Chapter 17

Mathematical Reasoning

Reasoning is the basis of mathematics. When we are called upon to solve a problem, we have to follow the chain of reasoning and thus from one thing to other, and still from that to another, we reach the solution

This chapter introduces the art of mathematical reasoning, its precision and rigour. Everything has been explained in great detail to help the learner understand the basics.

Mathematical Reasoning

Inductive (*Mathematical Induction*) | *Deductive*

FUNDAMENTALS OF DEDUCTIVE REASONING

Statements

A sentence is called a mathematically acceptable statement which is either true or false but not both.

A statement is an unambiguous declarative sentence which conveys a complete thought and which is either true or false but not both.

Example :

A.P.J. ABDUL KALAM WAS PRESIDENT OF INDIA.

Mr. 'X' will qualify IIT JEE entrance exam.

From the above two sentences, we can say first sentence is True and there is no confusion. 'In Mathematics such sentences are called STATEMENTS'.

On the other hand second sentence may be true or false i.e. there is ambiguity. Such a sentence is not mathematically acceptable.

Exclamatory sentence (How Sweet!), Imperative sentences (Bring a glass of water.), interogative sentences (what are you doing?), sentences involving variable time such as Yesterday and Tomorrow, Today are not mathematically acceptable statement. Also if pronouns do not refer to a particular person or it involves a variable places such as here, there etc, then such sentences are not mathematically acceptable statements.

Example :

He is an engineering graduate from IIT DELHI, IIT KHARAGPUR is away from here.

Statements are denoted by small letters p, q, r, ... and written as

p : Ram is an engineer.

NEGATION OF A STATEMENT

The denial of a statement is called the negation of the statement.

If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$, and read as 'not p'

Example :

p : everyone in India speaks Hindi

$\sim p$: everyone in India does not speak Hindi

Truth table for negation

p	~p
T	F
F	T

Compound Statements

It is a statement which is made up of two or more statements (using some connecting words like "and", 'or' etc). In this case each statements is called component statement.

SPECIAL WORDS/PHRASES (CONNECTIVES)

Some of the connecting words like 'And', 'or', etc are used in Mathematical statements. These are called connectives.

Rules Regarding the connective 'And'

(i) The compound statement with 'And' is true if all its component statements are true.

(ii) The compound statement with 'And' is false if any of its component statements is false

Note : *A statement with 'And' is not always a compound statement*

Example :

Bread and Butter is good for health. It is not a compound statement.

Truth table for conjunction (AND)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Rules Regarding the connective 'or'

(i) A compound statement with an 'or' is true when one component statement is true or both the component statements are true.

(ii) A compound statement with an 'or' is false when both the component statements are false.

'OR' is used in two ways in English.

Exclusive 'or' :

Example :

p : An ice cream or pepsi is available with a Thali in a restaurant

This means that a person who doesn't want ice cream can have a pepsi along with thali or one do not want a pepsi can have an ice cream along with thali. A person cannot have both ice cream and pepsi. It is called exclusive 'or'.

Inclusive 'or' :

Example :

p : A student who has taken biology or chemistry can apply for M.Sc microbiology program. Here it means that the students who have taken both biology and chemistry can apply for the microbiology program, as well as the students who have taken only one of these subjects. It is called Inclusive 'or'.

Truth table for Disjunction (OR)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The words "And" and "or" are called connectives and "There exists" and "for all" are called quantifiers.

COMMONLY USED SYMBOLS:

If p and q are two statements, then

$\sim p$: "Negation of p"

$p \wedge q$: "p and q"

$p \vee q$ = "p or q"

$p \veebar q$: "only one of p and q"

$p \downarrow q$: "Neither p nor q"

$p \to q$: "if p then q"

$p \leftrightarrow q$: "p if and only if q"

'$\wedge$' is called conjunction, '$\vee$' is called disjunction '$\veebar$' is called exclusive disjunction, '$\downarrow$' is called joint denial, '$\to$' is called conditional or implication and '$\leftrightarrow$' is called biconditional or equivalence.

IMPLICATIONS

("If-then ", "Only if" and "if and only if")

'If-then' : The statements with "if-then" are very common in mathematics.

Example :

r : If you are born in some country, then you are a citizen of that country.

It's corresponding two statements are p, q given by

p : you are born in some country

q : you are citizen of that country.

the sentence "If p, then q" says that in the event if p is true, then q must be true.

Truth table for Implication (If *p*, then *q*)

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

More interesting facts about the sentence "if p then q" is that it does not say any thing on q when p is false.

Quantifiers

Quantifiers are phrases like

"There exists" and "for all"

Example :

p : There exists a rectangle whose sides are equal. This means there is atleast one rectangle whose all sides are equal.

A closely connected word with "there exists" is 'for every' (or for all)

Example :

p : For every prime number, $\sqrt{p}$ is an irrational number.

This means that if S denotes the set of all prime numbers, then for all the members p of the set S, $\sqrt{p}$ is an irrational number.

Example :

r : If a number is a multiple of 8, then it is a multiple of 4.

Let p, q denote the statements

p : a number is a multiple of 8

q : a number is a multiple of 4

Then 'if p then q' is the same as the following:

(i) p Implies q is denoted by $p \Rightarrow q$.

(ii) p is sufficient condition for q.

i.e. a number as a multiple of 8 is sufficient to conclude that it is a multiple of 4

(iii) p only if q

This says that a number is a multiple of 8 only if it is multiple of 4

(iv) q is a necessary condition for p

This says that when a number is multiple of 8, it is necessarily a multiple of 4.

(v) $\sim q$ implies $\sim p$

This says that if a number is not a multiple of 4, then it is not a multiple of 8.

CONTRAPOSITIVE AND CONVERSE

There are certain other statements which can be formed from a given statement with "if-then"

Example :

The contrapositive of given statement "If p, then q" is "If $\sim q$, then $\sim p$".

If the physical environment changes, then the biological environment changes then contrapositive of this statement is

If the biological environment does not change then the physical environment does not change.

i.e. Both these statements convey the same meaning.

Converse

The converse of a given statement "if p, then q" is if q, then p.

The converse of the statement is

p : If a number is divisible by 20, then it is divisible by 10

q : If a number is divisible by 10, then it is divisible by 20

If and only if

It is represented by the symbol '$\Leftrightarrow$' means the following equivalent forms for the given statements p and q

(i) p if and only if q

(ii) q if and only if p

(iii) p is necessary and sufficient condition for q and vice-versa

(iv) $p \Leftrightarrow q$

Truth table for double Implication (*p* "If and only if *q*")

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

VALIDATING STATEMENTS

When a statement is true? To answer this question we have to answer all the following questions.

What does the statement mean?

What would it mean to say that this statement is true and when this statement is not true?

The answer of these question depends upon the special words and phrases ("and", "or etc") implications ("if and only if", if-then, etc) and the quantifiers ("for every", "there exists")

Some general rules for checking whether a statement is true or not

Rule 1 :

If p and q are mathematical statements then in order to show that the statements "***p* and *q***" is true.

Case 1: Show that the statement p is true

Case 2: Show that the statement q is true

Rule 2 :

Statements with "**or**"

To show that the mathematical statement "p or q" is true, consider the following.

Case 1: Assuming p false, show that q must be true

Case 2: Assuming q false, show that p must be true

Rule 3:

Statements with "if-then"

Case 1: By assuming that p is true, show q must be true (Direct method)

Case 2: By assuming that q is false, prove that p must be false (contrapositive method)

Rule 4 :

Statements with "if and only if"

To prove the statement "p if and only if q" we need to show

(i) If p is true, then q is true

(ii) If q is true, then p is true

Rule 5 :

By contradiction

To check whether a statement p is true, we assume that p is not true i.e. $\sim p$ is true

Then, we arrive at some result which contradicts our assumption then we conclude that p is true.

Rule 6 :

Giving a counter example, show that the following Statement is false

Example :

If n is an odd integer, then n is prime. To show it false, we take an example that 9 is an odd number but not prime.

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