Chapter 3

Matrices and Determinants

MATRICES

A set of *mn* numbers (real or complex) arranged in the form of a rectangular array having '*m*' rows and '*n*' columns is called an $m \times n$ matrix. It is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

It is represented as $A = [a_{ij}]_{m \times n}$. The numbers a_{11}, a_{12}, ... etc. are called elements of the matrix. The element a_{ij} belong to i^{th} row and j^{th} column.

e.g., $A = \begin{bmatrix} 3 & 2 & 7 \\ 5 & -4 & 6 \\ 4 & 8 & -12 \end{bmatrix}$ is a 3×3 matrix

TYPES OF MATRICES

1. **Square Matrix :** A matrix having equal number of rows and columns is called a square matrix. If the matrix *A* has *n* rows and *n* columns, it is said to be square matrix of order *n*.

Example : $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is a square matrix of order 3.

The diagonal of this matrix containing the elements 1, 3, 5 is called the **leading** or **principal diagonal.**

Trace of matrix : The sum of the diagonal elements of a square matrix *A* is called trace of *A*.

$$\text{Trace }(A) = \sum_{i=1}^{n} a_{ii}$$

$$= a_{11} + a_{22} + a_{33} \dots + a_{nn}$$

Example : $A = \begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$. Trace $(A) = 2 + 3 + 4 = 9$

2. **Diagonal Matrix :** A square matrix in which all the elements, except those in the leading diagonal are zero is called a diagonal matrix. Thus square matrix is called diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

$$e.g.,\ A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ or } \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Above are diagonal matrices of the type 3 × 3. These are in short written as

diag [2, 1, 4] or diag [d_1, d_2 d_3]

If the diagonal elements are d_1, d_2, ... d_n, then it is written as

diag [d_1, d_2, ..., d_n].

3. **Scalar Matrix :** A diagonal matrix in which all the elements in the leading diagonal are equal is called scalar matrix. It is equal to KI_n for some scalar K.

$$e.g.,\ A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}$$

In general for a scalar matrix, $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = K$ for $i = j$

4. **Unit or Identity Matrix :** A square matrix in which all the elements in the leading diagonal are 1 and remaining elements are zero is called a unit or Identity Matrix. A unit matrix of order n is usually denoted by I_n. Thus for an Identity Matrix

$a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$

$$e.g.,\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ \&\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5. **Row matrix :** A matrix which has only one row and n columns is called a row matrix. It is evidently $1 \times n$ matrix. It is also called a row vector.

e.g., $A = [1 \quad 2 \quad 3]$ is a row matrix of order 1 × 3.

6. **Column matrix :** A matrix which has m rows and only one column is called a column matrix. It is evidently $m \times 1$ matrix. It is also called a column vector.

$$e.g.,\ A = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix} \text{ is a column matrix of order } 4 \times 1$$

7. **Null Matrix or Zero Matrix :** A ($m \times n$) matrix in which all the elements are zero is called a zero matrix or null matrix of the type $m \times n$ and is denoted by $O_{m \times n}$ or O.

$$\text{Thus } \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All the above are zero or null matrices of type 3 × 2, 3 × 3 and 2 × 4 respectively.

8. Horizontal matrix : A $m \times n$ matrix is called a horizontal matrix if $m < n$

e.g., $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 3 \end{bmatrix}$

9. Vertical matrix : A $m \times n$ matrix is called a vertical matrix if $m > n$.

e.g., $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 3 & 5 \end{bmatrix}$

10. Triangular matrix : Let A be a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

Any element of A is a_{ij} and for $i = j$ we get the diagonal elements. All those elements which are above the principal diagonal are of the type $a_{12}, a_{13}, a_{23}, a_{34}, a_{3n}$ .... or in general a_{ij} where $i < j$ and all those elements which are below the principal diagonal are of the type $a_{21}, a_{31}, a_{n3}, \dots$ so on or in general a_{ij} where $i > j$.

(a) **Upper Triangular Matrix :** A square matrix (a_{ij}) is called an upper triangular matrix if $a_{ij} = 0$ when $i > j$. In other words, all elements below the principal diagonal be zero.

(b) **Lower Triangular Matrix :** A square matrix (a_{ij}) is called lower triangular matrix if $a_{ij} = 0$ when $i < j$. In other words, all the elements above the principal diagonal be zero.

for *e.g.,* $\begin{bmatrix} 3 & -2 & 4 \\ 0 & 2 & -3 \\ 0 & 0 & 7 \end{bmatrix}$ & $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 8 & 6 \end{bmatrix}$

are upper and lower triangular matrices respectively.

11. Transpose of a Matrix : A matrix obtained by interchanging rows and columns of a matrix A is called transpose of A and is denoted by A' or A^T.

Note : *If A is a m × n matrix, then A′ is an n × m matrix.*

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Properties of Transpose

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$

(iii) $(KA)' = KA'$ [where, k is a scalar]

(iv) $(AB)' = B' A'$

12. **Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is said to be symmetric when $a_{ij} = a_{ji}$ for all i and j. In other words, $A' = A$

$$e.g., \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & -6 \\ 5 & -6 & 3 \end{bmatrix} \text{ is } 3 \times 3 \text{ matrix}$$

Here $a_{12} = a_{21}$, $a_{13} = a_{31}$, $a_{23} = a_{32}$

13. **Skew-symmetric Matrix :** A square matrix is said to be skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j so that all the leading diagonal elements are zero.

In other words $A' = -A$

$$e.g., \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix} \text{ is } 3 \times 3 \text{ matrix}$$

here $a_{12} = -a_{21}$, $a_{13} = -a_{31}$, $a_{23} = -a_{32}$

$a_{11} = a_{22} = a_{33} = 0$

Every square matrix can be uniquely expressed as a sum of a symmetric and skew-symmetric matrices.

Let A be the given square matrix, then

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q$$

where P is a symmetric matrix and Q is a skew-symmetric matrix.

14. **Singular Matrix :** A square matrix A is said to be singular if $|A| = 0$.

15. **Non-singular matrix :** A square matrix A is said to be non-singular if $|A| \neq 0$.

16. **Orthogonal Matrix :** A matrix A is said to be orthogonal if $AA' = I$. (Identity matrix)

Remarks:

If a matrix A is orthogonal, then $|A| = \pm 1$ matrix A is called proper or improper orthogonal matrix if $|A| = +1$ or -1. $A^{-1} = A$ (for orthogonal matrix)

Equality of Matrices

Two matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ are said to be equal and written as $A = B$ if and only if

(a) they are of the same order and

(b) each element of A is equal to the corresponding element of B.

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \end{bmatrix}$$

Here A and B are equal matrices and we write $A = B$

ALGEBRA OF MATRICES

Let A and B be two matrices of same order $m \times n$. Then their sum is defined to be the matrix of order $m \times n$ obtained by adding the corresponding elements of A and B

e.g., If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

then $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$

Properties :

(i) **Matrix addition is commutative**

If A and B be two $m \times n$ matrices, then $A + B = B + A$

(ii) **Matrix addition is associative**

If A, B and C be three matrices each of order $m \times n$, then

$(A + B) + C = A + (B + C)$

(iii) **Existence of additive Identity :** If O be the $m \times n$ matrix each of whose elements is zero, then

$A + O = A = O + A$ for every $m \times n$ matrix A. O is called additive Identity.

(iv) **Existence of additive Inverse:**

Let $A = [a_{ij}]_{m \times n}$. Then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$.

$A + (-A) = 0$

e.g., If $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -5 \end{bmatrix}$ then $-A = \begin{bmatrix} -1 & 2 & 0 \\ -3 & -2 & 5 \end{bmatrix}$

(v) **Cancellation laws hold good for addition :** If A, B, C be any three $m \times n$ matrices, then

$A + B = A + C \Rightarrow B = C$

(vi) $k(A + B) = kA + kB$, where k is scalar.

Subtraction of Two Matrices

Let A and B be two $m \times n$ matrices. Then the difference of A and B is denoted by $A - B$ and is defined by $A - B = A + (-B)$

$A - B$ will also be a $m \times n$ matrix. In order to find $A - B$, the elements of B must be subtracted from the corresponding elements of A.

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 0 & 6 \end{bmatrix}$

then $A - B = \begin{bmatrix} 1-2 & 2+1 & 3-4 \\ 3-5 & -1-0 & 0-6 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ -2 & -1 & -6 \end{bmatrix}$

Multiplication of Matrix by a scalar

The product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A.

Thus $k \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$

Properties of Scalar Multiplication

(i) If A and B be any two $m \times n$ matrices and k is any scalar, then $k(A \pm B) = kA \pm kB$

(ii) If A is any $m \times n$ matrix and a and b are any two scalars, then $(a \pm b)A = aA \pm bA$

(iii) If A be any $m \times n$ matrix and k be any scalar, then $(-k)A = -(kA) = k(-A)$

Multiplication of Matrices

Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{jk}]$ be a $n \times p$ matrix such that number of columns in A is equal to number of rows in B. Then product of matrices A and B denoted by AB is a matrix of $m \times p$ matrix.

For instance, let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}_{4\times 3}$ and $B = \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}_{3\times 2}$

then matrix AB is a matrix of order 4×2

$$AB = \begin{bmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 \\ a_4l_1 + b_4m_1 + c_4n_1 & a_4l_2 + b_4m_2 + c_4n_2 \end{bmatrix}$$

In other words, product AB is a matrix $C = [C_{ik}]_{m \times p} = \sum_{j=1}^{n} a_{ij} b_{jk}$

Properties :

(i) If $AB = 0$ (null matrix), it does not necessarily imply that A or B is a null matrix.

For instance, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then the product AB is a null matrix, although neither A nor B is a null matrix.

(ii) **Multiplication of matrices is not always commutative**

AB may or may not be equal to BA

e.g., $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

then $AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Thus $AB \neq BA$

(iii) **Multiplication of matrices is associative**

$(AB)C = A(BC)$

If A, B, C are $m \times n$, $n \times p$ and $p \times q$ matrices respectively.

(iv) **Multiplication of matrices is distributive w.r.t. addition and subtraction of matrices:**

If A is a $m \times n$ matrix and B and C are both $n \times p$ matrices, then

$A(B \pm C) = AB \pm AC$

(v) **Multiplication of Matrix by Unit matrix, I**

$AI = IA = A$ if conformability for multiplication is satisfied.

(vi) **Multiplication of Matrix by Null Matrix :**

$OA = AO = O$ where O = null matrix, if conformability for multiplication is satisfied.

DETERMINANTS OF 2ND & 3RD ORDER

The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of order 2. It has two rows and two columns. $a_1b_2 - a_2b_1$ is called expansion of determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. This idea can be extended to higher order determinants.

Determinant of 3rd order

The symbol $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is called the determinant of 3rd order and its value is equal to the number:

$$(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \quad \text{...(A)}$$

(A) is called expansion of determinant Δ along its first row. In fact, expansion can be carried along any row or column in a similar way.

MINORS AND CO-FACTORS OF DETERMINANT

In the determinant Δ, the minor of the element a_{ij} (where i denotes number of row and j the number of columns, *i.e.,* element common to i^{th} row and j^{th} column) is obtained by removing the i^{th} row and j^{th} column and it is given by M_{ij}

For example, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and

cofactor of a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$, for example, $C_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

or $C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

We can also express Δ as follows :

$\Delta = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

or $\Delta = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$

or $\Delta = a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$

Notations for Operations Performed on Rows/columns of Determinants

R_i (i = 1, 2, 3) and C_j (j = 1, 2, 3) are used to denote i^{th} row and j^{th} column respectively.

(i) The interchange of i^{th} row and j^{th} row : $R_i \longleftrightarrow R_j$

(ii) The addition of m times the elements of i^{th} row to the corresponding elements of j^{th} row :

$R_j \longrightarrow R_j + mR_i$

(iii) The addition of m times the elements of j^{th} row and n times the elements of k^{th} row to the corresponding elements of i^{th} row :

$R_i \longrightarrow R_i + mR_j + nR_k$

(Similar operation can be performed with respect to columns of Δ)

PROPERTIES FOR EVALUATION OF DETERMINANTS

I. The value of determinant is not changed when all rows and columns are interchanged *i.e.,*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

II. If any two rows (or any two columns) of a determinant are interchanged, the sign of the determinant is changed, but its value remains same (numerically),

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

III. If all the elements of the same row (or the same column) are multiplied by a number, then the determinant (its value) becomes multiplied by the number

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

IV. If any two rows (or columns) are identical or their respective elements are proportional then the value of a determinant is zero.

$$\begin{vmatrix} a_1 & ma_1 & c_1 \\ a_2 & ma_2 & c_2 \\ a_3 & ma_3 & c_3 \end{vmatrix} = 0$$

V. The value of a determinant does not change when any row (or column) is multiplied by a number or an expression and is added to or subtracted from any other row (or column)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 + m_1c_1 & c_1 \\ a_2 & b_2 + m_1c_2 & c_2 \\ a_3 & b_3 + m_1c_3 & c_3 \end{vmatrix}$$

VI. If every element of some row (or column) is the sum of two numbers, then determinant can be expressed as the sum of two determinants of the same order; one containing only the first number in place of sum, the other only the second number. The remaining element of both determinant are the same as in the given determinant.

$$\begin{vmatrix} a_1+\alpha_1+\beta_1 & b_1 & c_1 \\ a_2+\alpha_2+\beta_2 & b_2 & c_2 \\ a_3+\alpha_3+\beta_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \beta_1 & b_1 & c_1 \\ \beta_2 & b_2 & c_2 \\ \beta_3 & b_3 & c_3 \end{vmatrix}$$

(In this example, we have shown it for sum of three numbers)

Remarks :

1. The value of determinant remains unaltered under an operation of the form $R_i \longrightarrow R_i + mR_j + nR_k$ ($j,\ k \neq i$). These operation enable us to convert maximum elements into zeros for making evaluation of determinant simpler.
2. If a determinant $\Delta(x)$ becomes zero on putting $x = \alpha$, then $x - \alpha$ is a factor of $\Delta(x)$.
3. Determinant which have all elements equal to zero except the diagonal elements is equal to the product of the diagonal elements *i.e.*

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Also $$\begin{vmatrix} a & \alpha & \beta \\ 0 & b & \gamma \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ \alpha & b & 0 \\ \beta & \gamma & c \end{vmatrix} = abc$$

4. Determinant $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$. (note that upper triangular is reflected along diagonal with opposite sign)

DETERMINANT OF CO-FACTORS

1. If $\Delta \neq 0$ be the determinant and Δ^c denotes determinant of cofactors then $\Delta^c = \Delta^{n-1}$ (where $n > 0$) and n is order of determinant Δ. In particular if Δ is determinant of 3rd order

and $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then $$\Delta^c = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$$

2. $\Delta = 0$ then $\Delta^c = 0$

Product of two Determinants

Let $D_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

and $D_2 = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$

then product of D_1 and D_2 is defined as follows :

$D_1 D_2 = \begin{vmatrix} \sum a_i\alpha_i & \sum a_i\beta_i & \sum a_i\gamma_i \\ \sum b_i\alpha_i & \sum b_i\beta_i & \sum b_i\gamma_i \\ \sum c_i\alpha_i & \sum c_i\beta_i & \sum c_i\gamma_i \end{vmatrix}$ [Row-wise multiplication]

(where $\sum$ runs from 1 to 3 i.e. $\sum a_i\alpha_i = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$)

Remark :

1. Since value of determinant remains same when rows and column are interchanged therefore, in finding product of two determinants, we can also multiply rows by columns or column by columns.
2. $D_1 D_2 = D_2 D_1$

Important Observation

If $D_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

and $D_2 = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

where A_i = cofactor of a_i

B_i = cofactor of b_i

and C_i = cofactor of c_i then $D_2 = (D_1)^2$

AREA OF TRIANGLE

If ABC is a triangle with $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ then area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

The area may come out to be either positive or negative, hence modulus of above would give the magnitude of area.

If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ turns out to be zero, it means points A, B & C are collinear and vice-versa.

ADJOINT OF MATRIX

Let $A = [a_{ij}]$ be a square matrix.

Let $B = [A_{ij}]$ where A_{ij} is the cofactor of the element a_{ij} in the det. A.

Then adjoint of matrix A is transpose B' of matrix B and is denoted by adj A.

INVERSE OF A SQUARE MATRIX

A square matrix A is said to be invertible if there exists a square matrix B such that $AB = BA = I_n$ (identity matrix)

Here $B = A^{-1}$ or $A = B^{-1}$.

The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

Inverse of matrix A is denoted by A^{-1} and $A^{-1} = \dfrac{\text{Adj}(A)}{|A|}$

***Note:** Non-singular matrix is invertible]*

SYSTEM OF LINEAR EQUATIONS

System of Linear equations :

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$\begin{matrix} \vdots & & \vdots \\ \vdots & & \vdots \end{matrix}$$

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

is said to be consistent, if it has at least one solution. Otherwise, it is said to be inconsistent.

I. System of homogeneous equations (linear) in three variables.

$a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$

Note that $x = y = z = 0$ (trivial solution) is always a solution. Therefore, this system is always consistent.

Case I : If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, there is unique solution $x = y = z = 0$ (trivial solution in this case).

Case II : If $\Delta = 0$, the system has infinite number of solutions (existence of non-trivial solutions)

CRAMER'S RULE

Consider the system of equations :

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$

We define

The denominator determinant $= \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ ...(1)

The numerator determinant of $x = \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ (Replace C_1 of (1) by d_1, d_2 , d_3)

The numerator determinant of $y = \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

and the numerator determinant of $z = \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Case I :

If $\Delta \neq 0$, then the system is called consistent and has a unique solution given by

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta}. (\Delta \neq 0)$$

Case II :

If $\Delta = 0$, but if Δ_x, Δ_y, Δ_z atleast one is non-zero, then the system is called inconsistent and has no solution.

Case III :

If $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$, then the system has infinite number of solutions or no solution.

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