}}$ $\Rightarrow$ put $x = a\cos\theta$ or $a\cos 2\theta$
5. $\sqrt{\dfrac{x-a}{x+a}}$ or $\sqrt{\dfrac{x+a}{x-a}}$ $\Rightarrow$ $x = a\sec\theta$ or $a\sec 2\theta$
6. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ $\Rightarrow$ put $x = a\sin 2\theta$
7. $\sqrt{2ax - x^2}$ $x = 2a\sin^2\theta$
8. $\sqrt{(x-\alpha)(\beta-x)}$ or $\sqrt{\dfrac{(x-\alpha)}{(\beta-x)}}$ $x = \alpha\cos^2\theta + \beta\sin^2\theta$ (Important)

   (Where $\alpha < x < \beta$)
9. $\sqrt{\dfrac{x-\alpha}{x-\beta}}$ or $\sqrt{(x-\alpha)(x-\beta)}$ $\Rightarrow$ Put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

Differentiation of Inverse Function

If $y = f(x)$ and $x = g(y)$ are inverse functions then $(g(f(x)) = x$ or $f(g(y)) = y)$, then their derivatives are reciprocal

$g'(y) = \dfrac{1}{f'(x)}$.

Differentiation of a Function with respect to a Function

This type of differentiation is similar to parametric form type differentiation. Let $y = f(x)$ and $z = \phi(x)$ be two functions of x and we have to differentiate $f(x)$ w.r.t. $\phi(x)$ i.e. we have to find $\frac{dy}{dz}$. Then first compute $\frac{dy}{dx} = f'(x)$ and $\frac{dz}{dx} = \phi'(x)$.

Now $$\frac{dy}{dz} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{f'(x)}{\phi'(x)}.$$

Differentiation of implicit function

If a differentiable function $y = y(x)$ satisfies the equation $f(x,y) = 0$, then we differentiate it w.r.t x considering y as a function of x and solve the equation $\frac{d}{dx}f(x,y) = 0$ for $\frac{dy}{dx}$

Note : *We will find this formula useful in case of implicit function*

$$\frac{dy}{dx} = -\frac{\text{differentiate } f \text{ w.r.t. } x \text{ keeping } y \text{ constant}}{\text{differentiate } f \text{ w.r.t. } y \text{ keeping } x \text{ constant}}.$$

Differentiation of Determinant whose Elements are Functions of Variable x

Let $\Delta(x) = \begin{vmatrix} f_{11}(x) & f_{12}(x)\ldots f_{1n}(x) \\ f_{21}(x) & f_{22}(x)\ldots f_{2n}(x) \\ \vdots & \\ f_{n1}(x) & f_{n2}(x)\ldots f_{nn}(x) \end{vmatrix}$, then

$$\frac{d}{dx}\Delta(x) = \begin{vmatrix} f'_{11}(x) & f'_{12}(x)\ldots f'_{1n}(x) \\ f_{21}(x) & f_{22}(x)\ldots f_{2n}(x) \\ \vdots & \\ f_{n1}(x) & f_{n2}(x)\ldots f_{nn}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}(x)\ldots f_{1n}(x) \\ f'_{21}(x) & f'_{22}(x)\ldots f'_{2n}(x) \\ \vdots & \\ f_{n1}(x) & f_{n2}(x)\ldots f_{nn}(x) \end{vmatrix} + \ldots + \begin{vmatrix} f_{11}(x) & f_{12}(x)\ldots f_{1n}(x) \\ f_{21}(x) & f_{22}(x)\ldots f_{2n}(x) \\ \vdots & \\ f'_{n1}(x) & f'_{n2}(x)\ldots f'_{nn}(x) \end{vmatrix}$$

i.e. $$\frac{d}{dx}\Delta(x) = \begin{vmatrix} \text{differentiate 1st row} \\ \text{keeping other rows as} \\ \text{they are.} \end{vmatrix} + \begin{vmatrix} \text{differentiate 2nd row} \\ \text{keeping remaining rows} \\ \text{as they are.} \end{vmatrix} + \ldots + \begin{vmatrix} \text{differentiate } n\text{th row} \\ \text{keeping remaining rows} \\ \text{as they are.} \end{vmatrix}$$

DERIVATIVE OF INFINITE SERIES

(i) If $y = \sqrt{g(x) + \sqrt{g(x) + \sqrt{g(x) + \ldots\infty}}}$

We can express

$$y = \sqrt{g(x) + y}$$

$\Rightarrow$ $y^2 - y = g(x)$

$\Rightarrow$ $(2y-1)\frac{dy}{dx} = g'(x)$

$$\frac{dy}{dx} = \frac{g'(x)}{2y-1}$$

(ii) If $y = g(x)^{(g(x))^{(g(x))^{\dots \text{ to } \infty}}}$

$y = (g(x))^y = e^{y \ln g(x)}$

$$\frac{dy}{dx} = e^{y \ln g(x)}\left[\frac{y}{g(x)}.g'(x) + \frac{dy}{dx}\ln g(x)\right]$$

$$\frac{dy}{dx} = y\left[\frac{y}{g(x)}g'(x) + \ln g(x)\frac{dy}{dx}\right]$$

$$[1 - y\ln g(x)]\frac{dy}{dx} = \frac{y^2 g'(x)}{g(x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 g'(x)}{g(x)\{1 - y\ln g(x)\}}$$

$$\frac{dy}{dx} = \frac{y^2 g'(x)}{g(x)\{1 - y\ln(g(x))\}}$$

HIGHER ORDER DERIVATIVES

When $y = f(x)$ is differentiable w.r.t. x, the differential coefficient $\frac{dy}{dx}$ or $f'(x)$ so obtained, being a function of x, can be differentiated again (if it is differentiable), to obtain a quantity, which is termed as 2nd differential coefficient of $f(x)$ and is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$. Similarly we can obtain third, fourth and higher order differential coefficient of $f(x)$, denoted by $\frac{d^3y}{dx^3} = f'''(x), \frac{d^4y}{dx^4} \equiv f^{(iv)}(x)$ and so on.

If u and v are functions differentiable n times, then

$(c_1u + c_2v)^{(n)} = c_1u^{(n)} + c_2v^{(n)}$ (where c_1 and c_2 are constants and $u^{(n)}$ denote derivative of n^{th} order)

Leibnitz formula : (n^{th} order derivative of product of two functions)

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{1.2}u^{(n-2)}v^{(2)} + \ldots + uv^{(n)}$$

$$= \sum_{r=0}^{n} C(n,r)u^{(n-r)}v^{(r)}$$

where $u^{(0)} = u$, $v^{(0)} = v$ and $C(n,r) = \frac{n!}{(n-r)!r!}$ (binomial coefficients)

Some basic formulae : (n) denote nth derivative.

(1) $(x^m)^{(n)} = m(m-1) \ldots\ldots (m-n+1)\, x^{m-n} = \dfrac{\lfloor m\, x^{m-n}}{\lfloor m-n} = {}^mP_n\, x^{m-n}$

(2) $(a^x)^{(n)} = a^x(\ln a)^n$; $(a > 0)$

(3) $(e^x)^{(n)} = e^x$

(4) $(\ln x)^{(n)} = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$

(5) $(\sin x)^{(n)} = \sin\left(x + \dfrac{n\pi}{2}\right)$

(6) $(\cos x)^{(n)} = \cos\left(x + \dfrac{n\pi}{2}\right)$

DETERMINING A FUNCTION WHICH SATISFIES A GIVEN RELATION

Let's consider a function satisfying a given condition. We must carefully note the steps to be followed in determining $f(x)$.

Let $f : R \to (0, \infty)$ satisfies the relation $f(x + y) = f(x)\, f(y)$. Let $f(x) \neq 0$ for any x and $f(x)$ be differentiable on R such that $f'(0) = 5$, then we shall determine $f(x)$.

We have

$f(x + y) = f(x)f(y)$

$\Rightarrow \quad f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{f(x)f(h)-f(x)}{h}$

$\Rightarrow \quad f'(x) = \lim\limits_{h\to 0} f(x)\dfrac{f(h)-1}{h}$

Now, $f(x+y) = f(x)\cdot f(y)$

$\therefore \quad f(1+0) = f(1)\cdot f(0)$

$f(1) - f(1)\cdot f(0) = 0$

$f(1)(1-f(0)) = 0$

$f(1) \neq 0 \quad \therefore\ f(0) = 1$

$\therefore \quad f'(x) = \lim\limits_{h\to 0} f(x)\cdot\left(\dfrac{f(h)-f(0)}{h}\right)$

$f'(x) = f(x)\cdot f'(0)$

$\Rightarrow \quad f'(x) = 5f(x)$

$$\frac{dy}{dx} = 5y$$

$$\frac{1}{y}\frac{dy}{dx} = 5$$

$\therefore \quad \ln y = 5x + C$

at $x = 0$, $y = 1$

$$\ln 1 = 5(0) + C$$

$\Rightarrow \quad C = 0$

Hence, $\ln y = 5x$

$\Rightarrow \quad y = e^{5x}$

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