Chapter 4

Permutations and Combinations

1. FUNDAMENTAL PRINCIPLE OF COUNTING

Multiplication theorem

If an operation can be performed in m ways and another operation can be performed in n ways, then the two operations in succession can be performed in $m \times n$ ways. For example: A person can go by rail from Delhi to Lucknow in two ways and from Lucknow to Varanasi in three ways then he can travel from Delhi to Varanasi via Lucknow in $2 \times 3 = 6$ ways

Addition Theorem

If an operation can be performed in 'm' ways and another independent operation can be performed in 'n' ways, then either of the two operations can be performed in '$m + n$' ways.

2. PERMUTATION AS AN ARRANGEMENT

Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.

Before using the concept of permutation we must know about the factorial notation.

(a) $\underline{|n}$ or $n!$ is read as 'n factorial'

(b) $n! = n(n-1)(n-2)$ 3.2.1

(c) $n! = n(n-1)!$

(d) $1! = 1$, $0! = 1$

(e) $(-r)!$ is not defined and $\dfrac{1}{(-r)!} = 0$

I. Meaning of ^nP_r : Number of permutations of n different things taken r at a time is denoted by ^nP_r or $P(n, r)$.

$$^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2).......(n-r+1)$$

II. Number of permutations of n different things taken all at a time is $^nP_n = n!$

III. Number of permutations of n things, taken all at a time, out of which p things are alike of one type, q things are alike of second type and rest are all different is $\dfrac{n!}{p!q!}$

IV. The number of permutations of n different things, taken r at a time, when each thing can be repeated any number of times is n^r.

V. Permutations under restrictions

1. Number of permutations of n different things taken r at a time, when one particular thing is to be always included is, $r.{}^{n-1}P_{r-1}$.
2. Number of permutations of n different things taken r at a time, when p particular things are to be always included, is $\frac{r!}{(r-p)!}.{}^{n-p}P_{r-p}$
3. Number of permutations of n different things taken r at a time, when p particular things are always together is
$$p!.[r-(p-1)]!\,{}^{n-p}P_{r-p}$$
4. Number of permutations of n different things, taken r at a time, when p particular things are not to be taken is ${}^{n-p}P_r$.
5. Number of permutations of n different things, taken all at a time, when p particular things always occur together is
$$p!(n-p+1)!$$
6. Number of permutations of n different things, taken all at a time, when m particular things all never occur together is $n! - [m!\,.\,(n-m+1)!]$
7. The number of permutations of n distinct objects taken all when r particular objects are in a specific order is $\frac{n!}{r!}$.

VI. Circular Permutations

1. **Arrangements round a circular table**

   When clockwise and anticlockwise arrangements are considered different then number of circular permutations of n different things taken all at a time is $(n-1)!$

2. **Arrangement of flowers in a garland**

   In these cases clockwise and anticlockwise arrangements are considered same, hence required permutations = $\frac{1}{2}(n-1)!$

3. **Number of circular permutations of *n* different things taken**

r at a time	$= ({}^nP_r)\,/\,r$	When clockwise is different than anticlockwise arrangement
	$= ({}^nP_r)\,/\,(2r)$	When clockwise and anticlockwise are considered same.

3. COMBINATION AS SELECTION

Combination is the selection of a group which can be made by some or all of number of things without reference to the order.

Hence just the selection is combination whereas selection cum arrangement is permutation.

Generally, number of permutations is larger than number of selections.

Meaning of ${}^nC_r$: ${}^nC_r$ is equal to the number of ways of selections of r objects out of n distinct objects.

${}^nC_r$ is also denoted by $C(n, r)$ or $\binom{n}{r}$ and ${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!}$

STANDARD RESULTS USED FOR COMBINATION

(i) $^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{^nP_r}{r!}$ (but if $r > n$ then $^nC_r = 0$)

(ii) $^nC_r \in N$

(iii) $^nC_0 = {}^nC_n = 1$

(iv) $^nC_1 = n$

(v) $^nC_r = {}^nC_{n-r}$

(vi) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(vii) $n.{}^{n-1}C_{r-1} = (n-r+1).{}^nC_{r-1}$

(viii) If ^nC_r is maximum then $r = \begin{cases} \dfrac{n}{2}; & \text{if } n \text{ is even integer} \\ \dfrac{n-1}{2}, \dfrac{n+1}{2}; & \text{if } n \text{ is odd integer} \end{cases}$

(ix) $^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1}$

(x) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

(xi) $C_0 + C_1 .... + C_n = 2^n$, where $^nC_r = C_r$

(xii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + ... = 2^{n-1}$, where $^nC_r = C_r$

(xiii) $^{2n+1}C_0 + {}^{2n+1}C_1 + ... + {}^{2n+1}C_n = 2^{2n}$

(xiv) $^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + ... + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

(xv) The product of r consecutive positive integers is divisible by r!

CONCEPT OF COMBINATION FOR DIFFERENT CASES

(a) The number of combinations of n different things taken r at a time is ^nC_r.

(b) Number of combination of n different things when k particular things always occur is $^{n-k}C_{r-k}$.

(c) Number of combination of n different things when k particular things never occur is $^{n-k}C_r$.

(d) All possible combination of n distinct objects = $^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

(e) Number of combination of n different things selecting atleast one of them is
$^nC_1 + {}^nC_2 + ...{}^nC_n = 2^n - 1$

(f) Number of combination of n identical things, taking r ($r \le n$) at a time is 1.

(g) Number of ways of selecting none, one or more things out of n alike things is (n + 1)
(*i.e.* 0, 1, 2,............... or n things)

(h) Out of ($p + q + r + s$) things, p are alike of one kind, q are alike of second kind, r are alike of third kind and s are different, then total number of combination of selecting at least one item is
$(p + 1)(q + 1)(r + 1)(2)^s - 1$

4. SIMPLE APPLICATIONS OF ^nC_r and ^nP_r

I. Divisors

Let $N = P_1^{\alpha_1}.P_2^{\alpha_2}.P_3^{\alpha_3}..............P_k^{\alpha_k}$ where P_1, P_2.......... P_k are distinct primes and α_1, α_2.......α_k are natural numbers; then

(a) Total number of divisors of N including 1 and N is

$(\alpha_1 + 1)(\alpha_2 + 1)\ldots\ldots(\alpha_k + 1)$

(b) Sum of these divisors including 1 and N is

$$= (P_1^0 + P_1^1 + \ldots\ldots + P_1^{\alpha_1})(P_2^0 + P_2^1 + \ldots\ldots + P_2^{\alpha_2})\ldots\ldots\ldots\ldots(P_k^0 + P_k^1 + \ldots\ldots + P_k^{\alpha_k})$$

$$= \left(\frac{P_1^{\alpha_1+1}-1}{P_1-1}\right)\left(\frac{P_2^{\alpha_2+1}-1}{P_2-1}\right)\ldots\left(\frac{P_k^{\alpha_k+1}-1}{P_k-1}\right)$$

(c) The number of ways in which N can be resolved into two factors which are coprime to each other is 2^{n-1} where n is the number of distinct prime factors of N.

(d) The number of ways in which N can be resolved as a product of two factors is

$\frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1)\ldots\ldots(\alpha_k + 1)]$, if N is not a perfect square

and $\frac{1}{2}[(\alpha_1+1)(\alpha_2+1)\ldots\ldots(\alpha_k+1)+1]$, if N is a perfect square.

II. Division into groups

(a) The number of ways in which $(m + n)$ distinct objects can be divided into two groups containing m and n objects $(m \neq n)$ respectively are

$$^{m+n}C_m.{}^nC_n = \frac{(m+n)!}{m!n!}$$

(b) If $m = n$ and group order is not important then number of ways $= \dfrac{(2n)!}{(2)!(n!)^2}$

(c) If $m = n$ and groups are distinct then number of ways $= \dfrac{(2n)!}{(n!)^2}$

Generalisation

The number of ways in which mn distinct objects can be divided equally among m groups each containing n elements.

(a) If order of group is not important is $\dfrac{(mn)!}{m!(n!)^m}$

(b) If order of group is important is $\dfrac{(mn)!}{(n!)^m}$

III. Arrangement into groups

(a) The number of ways to distribute n different things into r different groups if none of the groups is empty

$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n\ldots\ldots\ldots\ldots + (-1)^k\, {}^rC_k(r-k)^n$

(b) The number of ways to distribute n identical things into r different groups

If blank groups are allowed, is $^{n+r-1}C_{r-1}$

If blank groups are not allowed, is $^{n-1}C_{r-1}$

(c) The number of ways to distribute *n* identical things in *r* groups so that no group contains less than *l* and more than *m* things ($m > l$) is coefficient of x^n in the expansion of $(x^l + x^{l+1} ...... + x^m)^r$

(d) The number of ways to select *r* things from a group of *n* things havings *p* things identical is

$$\sum_{r=0}^{r} {}^{n-p}C_r \text{ if } r \le p$$

and $$\sum_{r=r-p}^{r} {}^{n-p}C_r \text{ if } r > p$$

IV. Derangements

If *n* things are arranged in a row, the number of ways in which they can be deranged so that no one of them occupies its original place or no object goes to its scheduled place, is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+...+(-1)^n\frac{1}{n!}\right)$$

Remark :

If *r* things go to wrong place out of *n* things, then $(n-r)$ things go to original place (Here $r < n$) D_n = Number of ways, if *r* things go to wrong place and $(n-r)$ things to correct place and D_r = Number of ways, if *r* things goes to wrong place. If *r* goes to wrong place out of *n*, then $(n-r)$ goes to correct places.

Then $D_n = {}^nC_{n-r}D_r$

where $D_r = r!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+...+(-1)^r\frac{1}{r!}\right)$

V. Applications of ${}^nC_r$ in formation of geometric figures

(a) The maximum number of lines made by *n* points is $= {}^nC_2$.

(b) The number of diagonals in a polygon of *n* sides $= \frac{n(n-3)}{2}$

(c) The maximum number of lines made by *n* points if *m* points out of these points are collinear $= {}^nC_2 - {}^mC_2 + 1$

(d) The maximum number of points of intersection of *n* lines $= {}^nC_2$ but if *m* lines out of these lines are concurrent then maximum number of points of intersections $= {}^nC_2 - {}^mC_2 + 1$.

(e) The maximum number of triangles made by *n* points $= {}^nC_3$ but if *m* points out of these points are collinear then total number of triangles $= {}^nC_3 - {}^mC_3$.

(f) The number of triangles made by joining all vertices of a polygon of *n* sides $= {}^nC_3$.

(g) The number of triangles in a polygon of *n* sides having one side, two sides, and no side common with the sides of polygon is $n(n-4)$, n and ${}^nC_3 - n - n(n-4)$ respectively.

(h) The maximum number of points of intersection of *n* distinct circles $= 2.{}^nC_2$

(i) The maximum number of points of intersection of *n* circles and *m* lines $= 2.{}^nC_2 + {}^mC_2 + 2.{}^nC_1.{}^mC_1$

(j) The number of parallelogram made by *n* parallel lines in one set and *m* parallel lines in another set $= {}^nC_2 \times {}^mC_2$.

(k) The number of rectangles (including squares) made in a rectangle of size $n \times p$ (where $n > p$) $= {}^{(n+1)}C_2 \times {}^{(p+1)}C_2 = \dfrac{np(n+1)(p+1)}{4}$ and the number of squares in this rectangle is $= \sum_{r=1}^{p}(n-r+1)(p-r+1)$.

(l) In a square of size $n \times n$, the number total rectangles $= {}^{n+1}C_2 \times {}^{n+1}C_2 = \left(\dfrac{n(n+1)}{2}\right)^2 = \sum n^3$

and squares $= \sum n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(m) The number of sections made of a plane by n distinct straight lines $= 1+\dfrac{n(n+1)}{2}$.

VI. Solutions of the Equations

$x_1 + x_2 + x_3 + ... + x_r = n$ ...(i)

(a) The number of non-negative integral solutions of the above equation $= {}^{n+r-1}C_{r-1}$

(b) The number of positive integral solutions of the above equation $= {}^{n-1}C_{r-1}$.

(c) $x_1x_2x_3...x_r = P_1^{\alpha_1}P_2^{\alpha_2}P_3^{\alpha_3}...P_n^{\alpha_n}$, (where $P_i \rightarrow$ prime number) ...(ii)

The positive integral solution of the above equation (ii) is

$\left({}^{(\alpha_1+r-1)}C_{r-1}\right)\left({}^{(\alpha_2+r-1)}C_{r-1}\right) \; ... \; \left({}^{(\alpha_n+r-1)}C_{r-1}\right)$

VII. Exponent : The exponent of any prime number p in $n!$ is given by

$E_p(n!) = \left[\dfrac{n}{p}\right]+\left[\dfrac{n}{p^2}\right]+\left[\dfrac{n}{p^3}\right]+...+\left[\dfrac{n}{p^s}\right], p^s \le n < p^{s+1}$, where [ ] represents greatest integer function.

Special Points about Exponent :

(a) $E_6(n!) = E_3(n!)$, $E_{10}(n!) = E_5(n!)$

(b) $E_p({}^nP_r) = E_p(n!) - E_p(n-r)!$

(c) $E_p({}^nC_r) = E_p(n!) - E_p(n-r)! - E_p(r!)$

(d) Number of zeroes at the end of $n! = E_{10}(n!) = E_5(n!)$

VIII. Sum of Numbers : The sum of the numbers of n digits formed by n distinct digits without repetition

$= (\text{sum of digits})\,(n-1)!\left(\dfrac{10^n-1}{9}\right)$.

IX. Rank of the word in dictionary : To find the rank of a word in a dictionary we use the rule that letters must be in alphabetic order.

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