Corporate Office : *AESL, 3rd Floor, Incuspaze Campus-2, Plot No. 13, Sector-18, Udyog Vihar, Gurugram, Haryana - 122015*

Practice Problem

Number System

1. Positive integers $a_1, a_2, \ldots, a_7, b_1, b_2, \ldots, b_7$ satisfy $2 \le a_i \le 166$ and $a_i^{b_i} \equiv a_{i+1}^2 \pmod{167}$ for each $1 \le i \le 7$ (where $a_8 = a_1$). Compute the minimum possible value of $b_1 b_2 \ldots b_7(b_1 + b_2 + \ldots + b_7)$.
2. Prove that there exists infinitely many positive integer k such that for every positive integer n, the number $k2^n + 1$ is composite.
3. Show that every positive integer can be written as $a_1 1^3 + a_2 2^3 + \ldots + a_k k^3$ for some positive integer k and $a_i \in \{\pm 1, \pm 2\}$.
4. Let $x_1, x_2, \ldots, x_n$ be positive integers. Assume that in their decimal representations no x_i "is an extension" of another x_j. For instance, 123 is an extension of 12, 459 is an extension of 4, but 134 is not an extension of 123. Prove that $\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} < 3$.
5. Let x, y, z be positive integers such that $gcd(x, y, z) = 1$ and

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$$

   is an integer. Prove that xyz is a perfect square.
6. The increasing sequence 1, 3, 4, 9, 10, 12, 13, … consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term of this sequence (where 1 is the 1st term, 3 is the 2nd term, and so on).
7. Let S be a set of integers (not necessarily positive) such that

   (A) there exist $a, b \in S$ with $gcd(a, b) = gcd(a - 2, b - 2) = 1$;

   (B) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S.

   Prove that S is the set of all integers.
8. Let k, m, n be positive integers such that $k > n > 1$ and the greatest common divisor of k and n is 1. Prove that if $k - n$ divides $k^m - n^{m-1}$, then $k \le 2n - 1$.
9. Let m and n be two positive integers. If there are infinitely many integers k such that $k^2 + 2kn + m^2$ is a perfect square, prove that $m = n$.

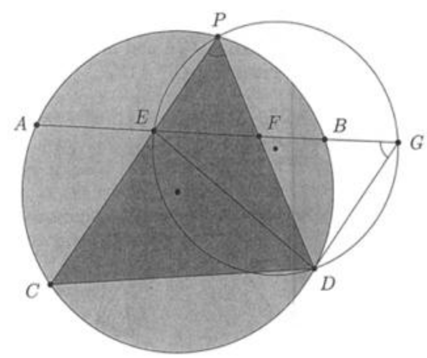
10. A mathematical frog jumps along the number line. The frog starts at 1 and jumps according to the following rule: if the frog is at integer n, then it can jump either to n + 1 or to $n+2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if k ≥ 2 is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.
11. Show that the equation $x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$ has infinitely many solutions in integers x, y, z.
12. Do there exist three distinct positive real numbers a, b, c such that the numbers a, b, c, b + c – a, c + a – b, a + b – c and a + b + c from a 7-term arithmetic progression in some order?
13. Find all primes p and q and even numbers n > 2, satisfying the equation $p^n + p^{n-1} + \ldots + p + 1 = q^2 + q + 1$.

Geometry

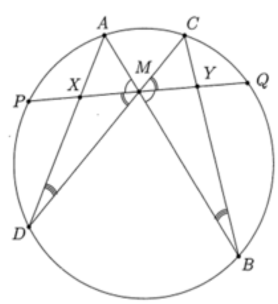
14. Given two non-intersecting chords *AB* and *CD* in a circle and a variable point *P* on the arc *AB* remote from points *C* and *D*, let *E* and *F* be the intersections of chords *PC*, *AB*, and of *PD*, *AB*, respectively. Prove that the value of

$$\frac{AE \cdot BF}{EF}$$

does not depend on the position of *P*.

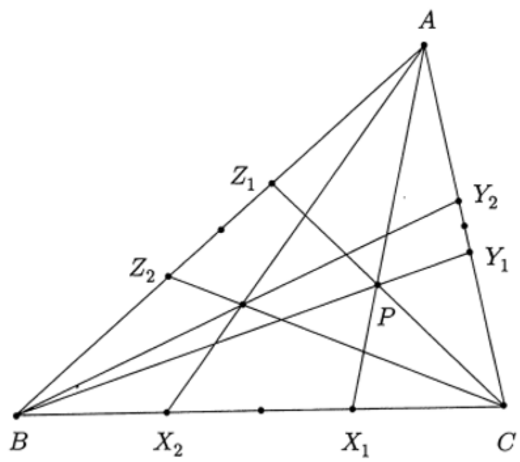


15. Let *M* be the midpoint of chord *PQ* of a given circle, through which two other chords *AB* and *CD* are drawn; *AD* cuts *PQ* at *X* and *BC* cuts *PQ* at *Y*. Then, *M* is also the midpoint of *XY*.

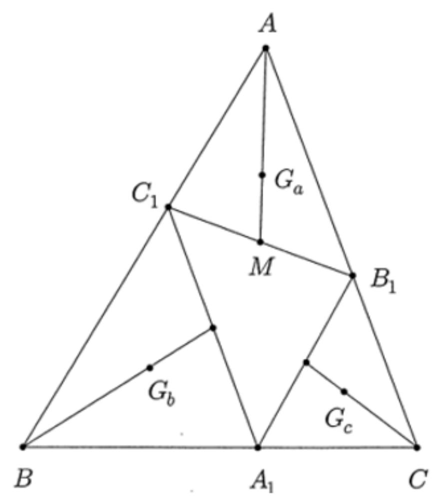


16. Let *ABC* be an acute triangle. Let the line through *B* perpendicular to *AC* meet the circle with diameter *AC* at points *P* and *Q*, and let the line through *C* perpendicular to *AB* meet the circle with diameter *AB* at points *R* and *S*. Prove that *P*, *Q*, *R*, *S* are concyclic.

17. Let ABC be a triangle and let P be points in its interior. Let X_1, Y_1, Z_1 be the intersections of AP, BP, CP with BC, CA and AB, respectively. Furthermore, let X_2, Y_2, Z_2 be the reflections of the points X_1, Y_1, Z_1 over the midpoints of BC, CA, and AB, respectively. Prove that the lines AX_2, BY_2, CZ_2 are concurrent.



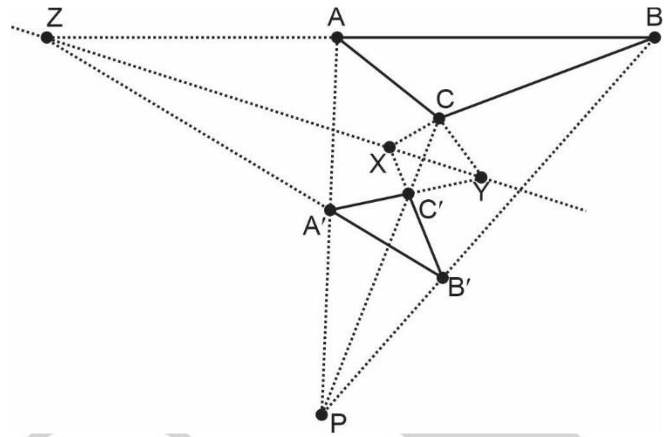
18. Points A_1, B_1, C_1 are chosen on the sides BC, CA, and AB, respectively of triangle ABC. Denote by G_a, G_b, G_c are the centroids of triangles AB_1C_1, BC_1A_1, CA_1B_1, respectively. Prove that the lines AG_a, BG_b, CG_c are concurrent if and only if lines AA_1, BB_1, CC_1 are concurrent.



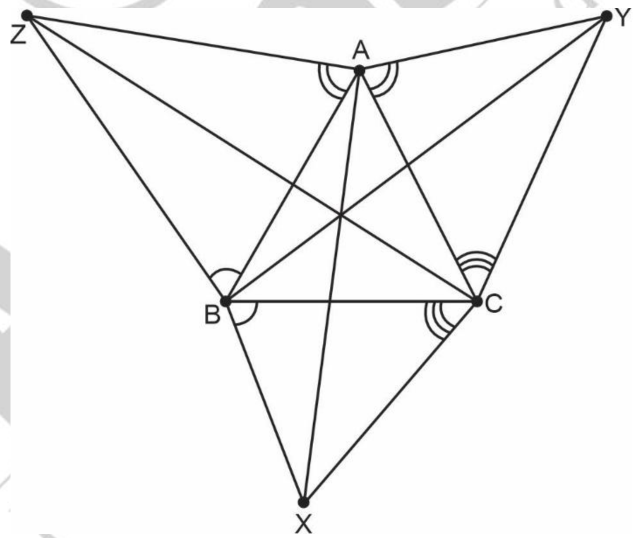
19. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

20. Let ABC be an non isosceles triangle which inscribed in a circle (O) and has incircle (I). Suppose that BI cuts AC at E and CI cuts AB at F. The circle passes through E, tangent to OB at B cuts (O) at M. The circle passes through F tangent to OC at C cuts (O) at N. The lines ME, NF respectively cut (O) second time at P, Q. Denote K as the intersection of EF and BC. The line PQ cuts BC, EF respectively at G, H. Prove that the median of the vertex G in triangle GHK is perpendicular to the line OI.

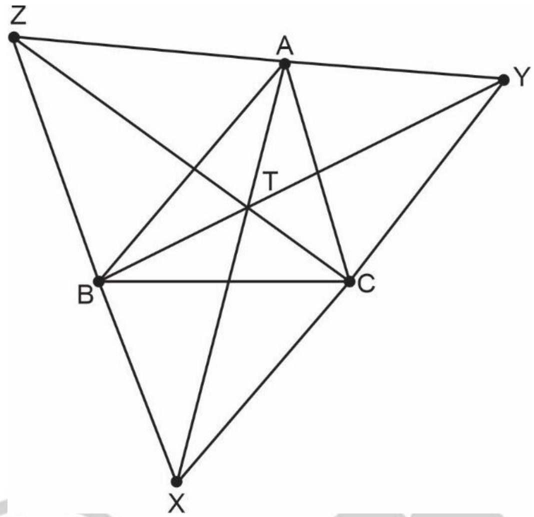
21. Let ABC and A’B’C’ be two triangles. Then, the lines AA′, BB′, CC′ are concurrent if and only if the intersections of BC and B′C′, of CA and C′A′ and of AB and A′B′ are collinear.



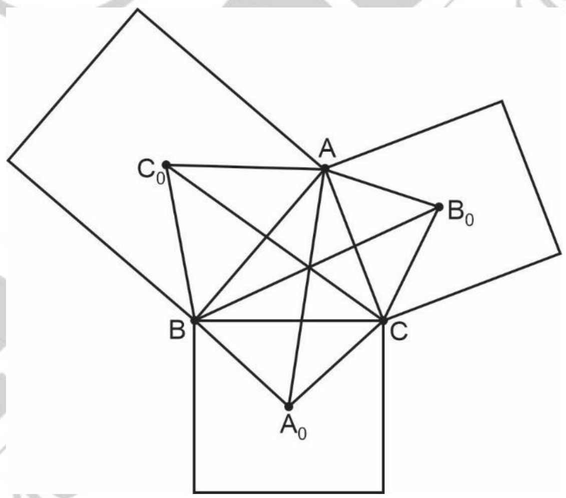
22. Let ABCDEF be a cyclic hexagon (with vertices not necessarily in this order on the circle). Then, the intersections AB ∩ DE, BC ∩ EF, CD ∩ FA are collinear.
23. Let ABC be a triangle and let X, Y, Z be three points in its plane such that ∠YAC = ∠BAZ, ∠ZBA = ∠CBX and ∠XCB = ∠ACY. Then, the lines AX, BY, CZ are concurrent.



24. Let XBC, YCA, ZAB be the equilateral triangles erected on the sides of ABC towards the exterior of the triangle. Then, the lines AX, BY, CZ are concurrent at a point that is usually denoted by T (the Torricelli point) or F_+ (the first Fermat point).



25. Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus, one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.



26. Let ABC be a triangle and let ACXY and ABRS be the squares erected on the sides AC and AB that are directed towards the exterior of triangle ABC. Let BX and CR intersect at T. Prove that AT is the A-altitude of triangle ABC.

Algebra

27. Prove that, for all positive real numbers *a, b, c*, $\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{a^3+c^3+abc}\le\frac{1}{abc}$

28. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_{i-1}\, a_{i+1} \le a_i^2$ for i = 1,2,3,…. (such a sequence is said to be long concave) Show that for each $n > 1$,

$$\frac{a_0+\ldots\ldots+a_n}{n+1}\cdot\frac{a_1+\ldots\ldots+a_{n-1}}{n-1}\ge\frac{a_0+\ldots\ldots+a_{n-1}}{n}\cdot\frac{a_1+\ldots\ldots+a_n}{n}$$

29. Prove that the zeroes of

    $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$

    cannot all be real if $2a^2 < 5b$

30. Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

31. Let P be a point in the plane of triangle ABC and γ a line passing through P. Let A′, B′, C′ be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB, respectively. Prove that A′, B′, C′ are collinear.

32. In triangle ABC, angle A is twice angle B, angle C is obtuse and the three side lengths a, b, c are integers. Determine, with proof, the minimum possible perimeter.

33. If a pint A_1 is in the interior of an equilateral triangle ABC and point A_2 is in the interior of ΔA_1BC, prove that I.Q. (A_1BC) > I.Q.(A_2BC), where the isoperimetric quotient of a figure F is defined by $\text{I.Q.}(F) = \dfrac{\text{Area}(F)}{[\text{Perimeter}(F)]^2}$.

34. Determine each real root of $x^4 - (2\cdot10^{10} + 1)x^2 - x + 10^{20} + 10^{10} - 1 = 0$ correct to four decimal places.

35. An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

36. Let R be the set of real numbers. Determine all functions $f : R \to R$ such that $f(x^2 - y^2) = xf(x) - yf(y)$ for all pairs of real numbers x and y.

37. Let $P(z) = z^n + c_1z^{n-1} + c_2z^{n-2} + \ldots + c_n$ be a polynomial in the complex variable z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that $P(a + bi) = 0$ and $(a^2 + b^2 + 1)^2 < 4b^2 + 1$.

38. Find all solutions to $(m^2 + n)(m + n^2) = (m - n)^3$, where m and n are non-zero integers.

39. Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k - 1$ for which $a_1 + a_2 + \ldots + a_k$ is divisible by k. For instance, when n = 9 the obtained sequence is 9, 1, 2, 0, 3, 3, 3, …. Prove that for any n the sequence $a_1, a_2, a_3, \ldots$ eventually becomes constant.

Combinatorics

40. A set of positive numbers has the triangle property if it has three distinct elements that are the lengths of the sides of a triangle whose area is positive. Consider sets {4, 5, 6, … *n*} of consecutive positive integers, all of whose ten-element subsets have the triangle property. What is the largest possible value of *n*?

41. There are at least four candy bars in $n(n \geq 4)$ boxes. Each time, Mr. Fat is allowed to pick two boxes, take one candy bar from each of the two boxes, and put those candy bars into a third box. Determine if it is always possible to put all the candy bars into one box.?

42. For any subset $S \subseteq \{1, 2, \ldots, 15\}$, a number n is called an "anchor" for S if n and $n + |S|$ are both members of S, where $|S|$ denotes the number of members of S. Find the average number of anchors over all possible subsets $S \subseteq \{1, 2, \ldots, 15\}$

43. Show that among any group of n people, where $n \geq 2$, there are at least two people who know exactly the same number of people in the group (assuming that "knowing" is a symmetry relation).

44. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

45. Alice, Betty and Carol took the same series of examinations. There was one grade of A, one grade of B and one grade of C for each examination, where A, B, C are different positive integers. The final test scores were

Alice	Betty	Carol
(20)	(10)	(9)

If betty placed first in the arithmetic examination, who placed second in the spelling examination?

46. Does there exist a natural number n for which the number $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ is divisible by 5?

47. Twenty five boys and twenty five girls sit around a table. Prove that it is always possible to find a person both of whose neighbors are girls.

48. Show that given any set A of 13 distinct real numbers, there exist $x, y \in A$ such that

$$0 < \frac{x-y}{1+xy} \leq 2-\sqrt{3}$$

49. Consider a standard 8 × 8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A zig-zag path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

50. The integer 9 can be written as a sum of two consecutive integers : 9 = 4 + 5. Moreover, it can be written as a sum of (more than one) consecutive positive integers in exactly two ways : 9 = 4 + 5 = 2 + 3 + 4. Is there an integer that can be written as a sum of 1990 consecutive integers and that can be written as a sum of (more than one) consecutive positive integers in exactly 1990 ways?

51. For every positive integer n determine the number of permutations $(a_1, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ with the following property :

    $2(a_1 + a_2 + \ldots + a_k)$ is divisible by k for $k = 1, 2, \ldots, n$.

52. Suppose that five points in a plane are situated so that no two of the straight lines joining them are parallel perpendicular or coincident. From each point perpendicular are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have?

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