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Practice Problem_Solutions

Number System

1. Solution

Let B = $b_1b_2 \ldots b_7 - 128$. Since $a_1^{b_1b_2\ldots b_7} \equiv a_2^{2b_2b_3\ldots b_7} \equiv a_3^{4b_3b_4\ldots b_7} \equiv \ldots \equiv a_1^{128} \pmod{167}$

We find that $a_1^B \equiv 1$. Similarly, $a_i^B \equiv 1 \pmod{167}$ for all i. Since 167 is a prime and $167 - 1 = 2\cdot 83$, we know that the order of each individual a_i (since $a_i \neq 1$) must be either 2 of a multiple of 83. If B is not a multiple of 83, then it follows that all the a_i must be –1, which implies that all the b_i must be even, meaning that the minimum possible value of $b_1b_2 \ldots b_7(b_1 + b_2 + \ldots b_7)$ is $2^7\cdot 14 > 1000$.

Oh the other hand, if B is a multiple of 83, then the smallest possible values for $b_1b_2 \ldots b_7$ are 45 and 128. If $b_1b_2 \ldots b_7 = 45$, then the smallest possible value for $b_1 + b_2 + \ldots b_7$ is 5 + 3 + 3 + 1 + 1 + 1 + 1 = 15, so the minimum possible value for $b_1b_2 \ldots b_7(b_1 + b_2 + \ldots b_7)$ is $45\cdot 15 = 675$. This can be achieved by letting g be an element of order 83 and setting $a_1 = g$, $a_2 = g^{1/2}$, $a_3 = g^{1/4}$, $a_4 = g^{1/8}$, $a_5 = g^{1/16}$, $a_6 = g^{3/32}$, $a_7 = g^{9/64}$ (all exponents are taken mod 83).

If $b_1b_2 \ldots b_7 \geq 128$, then by the AM-GM inequality we have

$b_1b_2 \ldots b_7(b_1 + b_2 + \ldots b_7) \geq 7(b_1b_2 \ldots b_7)^{8/7} \geq 7.2^8 > 1000$.

Therefore 675 is optional.

2. Solution

The idea here is to find a covering set $\{(a_i, b_i)\}$ of the integers, such that every integer $n \equiv a_i \pmod{b_i}$ for at least 1 pairs.

Then, for any prime p_i that divides $2^{b_i} - 1$, if $k \equiv -2^{b_i - a_i} \pmod{p_i}$, then $p_i \mid k2^n + 1$. If k is large enough relative to p_i (e.g., $k > p_i$), then this guarantees the term is composite.

By the Chinese remainder theorem, there exist infinitely many positive integers k such that

$k \equiv 1 \pmod 3$

$k \equiv 1 \pmod 5$

$k \equiv 3 \pmod 7$

$k \equiv 10 \pmod{13}$

$k \equiv 1 \pmod{17}$

$k \equiv -1 \pmod{241}$

If $n \equiv 1 \pmod 2$, then $k2^n + 1 \equiv 2 + 1 \equiv 0 \pmod 3$. Otherwise $2|n$. If $n \equiv 2 \pmod 4$, then $k2^n + 1 \equiv 2^2 + 1 \equiv 0 \pmod 5$. Otherwise $4|n$. If $n \equiv 4 \pmod 8$, then $k2^n + 1 \equiv 2^4 + 1 \equiv 0 \pmod{17}$. Otherwise $8|n$. Then we have three cases.

Case-1 : $n \equiv 8 \pmod{24}$

By Fermat's little theorem, $2^{24} = (2^{12})^2 \equiv 1 \pmod{13}$. So $2^n = 2^{8 + 24m} \equiv 256 \equiv -4 \pmod{13}$ and

$k2^n + 1 \equiv 10(-4) + 1 \equiv 0 \pmod{13}$.

Case-2 : $n \equiv 16 \pmod{24}$.

Since $2^{24} = (2^3)^8 \equiv 1 \pmod 7$, we have $2^n = 2^{16 + 24m} \equiv 2^{1 + 3(5 + 8m)} \equiv 2 \pmod 7$ and $k^{2n + 1} \equiv 3\cdot 2 + 1 \equiv 0 \pmod 7$.

Case-3 : $n \equiv 0 \pmod{24}$.

Since $2^{24} = (2^8)^3 \equiv 15^3 \equiv 225\cdot 15 \equiv -16\cdot 15 \equiv 1 \pmod{241}$. So, $2^n = 2^{24m} \equiv 1 \pmod{241}$ and then $k2^n + 1 \equiv -1 + 1 = 0 \pmod{241}$.

Comment :

We many wonder why modulo 3, 5, 7, 13, 17, 241 work. It may be that in dealing with $n \equiv 8, 16, 0 \pmod{24}$, we want $2^{24} \equiv 1 \pmod p$ for some useful primes p. Then we notice

$2^{24} - 1 = (2^3 - 1)(2^3 + 1)(2^6 + 1)(2^{12} + 1) = 7\cdot 3^2\cdot 5\cdot 13\cdot 17\cdot 241$.

3. Solution

Let A be the set of all numbers which can be expressed in the form

$a_1 1^3 + a_2 2^3 + \ldots a_k k^3$

For $a_i \in \{\pm 1, \pm 2\}$.

For simplicity, let us denote the sum $a_1 1^3 + a_2 2^3 + \ldots a_k k^3$ by $(a_1, a_2, a_3, \ldots a_k)$.

We can easily verify the following congruences modulo 18

$0 \equiv (2, 2)$	$1 = (1)$
$2 \equiv (2)$	$3 = (2, -1, 1)$
$4 \equiv (-2, 1, -1)$	$5 = (-1, 1, -1, 1)$
$6 \equiv (-2, 1)$	$7 = (-1, 1)$
$8 \equiv (1, 2, -1)$	$9 = (1, 1)$

(–1 to –8 can be obtained by reversing the sign of each number.)

We shall now show that if n is in A, then $n \pm 18$ is also in A.

Suppose n is in A. Then $n = (a_1, a_2, \ldots, a_k)$ for suitable $a_i \in \{\pm 1, \pm 2\}$

Using the identity $(k - 1)^3 - 2k^3 + (k + 1)^3 = 6k + 2$, we see that

$(a_i, a_2, \ldots, a_k, -1, 2, -1, 1, -2, 1)$

$= (a_1, a_2, \ldots, a_k) - 6(k + 2) - 2 + 6(k + 5) + 2$

$= n + 18$

Similarly, $(a_1, a_2, \ldots, a_k, 1, -2, 1, -1, 2, -1) = n - 18$

Therefore, if n is in A, $n + 18k$ is also in A for any integer k. For any number x, there is a number n in A congruent to x modulo 18. Thus $x = n + 18k$ for some integer k is in A.

4. Solution

Let us say the positive integer x reduces to the positive integer y, if $y = \left[\frac{x}{10}\right]$. we write $x \to y$ reduces to y. observe that the positive integer x" is an extension" of the positive integer y if and only if there is a chain $x \to \ldots \to y$ of reductions from x to y. We say that the finite set M of positive integers reduces to the set N of positive integers (and write $M \to N$) if N is obtained by the following construction. Put the greatest element a_1 of M and reduce it to, say, the positive integer b. include b, and remove all elements a_1, a_2, …, a_m that reduce to be (i.e. $a_1 \to b, a_2 \to b, \ldots, a_m \to b$)

$$N = \left(M \cup \{b\}\right) - \{a_1, a_2, \ldots a_m\} \qquad \ldots(1)$$

We say that a finite set M of positive integers is good if no element of M is an extension of another element of M. note that if M is good and $M \to N$, then N is also a good set. Call a good set M of positive integers irreducible if it does not reduce to any other set. It is clear that the irreducible sets are the subsets of {1, 2, …, 9}. For a finite set M of positive integers, let $f(M) = \sum_{x \in M} \frac{1}{x}$. For every irreducible set M, it holds

$$f(M) \le \sum_{k=1}^{9} \frac{1}{k} = 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) + \left(\frac{1}{4} + \frac{1}{5}\right) + \left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right)$$

$$< 1 + 1 + \frac{1}{2} + \frac{3}{7} < 3$$

If M is good and $M \to N$, then we have with the notation (1):

$$f(M) = f(N) + \frac{1}{a_1} + \ldots + \frac{1}{a_m} - \frac{1}{b}$$

$$< f(N) + \frac{1}{10b} + \frac{1}{10b+1} + \ldots + \frac{1}{10b+9} - \frac{1}{b} < f(N).$$

The set $M_1 = \{x_1, x_2, \ldots, x_n\}$ reduces in a finite chain $M_1 \to M_2 \to \ldots \to M_k$ to an irreducible set M_k, where $f(M1) < f(M2) < \ldots f(M_k) < 3$.

5. Solution

For a prime p and a positive integer n, we use the standard notation $\upsilon_p(n)$ to denote the largest integer k such that $p^k \mid n$. To show that xyz is a perfect square, It suffices to show that $\upsilon_p(xyz)$ is even for each prime $p \mid xyz$. Let p be a prime dividing xyz. Let a = $\upsilon_p(x)$, $b = \upsilon_p(y)$ and $c = \upsilon_p(z)$. without loss of generality, we may assume that $a \ge b \ge c$. Since $gcd(x, y, z) = 1$, it follows that $c = 0$. To finish the proof, it suffices to show that $a = b$. for the sake of contradiction, suppose otherwise that $a > b$. note that we have

$$\upsilon_p(xyz) = a + b, \upsilon_p\left(x^2y^2\right) = 2a + 2b, \upsilon_p\left(y^2z^2\right) = 2b, \upsilon_p\left(z^2x^2\right) = 2a.$$

Since $a > b$, it follows that $\upsilon_p\left(x^2y^2 + y^2z^2 + z^2x^2\right) = 2b < a + b = \upsilon_p(xyz)$, which contradict the assumption that $xyz \mid \left(x^2y^2 + y^2z^2 + z^2x^2\right)$

6. Solution

Note that a positive integer is a term of this sequence if and only if its base 3 representation consists only of 0's and 1's. Therefore, we can set up a one-to-one correspondence between the positive integers and the terms of this sequence by representing both with binary digits (0's and 1's), first in base 2 and then in base 3:

$1 = 1_{(2)} \quad \Leftrightarrow \quad 10_{(3)} = 1$

$2 = 10_{(2)} \quad \Leftrightarrow \quad 10_{(3)} = 3$

$3 = 11_{(2)} \quad \Leftrightarrow \quad 11_{(3)} = 4$

$4 = 100_{(2)} \quad \Leftrightarrow \quad 100_{(3)} = 9$

$5 = 101_{(2)} \quad \Leftrightarrow \quad 101_{(3)} = 10$

This is a correspondence between the two sequences in the order given, that is, the kth positive integer is made to correspond to the kth sum (in increasing order) of distinct powers of 3. This is because when the binary numbers are written in increasing order, they are still in increasing order when interpreted in any other base. (If you can explain why this is true when interpreted in base 10, you should be able to explain it in base 3 as well).

Therefore, to find the 100th term of the sequence, we need only look at the 100th line of the above correspondence

$100 = 11000100_{(2)} \quad \Leftrightarrow \quad 1100100_{(3)} = 981.$

7. Solution

In the solution below we use the expression S is stable under $x \mapsto f(x)$ to mean that if x belongs to S then $f(x)$ also belongs to S. if $c, d \in S$, then by(B), S is stable under $x \mapsto c^2 - x$ and $x \mapsto x$, hence stable under $x \mapsto c^2 - (d^2 - x) = x + (c^2 - d^2)$. Similarly, S is stable under $x \mapsto x + (d^2 - c^2)$. Hence S is stable under $x \mapsto x + n$ and

$x \mapsto x - n$ whenever n is an integer linear combination of numbers of the form $c^2 - d^2$ with $c, d \in S$. In particular, this holds for $n = m$, where $m = gcd\{c^2 - d^2 : c, d \in S\}$.

Since $S \neq \phi$ by (A), it suffices to prove that $m = 1$. For the sake of contradiction, assume that $m \neq 1$. Let p be a prime dividing m. then $c^2 - d^2 \equiv 0 \pmod p$ for all $c, d \in S$. In other words, for each $c, d \in S$, either $d \equiv c \pmod p$ or $d \equiv -c \pmod p$. given $c \in S$, $c^2 - c \in S$ by (B), so $c^2 - c \equiv c \pmod p$ or $c^2 - c \equiv -c \pmod p$. hence

For each $c \in$ S, either $c \equiv 0 \pmod p$ or $c \equiv 2 \pmod p$. (*)

By (A), there exist some a and b in S such that $gcd(a, b) = 1$, that is, at least one of a or b cannot be divisible by p. denote such an element of S by α : thus, $\alpha \not\equiv 0 \pmod p$. similarly, by (A) $gcd(a - 2, b - 2) = 1$, so p cannot divide both $a - 2$ and $b - 2$. Thus, there is an element of S, call it β, such that $\beta \not\equiv 2 \pmod p$. by (*), $\alpha \equiv 2 \pmod p$ and $\beta \equiv 0 \pmod p$. By (B) $\beta^2 - \alpha \in S$. Taking $c = \beta^2 - \alpha$ in (*) yields either $-2 \equiv 0 \pmod p$ or $-2 \equiv 2 \pmod p$, so $p = 2$. Now (*) says that all elements of S are even, contradicting (A). Hence our assumption is false and S is the set of all integers.

8. Solution

If k – n divides $k^m - n^{m-1}$, then we have

$k^m \equiv n^{m-1} \pmod{k - n}$.

Because k = (k – n) + n, $k \equiv n \pmod{k-n}$. That means $k^m \equiv k^{m-1} \pmod{k-n}$

Since gcd(k, n) = 1, k – n does not share any factors with k. We can multiply both sides by the modular multiplicative inverse, so $k \equiv 1 \pmod{k-n}$.

Thus, k = (k – n)a + 1, where a is an integer. After rearranging terms, we have k(a – 1) = an – 1

If a = 1, then n = 1. This can not happen, so we can divide both sides by a – 1 without losing any solutions.

$$k = \frac{an-1}{a-1}$$

$$= \frac{a}{a-1}n - \frac{1}{a-1}$$

To finish the proof that $k \le 2n - 1$, we will use induction. For the base case, letting a = 2 results in k = 2n – 1, which satisfies the inequality.

For the inductive step, assume $\frac{a}{a-1}n - \frac{1}{a-1} \le 2n-1$. Multiplying both sides by a – 1 and dividing both sides by a results in

$$\frac{a}{a}n - \frac{1}{a} \le (2n-1)\frac{a-1}{a}$$

Adding both sides by $\frac{n}{a}$ results in

$$\frac{a+1}{a}n - \frac{1}{a} \le \frac{2an-2n-a+1}{a} + \frac{n}{a} \le \frac{2an-n-a+1}{a} \le 2n-1+\frac{1-n}{a}$$

Because n, a > 1, the value $\frac{1-n}{a}$ is less than 0. Thus, $2n-1+\frac{1-n}{a} < 2n-1$, so we have

$$\frac{a+1}{a}n - \frac{1}{a} < 2n-1$$

The inductive step is complete, so we have proven that $k \le 2n - 1$.

9. Solution

Note that we can complete the square to get $k^2 + 2kn + n^2 - n^2 + m^2$, which equals $(k + n)^2 + m^2 - n^2$

Assume that m > n. Since m, n are positive we know that $m^2 - n^2 > 0$. In order to prove that $(k + n)^2 + m^2 - n^2$ is not a perfect square, we can show that there are values of k where $(k + n)^2 < (k + n)^2 + m^2 - n^2 < (k + n + 1)^2$.

Since $m^2 - n^2 > 0$, we know that $(k + n)^2 < (k + n)^2 + m^2 - n^2$. In the case where $(k + n)^2 + m^2 - n^2 < (k + n + 1)^2$, we can expand and simplify to get

$k^2 + 2kn + n^2 + m^2 - n^2 < k^2 + 2kn + n^2 + 2k + 2n + 1$

$m^2 - n^2 < 2k + 2n + 1$

$$\frac{m^2 - n^2 - 2n - 1}{2} < k$$

All steps are reversible so there are values of k where $(k + n)^2 < (k + n)^2 + m^2 - n^2 < (k + n + 1)^2$, so there are no values of m, n where m > n that results in infinite number of integers k that satisfy the original conditions.

Now assume that m < n. Since m, n are positive, we know that $m^2 - n^2 < 0$. In order to prove that $(k + n)^2 + m^2 - n^2$ is not a perfect square, we can show that there are values of k where $(k + n - 1)^2 < (k + n)^2 + m^2 - n^2 < (k + n)^2$.

Since $m^2 - n^2 < 0$, we know that $(k + n)^2 + m^2 - n^2 < (k + n)^2$. In the case where $(k + n - 1)^2 < (k + n)^2 + m^2 - n^2$, we can expand and simplify to get

$k^2 + 2kn + n^2 - 2k - 2n + 1 < k^2 + 2kn + n^2 + m^2 - n^2 - 2k - 2n + 1 < m^2 - n^2$

$$k > -\frac{m^2 - n^2 + 2n - 1}{2}$$

All steps are reversible, so there are value of k where $(k + n - 1)^2 < (k + n)^2 + m^2 - n^2$, so there are no values of m, n where m < n that results in infinite number of integers k that satisfy the original conditions.

Now we need to prove that if m = n, there are an infinite number of integers k that satisfy the original conditions. By the substitution property, we find that $k^2 + kn + m^2 = k^2 + 2kn + n^2$. The expression can be factored into $(k + n)^2$. Since the expression is a perfect square, for all integer values of n, k there are an infinite number of integer k that satisfies the original conditions when m = n.

10. Solution

For $i \geq 0$ and $k \geq 1$, let $x_{i,k}$ denote the minimum number of jumps needed to reach the integer $n_{i,k} = 2^i k$. We must prove that $x_{i,k} > x_{i,1}$ … (i)

For all $i \geq 0$ and $k \geq 2$. We prove this using the method of descent.

First note that (i) holds for i = 0 and all $k \geq 2$, because it takes 0 jumps to reach the starting value $n_{0,1} = 1$ and at least one jump to reach $n_{0,k} = k \geq 2$. Now assume that (i) is not true for all choices of i and k. Let i_0 be the minimal value of i for which (i) fails for some k, let k_0 be the minimal value of k > 1 for which $x_{i_0,k} \leq x_{i_0,1}$. Then it must be the case that $i_0 \geq 1$ and $k_0 \geq 2$. Let J_{i_0,k_0} be a shortest sequence of $x_{i_0,k_0} + 1$ integers that the frog occupies in jumping from 1 to $2^{i_0}k_0$. The length of each jump that is the difference between consecutive integers in J_{i_0,k_0} is either 1 or a positive integer power of 2. The sequence J_{i_0,k_0} cannot contain 2^{t_0} because it takes more jumps to reach $2^{t_0}k_0$ than it does to reach 2^{t_0}. Let 2^{M+1}, $M \geq 0$ be the length of the longest jump made in generating J_{i_0,k_0}. Such a jump can only be made from a number that is divisible by 2^M (and by no higher power of 2). Thus we must have $M < i_0$, since otherwise a number divisible by 2^{i_0} is visited before $2^{i_0}k_0$ is reached, contradicting the definition of k_0.

Let 2^{m+1} be the length of the jump when the frog jumps over 2^{t_0}. If this jump starts at 2m(2t – 1) for some positive integer t, then it will end at 2m(2t – 1) + 2m + 1 = 2m(2t + 1). Since it goes over 2^{i_0} we see $2^m(2t-1) < 2^{i_0} < 2^m(2t+1)$ or $\frac{(2^{i_0-m}-1)}{2} < t < \frac{(2^{i_0-m}+1)}{2}$. Thus $t = 2^{i_0-m-1}$ and the jump over 2^{i_0} is from $2^m(2^{i_0-m}-1) = 2^{i_0} - 2^m$ to $2^m(2^{i_0-m}+1) = 2^{i_0} + 2^m$.

Considering the jumps that generate J_{i_0,k_0} let N_1 be the number of jumps from 1 to $2^{i_0}+2^m$ and let N_2 be the number of jumps from $2^{i_0}+2^m$ to $2^{i_0}k$. By definition of i_0, it follows that 2^m can be reached from 1 in less than N_1 jumps. On the other hand, because $m < i_0$ the number $2^{i_0}(k_0-1)$ can be reached from 2^m in exactly N_2 jumps by using the same jump length sequence as in jumping from $2^m+2^{i_0}$ to $2^{i_0}k_0 = 2^{i_0}(k_0-1)+2^{i_0}$. The key point here is that the shift by 2^{i_0} does not affect any of divisibility conditions needed to make jumps of the same length. In particular, with the exception of the last energy, $2^{i_0}k_0$ all of the elements of J_{i_0,k_0} are of the form $2^p(2t + 1)$ with $p < i_0$, again because of the definition of k_0. Because $2^p(2t+1)-2^{i_0} = 2^p\left(2t-2^{i_0-p+1}\right)$ and the number $2t+2^{i_0-p}+1$ is odd, a jump of size 2^{p+1} can be made from $2^p(2t+1)-2^{i_0}$ just as it can be made from $2^p(2t + 1)$.

Thus the frog can reach 2^m from 1 in less than N_1 jumps and can the reach $2^{i_0}(k_0-1)$ from 2^m in N_2 jumps. Hence the frog can reach $2^{i_0}(k_0-1)$ from 1 in less $N_1 + N_2$ jumps, that is in fewer jumps than needed to get to $2^{i_0}k_0$ and hence in fewer jumps than required to get to 2^{i_0}. This contradicts the definition of k_0.

Solution-2

Suppose $x_0 = 1, x_1, \ldots, x_t = 2^ik$ are the integers visited by the frog on his trip from 1 to 2^ik, $k \geq 2$. Let $s_j = x_j - x_{j-1}$ be the jump sizes. Define a reduced path y_j inductively by

$$y_j = \begin{cases} y_{j-1}+s_j & \text{if } y_{j-1}+s_j \leq 2^i \\ y_{j-1} & \text{otherwise} \end{cases}$$

Say jump s_j is deleted in the second case. We will show that the distinct integers among the y_j give a shorter path from 1 to 2_i. Clearly $y_j \leq 2^i$ for all j. Suppose $2^i - 2^{r+1} < y_j \leq 2^i - 2^r$ for some $0 \leq r \leq i - 1$. Then every deleted jump before y_j must have length greater than 2^r, hence must be a multiple of 2^{r+1}. Thus $y_j \equiv x_j \pmod{2^{r+1}}$. If $y_{j+1} > y_j$, then either $s_{j+1} = 1$ (in which case this is a valid jump) or $\frac{s_{j+1}}{2} = 2^m$ is the exact power of 2 dividing x_j. In the second case, since $2^r \geq s_{j+1} > 2^m$, the congruence says 2m is also the exact power of 2 dividing y_j, thus again this is a valid jump. Thus the distinct y_j form a valid path for the frog. If j = t the congruence gives $y_t \equiv x_t \equiv 0 \pmod{2^{r+1}}$, but this is impossible for $2^i - 2^{r+1} < y_t \leq 2^i - 2^r$. Hence we see $y^t = 2^i$, that is the reduced path ends at 2i. Finally since the reduced path ends at $2^i < 2^ik$ at least one jump must have been deleted and it is strictly shorter than the original path.

11. Solution

We seek solutions (x, y, z) which are in arithmetic progression. Let us put $y - x = z - y = d > 0$ so that the equation reduces to the form $3y^2 + 2d^2 = 2d^3$.

Thus we get $3y^2 = 2(d - 1)d^2$. We conclude that $2(d - 1)$ is 3 times a square. This is satisfied if $d - 1 = 6n^2$ for some n. Thus $d = 6n^2 + 1$ and $3y^2 = d^2. 2(6n^2)$ giving us $y^2 = 4d^2n^2$. Thus we can take $y = 2dn = 2n(6n^2 + 1)$. From this we obtain $x = y - d = (2n - 1)(6n^2 + 1)$, $z = y + d = (2n + 1)(6n^2 + 1)$. It is easily verified that (x, y, z)

= ((2n – 1)(6n^2 + 1), 2n(6n^2 + 1), (2n + 1)(6n^2 + 1)), is indeed a solution for a fixed n and this gives an infinite set of solutions as n varies over natural numbers.

12. Solution

We show that that answer is NO. Suppose, if possible, let a, b, c be three distinct positive real numbers such that a, b, c, b + c – a, c + a – b, a + b – c and a + b + c from a 7-term arithmetic progression in some order. We may assume that a < b < c. Then there are only two cases we need to check : (I) a + b – c < a < c + a – b < b < c < b + c – a < a + b + c and (II) a + b – c < a < b < c + a – b < c < b + c – a < a + b + c.

Case-I: Suppose the chain of inequalities a + b – c < a < c + a – b < b < c < b + c – a < a + b + c holds good. Let d be the common difference. Thus we see that c = a + b + c – 2d, b = a + b + c – 3d, a = a + b + c – 5d. Additing these, we see that a + b + c = 5d. But then a = 0 contradicting the positively of a.

Case-II: Suppose the inequalities a + b – c < a < b < c + a – b < c < b + c – a < a + b + c are true. Again we see that c = a + b + c – 2d, b = a + b + c – 4d, a = a + b + c – 5d.

We thus obtain $a+b+c=\left(\frac{11}{2}\right)d$. This gives $a=\frac{1}{2}d,\ b=\frac{3}{2}d, c=\frac{7}{2}d$.

Note that a + b – c = a + b + c – 6d = $-\left(\frac{1}{2}\right)d$.

However we also get a + b – c = $\left[\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)-\left(\frac{7}{2}\right)d=-\left(\frac{3}{2}\right)d\right]=-\left(\frac{3}{2}\right)d$. It follows that 3e = e giving d = 0. But this is impossible. Thus there are no three distinct positive real numbers a, b, c such that a, b, c, b + c – a, c + a – b, a + b – c and a + b + c form a 7-term arithmetic progression in some order.

13. Solution

Obviously p ≠ q. We write this in the form p(p^{n-1} + p^{n-2} + … + 1) = q(q + 1). If q ≤ p$^{n/2}$ – 1, then q < p$^{n/2}$ and hence we see that q^2 < p^n. Thus we obtain q^2 + q < p^n + p$^{n/2}$ < p^n + p^{n-1} + … + p, since n > 2. It follows that q ≥ p$^{n/2}$. Since n > 2 and is an even number, $\frac{n}{2}$ is a natural number larger than 1. This implies that q ≠ p$^{n/2}$ by the given condition that q is a prime. We conclude that q ≥ p$^{n/2}$ + 1. We may also write the above relation in the form p(p$^{n/2}$ – 1)(p$^{n/2}$ + 1) = (p – 1)q(q + 1). This shows that q divides (p$^{n/2}$ – 1)(p$^{n/2}$ + 1). But q ≥ p$^{n/2}$ + 1 and q is a prime. Hence the only possibility is q = p$^{n/2}$ + 1. This gives p(p$^{n/2}$ – 1) = (p – 1)(q + 1) = (p – 1)(p$^{n/2}$ + 2). Simplification leads to 3p = p$^{n/2}$ + 2. This shows that p divides 2. Thus p = 2 and hence q = 5, n = 4. It is easy to verify that these indeed satisfy the given equation.

Geometry

14. Solution

The proof relies on the fact that the angle ∠*CPD* is constant. We begin by constructing the circumcircle of triangle *PED*. Define point *G* to be the intersection of this circle with the line *AB*. Note that ∠*EGD* = ∠*EPD* as they are subtended by the same chord *ED* of the circumcircle of triangle *PED*; these angles remain constant as *P* varies on the arc *AB*. Hence, for all positions of *P*, ∠*EGD* remains fixed and, therefore, point *G* remains

fixed on the line AB. It follows that BG is constant. On the other hand, by Power of Point, we have that $AF \cdot FB = PF \cdot FD$ and $EF \cdot FG = PF \cdot FD$. Hence,

$(AE + EF) \cdot FB = EF \cdot (FB + BG)$,

And $AE \cdot FB = EF \cdot BG$. Therefore, we conclude that

$$\frac{AE \cdot BF}{EF} = BG,$$

15. Solution

We think of A and C as being two positions of the variable point traversing the circle. Then, Haruki's lemma tells us that

$$\frac{XP \cdot MQ}{XM} = \frac{MP \cdot YQ}{YM},$$

Which, because of $MP = MQ$, is simplified to

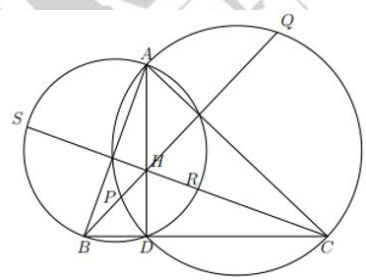
$$\frac{XP}{XM} = \frac{YQ}{YM}.$$

Adding 1 to both sides gives

$$\frac{XP + XM}{XM} = \frac{YQ + YM}{YM}$$

Applying $MP = MQ$ again, we obtain the required $XM = YM$. This completes the proof.

16. Solution



Let D be the foot of the perpendicular from A to BC, and let H be the orthocenter of ABC. Since $\angle ADB = 90°$, the circle with diameter AB passes through D, so $HS \cdot HR = HA \cdot HD$ by power of a point. Similarly, the circle with diameter AC passes through D as well, so $HP \cdot HQ = HA \cdot HD$ as well. Hence $HP \cdot HQ = HR \cdot HS$, and therefore by the converse of power of a point, P, Q, R, S are concyclic.

17. Solution

We have that $X_1B = X_2C$ and $X_1C = X_2B$ (since the segments X_1X_2 and BC share the same midpoint). Hence,

$\frac{X_2B}{X_2C} = \frac{X_1C}{X_1B}$, and similarly $\frac{Y_2C}{Y_2A} = \frac{Y_1A}{Y_1C}$ *and* $\frac{Z_2A}{Z_2B} = \frac{Z_1B}{Z_1A}$.

Therefore, $$\frac{X_2B}{X_2C} \cdot \frac{Y_2C}{Y_2A} \cdot \frac{Z_2A}{Z_2B} = \frac{X_1C}{X_1B} \cdot \frac{X_1A}{Y_1C} \cdot \frac{Z_1B}{Z_1A}$$

$$=\left(\frac{X_1B}{X_1C}\cdot\frac{Y_1C}{Y_1A}\cdot\frac{Z_1A}{Z_1B}\right)^{-1}$$

$=1,$

Where the last equality holds because of the direct implication of Ceva's Theorem, since the lines AX_1, BY_1, CZ_1 are concurrent at P. Thus, the converse of Ceva's theorem allows us to conclude that the lines AX_2, BY_2, CZ_2 are concurrent!

18. Solution

Let M be the midpoint of B_1C_1. Then, the Ratio Lemma yields that

$$=\frac{AB_1}{AC_1}=\frac{\sin C_1AM}{\sin B_1AM}=\frac{\sin C_1AG_a}{\sin B_1AG_a}$$

Doing the same for the rest and multiplying yields that

$$\frac{\sin C_1AG_a}{\sin B_1AG_a}\cdot\frac{\sin B_1CG_c}{\sin A_1CG_c}\cdot\frac{\sin A_1BG_b}{\sin C_1BG_b}=\frac{AB_1}{AC_1}\cdot\frac{A_1C}{BC_1}\cdot\frac{BC_1}{A_1B}$$

Thus, one of these is equal to one if and only if the other one is equal to one as well. By the converses of Trig Ceva and regular Ceva, we see that AA_1, BB_1, and CC_1 concur if and only if Aga, BG_a and CG_c concur as desired.

19. Solution

Let $I = BE \cap AD$. Note that I is the incenter of triangle ABC. Letting $x = \angle EBC$ and angle chasing we have that $\angle EIC = 2x$ and $\angle CID = 90° - x$ and $\angle CEK = 135° - 3x$. by quadrilateral Ceva on quadrilateral $IECD$ with interior point K we obtain

$\sin 45° \sin(90° - x) = \sin 2x \sin x(135° - 3x)$

$\Rightarrow$ $\sin 45° = 2\sin x \sin(135° - 3x) = \cos(135° - 4x) - \cos(135° - 2x)$,

which rearranges as

$\cos 135° + \cos(135° - 4x) = \cos(135° - 2x)$

$\Rightarrow$ $2\cos(135 - 2x)\cos 2x = \cos(135 - 2x)$.

Consequently, we either have $\cos(135° - 2x) =$ or $\cos 2x = \frac{1}{2}$. This means that $2x$ is equal to either 45° or 60° and so $\angle CAB = 180° - 4x$ is either 60° or 90°.

20. Solution

First, we need to prove the following lemma

Lemma. Let ABC be an acute, non-isosceles triangle with altitudes AD, BE, CF. DE cuts CF at X and DF cuts BE at Y. Denote N as the nine-point center then we have $AN \perp XY$. Proof. Denote H, O as the orthocenter and circumcenter of triangle ABC. Let O' is the reflection of O through BC then O' is the circumcenter of (BHC). Since N is the midpoint of OH, then AH, so $AHO'O$ is the parallelogram. So A, O', N are collinear.

21. Solution

Let us first prove the direct implication i.e., assume that the lines AA′, BB′, CC′ are concurrent at a point, say P. Let X be the intersection of BC and B′C′, Y the intersection of CA and C′A′ and Z the intersection of AB and

A′B′. To show that X, Y, Z are collinear, we will use Menelaus in triangle ABC. So, in other words, we want to show that

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=1$$

Hence, we need to find the ratios $\frac{XB}{XC}$ etc. Now, the points B′, C′, X are collinear, so Menelaus in triangle PBC gives us that

$$\frac{XB}{XC}\cdot\frac{C'C}{C'P}\cdot\frac{B'P}{B'B}=1$$

Similarly, Menelaus for C′, A′, Y and A′, B′, Z in triangles PCA and PAB, respectively, tells us that

$$\frac{YC}{YA}\cdot\frac{A'A}{A'P}\cdot\frac{C'P}{C'C}=1 \text{ and } \frac{ZA}{ZB}\cdot\frac{B'B}{B'P}\cdot\frac{A'P}{A'A}=1$$

Therefore, multiplying these last three equations, we get that

$$\frac{XB}{XC}\cdot\frac{C'C}{C'P}\cdot\frac{B'P}{B'B}=1$$

As desired so X, Y, Z are indeed collinear.

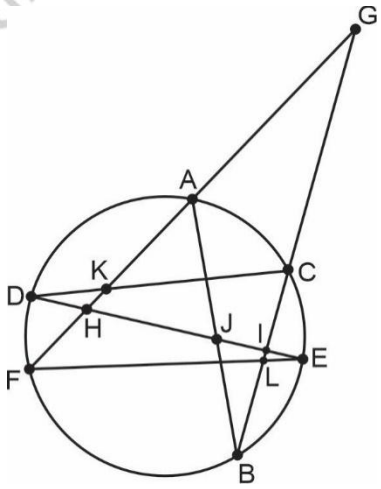
22. Solution

Let J = AB ∩ DE, L = BC ∩ EF, K = CD ∩ FA, G = BC ∩ FA, H = DE ∩ FA and I = BC ∩ DE. By Menelaus's theorem on triangle GHI with points D, K, C we find that

$$\frac{DI}{DH}\cdot\frac{CG}{CI}\cdot\frac{KH}{KG}=1$$

By Menelaus' theorem on the same triangle with points A, J, B and then with points E, L, F we obtain two similar equations and multiplying them together yields

$$\frac{KH}{KG}\cdot\frac{LG}{LI}\cdot\frac{JI}{JH}\cdot\left(\frac{ID\cdot IE}{IB\cdot IC}\right)\cdot\left(\frac{HF\cdot HA}{HD\cdot HE}\right)\cdot\left(\frac{GC\cdot GB}{GF\cdot GA}\right)=1$$



By power of point, the expressions in parentheses are each equal to 1 and now Menelaus on triangle GHI with points J, L, K yields the desired collinearity.

23. Solution

The proof is very simple. To avoid complications, we use directed angles taken modulo 180°. Denote by A, B, C, x, y, z the magnitudes of the angles ∠CAB, ∠ABC, ∠BCA, ∠YAC, ∠ZBA and ∠XCB, respectively. Since the lines AX, BX, CX are (obviously) concurrent (at X), Trig Ceva yields

$\angle ABX = \angle ABC + \angle CBX = B + y,\ \angle XBC = -\angle CBX = -y$

We now notice that

$\angle ABX = \angle ABC + \angle CBX = B + y,\ \angle XBC = -\angle CBX = -y$

$\angle BCX = -\angle XCB = -z,\ \angle XCA = \angle XCB + \angle BCA = z + C$

Hence, we get

$$\frac{\sin CAX}{\sin XAB}\cdot\frac{\sin(B+y)}{\sin(-y)}\cdot\frac{\sin(-z)}{\sin(C+z)} = 1$$

Similarly, we can find

$$\frac{\sin ABY}{\sin YBC}\cdot\frac{\sin(C+z)}{\sin(-z)}\cdot\frac{\sin(-x)}{\sin(A+x)} = 1$$

$$\frac{\sin BCZ}{\sin ZCA}\cdot\frac{\sin(A+x)}{\sin(-x)}\cdot\frac{\sin(-y)}{\sin(B+y)} = 1$$

Multiplying all these three equations and canceling similar terms, we get

$$\frac{\sin CAX}{\sin XAB}\cdot\frac{\sin ABY}{\sin YBC}\cdot\frac{\sin BCZ}{\sin ZCA} = 1$$

And using Trig Ceva once more, we find that the lines AX, BY, CZ are concurrent, which completes the proof.

24. Solution

We have that ∠YAC = ∠BAZ = 60°, ∠ZBA = ∠CBX = 60° and ∠XCB = ∠ACY = 60°; thus by Jacobi's theorem the lines AX, BY, CZ are indeed concurrent.

25. Solution

Let BCX_1X_2 be the square constructed on the side BC in the exterior of the triangle and let A_0 be the center of this square. The points A, A_1, A_0 are collinear, by homothety. Similarly, if we define B_0, C_0 to be the centers of the squares erected on the sides CA, AB which are in the exterior of ABC, we get that B, B_0, B_1 and C, C_0, C_1 are collinear. But $\angle C_0AB = \angle B_0AC = 45°$, $\angle C_0BA = \angle A_0BC = 45°$, $\angle B_0CA = \angle A_0CB = 45°$, so by Jacobi's theorem, the lines AA_0, BB_0, CC_0 are collinear, which settles the proof.

26. Solution

Consider the point at infinity A_∞ on the A-altitude of triangle ABC. We have that ∠RAB = ∠XAC = 45° and $\angle RBA = \angle A_\infty BC = 90°$ and $\angle XCA = \angle A_\infty CB = 90°$, so by Jacobi's theorem, we have that lines BX and CR intersect on the A-altitude of triangle ABC as desired.

Algebra

27. Solution

$\because a^3+b^3=(a+b)(a^2+b^2-ab)$

$\because a^2+b^2\ge 2ab \Rightarrow a^2+b^2-ab\ge ab$

$\Rightarrow (a^2+b^2-ab)(a+b)\ge ab(a+b) \Rightarrow a^3+b^3\ge ab(a+b)$

$\Rightarrow a^3+b^3+abc\ge ab(a+b)+abc$

$$\Rightarrow \frac{1}{a^3+b^3+abc}\le \frac{1}{ab(a+b+c)}$$

$$\Rightarrow \frac{1}{a^3+b^3+abc}\le \frac{c}{abc(a+b+c)}$$

Similarly $\frac{1}{b^3+c^3+abc}\le \frac{a}{abc(a+b+c)}$, $\frac{1}{c^3+a^3+abc}\le \frac{b}{abc(a+b+c)}$

$$\Rightarrow \frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le \frac{1}{abc}$$

28. Solution

Notice that because

$(a_0+a_1+....+a_{n-1})(a_1+a_2+....+a_n)=(a_0+(a_1+...+a_{n-1}))(-a_0+(a_0+a_1+...+a_n))$

$= -a_0^2+a_0((a_0+a_1+....+a_n)-(a_1+a_2+....+a_{n-1}))+(a_1+a_2+...+a_{n-1})(a_0+a_1+...+a_n)$

$= a_0a_n+(a_1+a_2+...+a_{n-1})(a_0+a_1+....+a_n)$,

We may subtract $\frac{(a_1+a_2+....+a_{n-1})(a_0+a_1+....+a_n)}{n^2}$ from both sides of the inequality and observe that it is sufficient to prove that

$$\frac{(a_1+a_2+...+a_{n-1})(a_0+a_1+....+a_n)}{(n^2-1)n^2}\ge \frac{a_0a_n}{n^2},$$

Or $(a_1+a_2+...+a_{n-1})(a_0+a_1+.....+a_n)\ge (n^2-1)a_0a_n$

Fortunately this is an easy inequality. Indeed from AM-GM applied on each group of terms we have $(a_1+a_2+...+a_{n-1})(a_0+a_1+...+a_n)\ge (n^2-1)\sqrt[n-1]{a_1a_2....a_{n-1}}\sqrt[n+1]{a_0a_1....a_n}$ and so it suffices to prove

$\sqrt[n-1]{a_1a_2....a_{n-1}}\sqrt[n+1]{a_0a_1....a_n}\ge a_0a_n$

Or after taking both sides to the (n^2-1) power simplifying and taking the nth root of both sides to prove

$a_1^2a_2^2a_3^2.....a_{n-1}^2\ge a_0^{n-1}a_n^{n-1}$

This easily follows from the Fact that $a_0a_n\le a_ia_{n-i}$ for $1\le i\le n-1$. Indeed we are given that

$a_0a_2\le a_1^2$

$a_1a_3\le a_2^2$

$a_2a_4\le a_3^2$

. . . .

$a_{n-3}a_{n-1} \le a_{n-2}^2$

$a_{n-2}a_n \le a_{n-1}^2$

Multiply equalities together and cancel $a_1, a_2^2, a_3^2, \ldots\ldots, a_{n-2}^2, a_{n-1}$ to give $a_0 a_n \le a_1 a_{n-1}$. to give $a_0 a_n \le a_1 a_{n-1}$. Similarly, by multiplying all inequalities except the first and the last, we deduce that $a_2 a_{n-2} \ge a_1 a_{n-1} \ge a_0 a_n$, and a simple induction argument proves the verity of the fact for $1 \le i \le \frac{n}{2}$ and so by the Commutative Property the fact is true for all $1 \le i \le n-1$, as desired. Now multiply each inequality of the Fact for i = 1,2,3, …., n –1 to give the desired result.

29. Solution

We prove the contrapositive: if the polynomial in question has the five real roots

x_1, x_2, x_3, x_4, x_5, then $5b \le 2a^2$

because $a = -(x_1 + x_2 + x_3 + x_4 + x_5)$ and $b = x_1x_2 + x_1x_3 + \ldots\ldots + x_4x_5$ by Vieta's formulae, we have

$$2b = 2x_1x_2 + 2x_1x_3 + \ldots. + 2x_4x_5 = \left(x_1 + x_2 + x_3 + x_4 + x_5\right)^2 - \left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\right)$$

$$= a^2 - \frac{(1+1+1+1+1)\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\right)}{5} \le a^2 - \frac{\left(x_1 + x_2 + x_3 + x_4 + x_5\right)^2}{5}$$

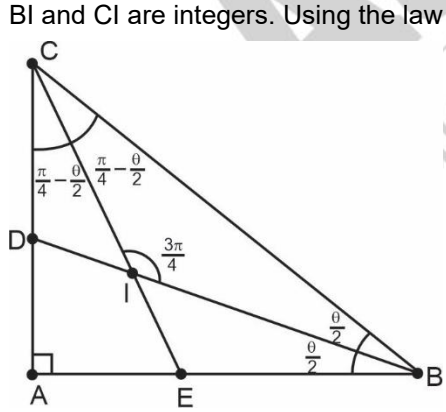
(By Cauchy-Schwarz)

$$= \frac{4a^2}{5}$$

So $5b \ge 2a^2$, as desired.

30. Solution

We know that angle BIC = 135°, as the other two angles in triangle BIC add to 45°. Assume that only AB, AC, BI and CI are integers. Using the law of cosines on triangle BIC



$BC^2 = BI^2 + CI^2 - 2BI \cdot CI \cdot \cos 135°$. Observing that $BC^2 = AB^2 + AC^2$ is an integer and that $\cos 135° = -\frac{\sqrt{2}}{2}$, we have

$$BC^2 - BI^2 - CI^2 = BI \cdot CI \cdot \sqrt{2}$$

And therefore, $\sqrt{2} = \frac{BC^2 - BI^2 - CI^2}{BI \cdot CI}$

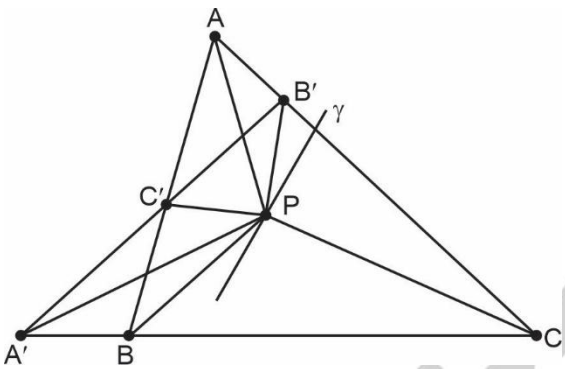
The LHS $\left(\sqrt{2}\right)$ is irrational, while the RHS is the quotient of the division of two integers and thus is rational. Clearly, there is a contradiction. Therefore, it is impossible for AB, AC, BI, and CI to all the integers, which invalidates the original claim that all six lengths are integers, and we are done.

31. Solution

By the sine law on triangle AB′P

$$\frac{AB'}{\sin\angle APB'} = \frac{AP}{\sin\angle AB'P}$$

So, $AB' = AP.\dfrac{\sin\angle APB'}{\sin\angle AB'P}$



Similarly, $B'C = CP.\dfrac{\sin\angle CPB'}{\sin\angle CB'P}$

$$CA' = CP.\frac{\sin\angle CPA'}{\sin\angle CA'P}$$

$$A'B = BP.\frac{\sin\angle BPA'}{\sin\angle BA'P}$$

$$BC' = BP.\frac{\sin\angle BPC'}{\sin\angle BC'P}$$

$$C'A = AP.\frac{\sin\angle APC'}{\sin\angle AC'P}$$

Hence, $\dfrac{AB'}{B'C}.\dfrac{CA'}{A'B}.\dfrac{BC'}{C'A} = \dfrac{\sin\angle APB'}{\sin\angle AB'P}.\dfrac{\sin\angle CB'P}{\sin\angle CPB'}.\dfrac{\sin\angle CPA'}{\sin\angle CA'P}.\dfrac{\sin\angle BA'P}{\sin\angle BPA'}.\dfrac{\sin\angle BPC'}{\sin\angle BC'P}.\dfrac{\sin\angle AC'P}{\sin\angle APC'}$

Since angles ∠AB′P and ∠CB′P are supplementary or equal depending on the position of B′ on AC,

Sin ∠AB′P = sin∠CB′P

Similarly,

sin∠CA′P = sin∠BA′P

sin∠BC′P = sin∠AC′P

By the reflective property, ∠APB′ and ∠PBA′ are supplementary or equal so

sin∠APB′ = sin∠BPA′

Similarly,

sin∠CPA′ = sin∠APC′

$\sin\angle BPC' = \sin\angle CPB'$

Therefore,

$$\frac{AB'}{B'C}.\frac{CA'}{A'B}.\frac{BC'}{C'A} = 2$$

So by Menelaus's theorem, A′, B′ and C′ are collinear.

32. Solution

In ΔABC let $\angle B = \beta$, $\angle A = 2\beta$, $\angle C = 180° - 3\beta$. From the law of sines, we have

Thus the ratio

$b : a : c = \sin\beta : \sin 2\beta : \sin 3\beta$

We can simplify $\frac{\sin 2\beta}{\sin\beta} = \frac{2\sin\beta\cos\beta}{\sin\beta} = 2\cos\beta$

Likewise, $\frac{\sin 3\beta}{\sin\beta} = \frac{\sin 2\beta\cos\beta + \sin\beta\cos 2\beta}{\sin\beta} = \frac{2\sin\beta\cos^2\beta + \sin\beta\left(\cos^2\beta - \sin^2\beta\right)}{\sin\beta}$

$= 2\cos^2\beta + \cos^2\beta - \sin^2\beta = 4\cos^2\beta - 1$

Letting $\gamma = \cos\beta$, rewrite

$b : a : c = 1 : 2\gamma : 4\gamma^2 - 1$

We find that to satisfy the conditions for an obtuse triangle, $\beta \in (0°, 30°)$ and therefore $\gamma \in \left(\frac{\sqrt{3}}{2}, 1\right)$. The rational number with minimum denominator (in order to minimize scaling to obtain integer solution) above $\frac{\sqrt{3}}{2}$ is $\frac{7}{8}$, which also has a denominator divisible by 2 (to take advantage of the coefficients of 2 and 4 in the ratio and further minimize scaling).

Inserting $\gamma = \frac{7}{8}$ into the ratio, we find $b : a : c = 1 : \frac{7}{4} : \frac{33}{16}$. When scaled minimally to obtain integer side lengths, we find b, a, c = 16, 28, 33 and that the perimeter is 77.

33. Solution

Find an arbitrary triangle ABC has isoperimetric quotient (using the notation [ABC] for area and $s = \frac{a+b+c}{2}$)

$$\frac{[ABC]}{4s^2} = \frac{[ABC]^3}{4s^2[ABC]^2} = \frac{r^3s^3}{4s^2.s(s-a)(s-b)(s-c)} = \frac{r^3}{4(s-a)(s-b)(s-c)}$$

$$= \frac{1}{4}.\frac{r}{s-a}.\frac{r}{s-b}.\frac{r}{s-c} = \frac{1}{4}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$$

Lemma. tanx tan(A – x) is increasing on $0 < x < \frac{A}{2}$, where 0 < A < 90°.

Proof, $\tan x\tan(A-x) = \tan x.\frac{\tan A - \tan x}{1+\tan A\tan x} = 1 - \frac{1}{\cos^2 x(1+\tan A\tan x)}$

$$= 1-\frac{2}{1+\cos 2x+\tan A\sin 2x}=1-\frac{2}{1+\sec A\cos(A-2x)}$$

Is increasing on the desired interval, because cos(A – 2x) is increasing on $0<x<\frac{A}{2}$. Let x_1, y_1, z_1 and x_2, y_2, z_2 be half of the angles of triangles A_1BC and A_2BC in that order, respectively. Then it is immediate that $30° > y_1 > y_2$, $30° > z_1 > z_2$ and $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 90°$. Hence, by Lemma it follows that

$$\tan x_1\tan y_1\tan z_1=\tan(90°-y_1-z_1)\tan y_1\tan z_1>\tan(90°-y_1-z_2)\tan y_1\tan z_2>\tan(90°-y_2-z_2)$$

$$\tan y_2\tan z_2=\tan x_2\tan y_2\tan z_2.$$

Multiplying this inequality by $\frac{1}{4}$ gives that I.Q[A_1BC] > I.Q[A_2BC] as desired.

34. Solution

The equation can be re-written as

$(x + 10^5)^2(x – 10^5)^2 – (x + 10^5)(x – 10^5) – x – 1 = 0$ … (i)

We first prove that the equation has no negative roots. Let $x \le 0$. The equation above can be further re-arranged as

$[(x + 10^5)(x – 10^5) + 1][(x + 10^5)(x – 10^5) – 2] = x – 1$.

The right hand side of the equation is negative. Therefore

$[(x + 10^5)(x – 10^5) + 1][(x + 10^5)(x – 10^5) – 2] < 0$, and we have $–1 < (x + 10^5)(x – 10^5) < 2$. Then the left hand side of the equation is bounded by

$|[(x + 10^5)(x – 10^5) + 1][(x + 10^5)(x – 10^5) – 2]| \le 3 \times 3$.

However since $|(x + 10^5)(x – 10^5)| \le 2$ and $x < 0$, if follows that $\left|x+10^5\right|<\frac{2}{\left|x-10^5\right|}<2\times10^{-5}$ for negative x. then $x < 2 \times 10^{-5} – 10^5$. The right hand side of the equation is then a large negative number. It cannot be equal to the left hand side which is bounded by 9.

Now let $x > 0$. When $x = 10^5$, the left hand side of equation (i) is negative. Therefore the equation has real roots on both side of 10^5 as its leading coefficient is positive. We will prove that $x = 10^5$ is a good approximation of the roots (within 10^{-2}). In fact we can solve the 'quadratic' equation (i) for $(x + 10^5)(x – 10^5)$

$$\left(x+10^5\right)\left(x-10^5\right)=\frac{1\pm\sqrt{1+4(x+1)}}{2}$$

Then, $x-10^5=\frac{1\pm\sqrt{1+4(x+1)}}{2\left(x+10^5\right)}$ easy to see that $|x – 10^5| < 1$ for positive x. Therefore, $10^5 – 1 < x < 10^5 + 1$.

Then

$$\left|x-10^5\right|=\left|\frac{1\pm\sqrt{1+4(x+1)}}{2\left(x+10^5\right)}\right|\le\left|\frac{1}{2\left(x+10^5\right)}\right|+\left|\frac{\sqrt{1+4(x+1)}}{2\left(x+10^5\right)}\right|\le\frac{1}{2\left(10^5-1+10^5\right)}+\frac{\sqrt{1+4\left(10^5+1+1\right)}}{2\left(10^5-1+10^5\right)}<10^{-2}$$

Let x_1 be a root of the equation with $x_1 < 10^5$. Then $0 < 10^5 - x_1 < 10^{-2}$ and $x_1 - 10^5 = \dfrac{1-\sqrt{1+4(x_1+1)}}{2(x_1+10^5)}$

An approximation of x_1 is defined as follows:

$$\overline{x_1} = 10^5 + \frac{1-\sqrt{1+4(10^5+1)}}{2(10^5+10^5)}$$

We check the error of the estimate :

$$\left|\overline{x_1} - x_1\right| = \left|\frac{1-\sqrt{1+4(10^5+1)}}{2(10^5+10^5)} - \frac{1-\sqrt{1+4(x_1+1)}}{2(x_1+10^5)}\right| \le \left|\frac{1}{2(10^5+10^5)} - \frac{1}{2(x_1+10^5)}\right| + \left|\frac{\sqrt{1+4(10^5+1)}}{2(10^5+10^5)} - \frac{\sqrt{1+4(x_1+1)}}{2(x_1+10^5)}\right|$$

The first absolute value

$$\left|\frac{1}{2(10^5+10^5)} - \frac{1}{2(x_1+10^5)}\right| = \frac{\left|x_1 - 10^5\right|}{2(10^5+10^5)(x_1+10^5)} < 10^{-12}$$

The second absolute value

$$\left|\frac{\sqrt{1+4(10^5+1)}}{2(10^5+10^5)} - \frac{\sqrt{1+4(x_1+1)}}{2(x_1+10^5)}\right| \le \left|\frac{\sqrt{1+4(10^5+1)}}{2(10^5+10^5)} - \frac{\sqrt{1+4(x_1+1)}}{2(10^5+10^5)}\right| + \left|\frac{\sqrt{1+4(x_1+1)}}{2(10^5+10^5)} - \frac{\sqrt{1+4(x_1+1)}}{2(x_1+10^5)}\right| \le 10^{-7} + 10^{-9}$$

Through a rationalized numerator. Therefore $\left|\overline{x_1} - x_1\right| \le 10^{-6}$

For a real root x_2 with $x_2 > 10^5$, we choose

$$\overline{x_2} = 10^5 + \frac{1+\sqrt{1+4(10^5+1)}}{2(10^5+10^5)}$$

We can similarly prove it has the desired approximation.

35. Solution

Given any binary sequence $B = (b_1, b_2, b_3, \ldots, b_k)$, define $f(B) = (|b_2 - b_1|, |b_3 - b_2|, \ldots, |b_k - b_{k-1}|)$. The operator f basically takes pairs of consecutive terms and returns 0 if the terms are the same and 1 otherwise. Note that for every sequences S of length n there exist exactly two binary sequences B of length n + 1 such that f(B) = S.

If f(B) does not contain the string 0, 1, 0, B cannot contain either of the strings 0, 0, 1, 1 or 1, 1, 0, 0. Conversely, if B does not contain the sequences 0, 0, 1, 1 or 1, 1, 0, 0, f(B) cannot contain 0, 1, 0. There are a_n such f(B) and b_{n+1} such B. Since each S corresponds with two B, there are twice as many such B as such S; thus $b_{n+1} = 2a_n$.

36. Solution

We first prove that f is odd.

Note that $f(0) = f(x^2 - x^2) = xf(x) - xf(x) = 0$ and for nonzero y, $xf(x) + yf(-y) = f(x^2 - y^2) = xf(x) - yf(y)$, or $yf(-y) = -yf(y)$, which implies $f(-y) = -f(y)$. Therefore, f is odd. Henceforth, we shall assume that all variables are non-negative.

If we let y = 0, then we obtain $f(x^2) = xf(x)$. Therefore the problems' condition becomes $f(x^2 - y^2) + f(y^2) = f(x^2)$

But for any a, b, we may set $x = \sqrt{a}$, $y = \sqrt{b}$ to obtain

$f(a - b) + f(b) = f(a)$

(It is well know that the only continuous solutions to this functional equation are fo the form f(x) = kx, but there do exist other solutions to this which are not solutions to the equation of this problem)

We may let a = 2t, b = t to obtain 2f(t) = f(2t)

Letting x = t + 1 and y = t in the original condition yields

$f(2t + 1) = (t + 1)f(t + 1) - tf(t)$

$= (t + 1)[f(t) + f(1)] - tf(t)$

$= f(t) + (t + 1) f(1)$

But we know $f(2t + 1) = f(2t) + f(1) = 2f(t) + f(1)$, so we have $2f(t) + f(1) = f(t) + tf(1) + f(1)$ or $f(t) = tf(1)$

Hence all solutions to our equation are of the form f(x) = kx. It is easy to see that real value of k will suffice.

37. Solution

Let $z_1, \ldots, z_n$ be the (not necessarily distinct) roots of P, so that

$$P(z) = \prod_{j=1}^{n}(z - z_j)$$

Since all the coefficients of P are real, it follows that if w is a root of P, then $P(\overline{w}) = \overline{P(w)} = 0$, so $\overline{w}$, the complex conjugate of w, is also a root of P.

Since $|i - z_1|\cdot|i - z_2| \ldots |i - z_n| = |P(i)| < 1$

If follows that for some (not necessarily distinct) conjugates z_i and z_j, $|z_i - i|\ |z_j - i| < 1$

Let $z_i = a + bi$ and $z_j = a - bi$, for real a, b. We note that

$(a + b + 1)^2 - (a + b - 1)^2 = 4a + 4b$

Thus, $(a^2 + b^2 + 1)^2 = (a^2 + b^2 - 1)^2 + 4a^2 + 4b^2 = |a^2 + b^2 - 1 - 2ai|^2 + 4b^2$

$= |(a - i)^2 - (bi)^2|^2 + 4b^2$

$= (|a + bi - i|\cdot|a - bi - i|)^2 + 4b^2$

$= (|z_i - i|\cdot|z_j - i|)^2 + 4b^2 < 1 + 4b^2$

Since $P(a + bi) = P(z_i) = 0$, these real numbers a, b satisfy the problems' conditions.

38. Solution

Expanding both sides,

$m^3 + mn + m^2n^2 + n^3 = m^3 - 3m^2n + 3mn^2 - n^3$

Note that m^3 can be canceled and as $n \neq 0$, n can be factored out. Writing this as a quadratic equation in n:

$2n^2 + (m^2 - 3m)n + (3m^2 + m) = 0$

The discriminant $b^2 - 4ac$ equals

$(m^2 - 3m)^2 - 8(3m^2 + m)$

$= m^4 - 6m^3 - 15m^2 - 8m$

Which we want to be a perfect square. Miraculously, this factors as $m(m-8)(m+1)^2$. This is square iff (if and only if) $m^2 - 8m$ is square or $m + 1 = 0$. It can be checked that the only nonzero m that work are –1, 8, 9. Finally plugging this in and discarding extraneous roots gives all possible ordered pairs (m, n) as

{(–1, –1), (8, –10), (9, –6), (9, –21)}

39. Solution

Let $S_k = a_1 + a_2 + \ldots + a_k$ and $b_k = \frac{S_k}{k}$. Thus, because $S_{k+1} = S_k + a_{k+1}$

$$b_{k+1} = \frac{b_k.k + a_{k+1}}{k+1} = \left(\frac{k}{k+1}\right).b_k + \frac{a_{k+1}}{k+1}$$

$\frac{k}{k+1} < 1$ and by definition, $\frac{a_{k+1}}{k+1} < 1$. Thus, $b_{k+1} < b_k + 1$. Also, both b_k and b_{k+1} are integers so, $b_{k+1} \le b_k$. As the b_k's form a non-increasing sequence of positive integers, they must eventually become constant. Therefore, $b_k = b_{k+1}$ for some sufficiently large value of k. Then $a_{k+1} = S_{k+1} - S_k = b_k(k+1) - b_k(k) = b_k$, so eventually the sequence a_k becomes constant.

Combinatorics

40. Solution

The set $\{4,5,9,14,23,37,60,97,157,254\}$ is a ten-element subset of $\{4,5,6,\ldots,254\}$ that does not have the triangle property. Let N be the smallest integer for which $\{4,5,6,\ldots,N\}$ has a ten-element subset that lacks the triangle property. Let $\{a_1, a_2, a_3, \ldots, a_{10}\}$ be such a subset, with $a_1 < a_2 < a_3 < \cdots < a_{10}$. Because none of its three-element subsets define triangles, the following must be true:

$$\begin{aligned} N &\ge a_{10} \ge a_9 + a_8 \ge (a_8 + a_7) + a_8 \\ &= 2a_8 + a_7 \ge 2(a_7 + a_6) + a_7 = 3a_7 + 2a_6 \\ &\ge 3(a_6 + a_5) + 2a_6 = 5a_6 + 3a_5 \ge 8a_5 + 5a_4 \\ &\ge 13a_4 + 8a_3 \ge 21a_3 + 13a_2 \ge 34a_2 + 21a_1 \\ &\ge 34 \cdot 5 + 21 \cdot 4 = 254 \end{aligned}$$

Thus, the largest possible value of n is $N - 1 = 253$. This is yet another application of the Fibonacci sequence.

41. Solution

It is always possible to put all the candy bars into one box. We will prove our statement by induction on m, the number of candy bars.

For the base case $m = 4$, there are at most 4 nonempty boxes. We disregard all the other empty boxes and consider all the possible initial distributions:

(1) $(1,1,1,1)$

(2) $(1,2,1,0)$

(3) $(2,2,0,0)$

(4) $(1,3,0,0)$

For distribution (1), we proceed as follows:

$$(1,1,1,1) \to (3,1,0,0) \to (2,0,2,0) \to (1,0,1,2) \to (0,0,0,4).$$

It is easy to see that all the other initial distributions are covered in the above sequence of operations. Thus the base case is proved

Now we assume that the statement is true for some positive integer $m \geq 4$. If we are given $m+1$ candy bars, we mark one of them and called it special. We first ignore the special candy bar and consider only the other m candy bars. By the induction hypothesis, we can put all m candy bars into one box. If this box also contains the special piece, we are done. If not, we pick two empty boxes and proceed as follows:

$$(1,m,0,0) \to (0,m-1,2,0) \to (0,m-2,1,2)$$
$$\to (2,m-3,0,2) \to (1,m-1,0,1) \to (0,m+1,0,0).$$

Now all the candy bars are in one box and our induction is complete.

42. Solution

We first find the sum of the numbers of anchors of all subsets S; this is equivalent to finding, for each n, the number of sets for which n is an anchor, and then summing over all n. Suppose that n is an anchor for S, and S has k elements. Then $n, n+k \in S \Rightarrow k \geq 2$, and also $n+k \leq 15$, or $k \leq 15-n$. The remaining $k-2$ elements of S (other than n and $n+k$) may be freely chosen from the remaining 13 members of $\{1,2,\ldots,15\}$, so we get $\binom{13}{k-2}$ possible sets S. Summing over all allowed values of k, we then have $\binom{13}{0}+\binom{13}{1}+\binom{13}{2}+\cdots+\binom{13}{13-n}$ sets with n as an anchor. If we sum over all $n=1,2,\ldots,13$ (since there are no possible values of k when $n>13$), we get a total of

$$13\binom{13}{0}+12\binom{13}{1}+11\binom{13}{2}+\ldots+\binom{13}{12}$$

If we call this quantity A, then, by symmetry, $2A$ equals

$$13\binom{13}{0}+12\binom{13}{1}+11\binom{13}{2}+\ldots+\binom{13}{12}+\binom{13}{1}+2\binom{13}{2}+\ldots+12\binom{13}{12}+13\binom{13}{13}$$

$$=13\left[\binom{13}{0}+\binom{13}{1}+\binom{13}{2}+\ldots+\binom{13}{12}+\binom{13}{13}\right]=13.2^{13}$$

So $A=13\cdot 2^{12}$ is the total number of anchors over all possible sets S. Finally, to find the average number of anchors, we divide by the number of sets, which is 2^{15}; thus, the answer is $13\cdot 2^{12}/2^{15}=13/8$.

43. Solution

Let A = {a_1, a_2, … a_n} be a group of n people and for each i = 1, 2, … n. Let us denote the number of people that a_i knows by f(i). Then it is clear that $0 \leq f(i) \leq n-1$ for all i = 1, 2, … n. We consider the following cases.

Case-1 : There exist distinct i, j $\in$ {1, 2, …, n} such that f(i) = f(j) = 0. The result holds trivially in this case.

Case-2 : There is exactly one $i \in \{1, 2, \ldots, n\}$ such that $f(i) = 0$. In this case, we have $1 \le f(j) \le n - 2$ for all $j \in \{1, 2, \ldots, n\}\backslash\{i\}$. By (PP), we have distinct $j, k \in \{1, 2, \ldots n\}\backslash\{i\}$ such that $f(j) = f(k)$.

Case-3 : We have $f(i) > 0$ for all $I = 1, 2, \ldots, n$. In this case, we have $1 \le f(i) \le n - 1$ for all $i = 1, 2, \ldots n$ and the (PP) shows that there are distinct $i, j \in \{1, 2, \ldots, n\}$ such that $f(i) = f(j)$. This completes the proof.

44. Solution

Let f(n) be the number of ways of doing so i.e., left side is always heavy. Now, the key observation is that smaller weights can never add up to the weight of a larger weight, i.e. which side is heavier is determined completely by the heaviest weight currently placed. If follows, therefore, that the number of ways to place n weights on the balance according to the rule is the same no matter which n distinct powers of two are the weights, as each weight completely overpowers any smaller weight and is completely overpowered by any larger weight.

Assume $n \ge 1$. Suppose that weight 2^{n-1} is placed on the balance in the ith move with $1 \le i \le n$. This weight has to be put on the left pan.

For the previous moves we have $\binom{n-1}{i-1}$ choices of the weights and from comment 2 there are $f(i - 1)$ valid ways of placing them on the balance. For later moves there is no restriction on the way in which the weights are to be put on the pans. Therefore, all $(n - i)!2^{n-i}$ ways are possible. This gives

$$f(n) = \sum_{i=1}^{n-1}\binom{n-1}{i-1} f(i-1)(n-i)!2^{n-i} = \sum_{i=1}^{n}\frac{(n-1)!f(i-1)2^{n-i}}{(i-1)!} \quad \ldots \text{(ii)}$$

Now we are ready to prove (i). Using n – 1 instead of n in (ii) we get

$$f(n-1) = \sum_{i=1}^{n-1}\frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}$$

Hence, again from (ii) we get

$$f(n) = 2(n-1)\sum_{i=1}^{n-1}\frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!} + f(n-1)$$

Therefore

$F(n) = 1\cdot3\cdot5\cdot \ldots \cdot (2n - 1)$

45. Solution

Denote by n the number of exams. We have $n(A + B + C) = 20 + 10 + 9 = 39$ and since A, B, C are distinct, their sum is at least 6: therefore $n = 3$ and $A + B + C = 13$. Assume w.l.o.g. that $A > B > C$. Since Betty gained A points in arithmetic, but fewer than 13 points in total, she had C points in both remaining exams (in spelling as well). Furthermore, Carol also gained fewer than 13 points, but with at least B points on two examinations (on which Betty scored C), including spelling. If she had A in spelling, then she would have at least $A + B + C = 13$ points in total, a contradiction. Hence, Carol scored B and placed second in spelling.

Remarks : Moreover, it follows that Alice, Betty and Carol scored $B + A + A$, $A + C + C$ and $C + B + B$ respectively and that $A = 8$, $B = 4$, $C = 1$.

46. Solution

We set

$$x=\sum_{k=0}^{n}\binom{2n+1}{2k+1}2^{3k}=\frac{1}{\sqrt{8}}\sum_{k=0}^{n}\binom{2n+1}{2k+1}\sqrt{8}^{2k+1}$$

$$y=\sum_{k=0}^{n}\binom{2n+1}{2k}2^{3k}=\sum_{k=0}^{n}\binom{2n+1}{2k}\sqrt{8}^{2k}$$

Both x and y are positive integers. Also, from the binomial formula we obtain

$$y+x\sqrt{8}=\sum_{i=0}^{2n+1}\binom{2n+1}{i}\sqrt{8}^{i}=\left(1+\sqrt{8}\right)2n+1$$

And similarly $y-x\sqrt{8}=\left(1-\sqrt{8}\right)^{2n+1}$

Multiplying these equalities, we get $y^2-8x^2=\left(1+\sqrt{8}\right)^{2n+1}\left(1-\sqrt{8}\right)^{2n+1}=-7^{2n+1}$. Reducing modulo 5 gives us $3x^2 - y^2 \equiv 2^{n+1} \equiv 2\cdot(-1)^n$

Now, we see that if x is divisible by 5, then $y^2 \equiv \pm 2 \pmod 5$, which is impossible. Therefore, x is never divisible by 5.

47. Solution

First Solution :

For the sake of contradiction we assume that there is a seating arrangement such that there is no one sitting in between two girls. We call a block any group of girls (boys) sitting next to each other and sandwiched by boys(girls) from both sides. By our assumption, each girl block has at most 2 girls and there are at least 2 boys in the gap between two consecutive girl blocks. Hence there are at least $\left[\frac{25}{2}\right]=13$ girls blocks and at least 2 × 13 boys sitting in between the 13 gaps between girls blocks. But we only have 25 boys a contradiction. Therefore, our assumption was wrong and it is always possible to find someone sitting between two girls.

Second Solution :

We again approach indirectly by assuming that there is a seating arrangement such that no one is sitting in between two girls. We further assume they are sitting is positions $a_1, a_2, \ldots, a_{50}$ in a counter clockwise order (so a_{50} is next to a_1). Now we split them into two tables with seating orders ($a_1, a_3, a_5, \ldots, a_{49}$) and ($a_2, a_4, a_6, \ldots, a_{50}$), each in counterclockwise order. Then by our assumption, no girls are next to each other in the resulting two-seating arrangements. So there are at most 12 girls sitting around each new table for a total of at most 24 girls, a contradiction. Therefore, our assumption was wrong and it is always possible to find someone sitting in between two girls.

48. Solution

Firstly, we note that $\frac{\pi}{6}=\frac{\sqrt{3}}{3}$. Let us show that $\alpha=\tan\frac{\pi}{12}=2-\sqrt{3}$. To this end, we first note from the tangent double angle formula that $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$ for all real numbers x. This implies that

$$\frac{\sqrt{3}}{3} = \tan\frac{\pi}{6} = \frac{2\alpha}{1-\alpha^2}$$

Or equivalently, $\alpha^2 + 2\sqrt{3}\alpha - 1 = 0$. Solving for the roots of this quadratic equation yields $\alpha = 2-\sqrt{3}$ or $\alpha = -2-\sqrt{3}$. As $\alpha > 0$, we must $\alpha = 2-\sqrt{3}$ as desired.

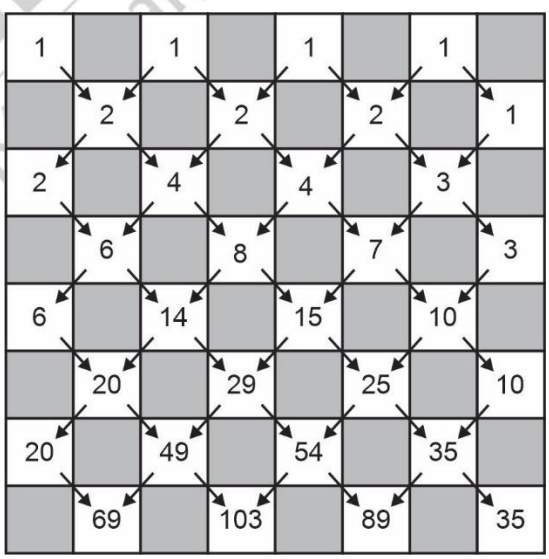
Now, for each $\alpha \in A$, let θ_a denote the unique real number in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that satisfy $\tan\theta_a = a$ (note that such a θ_a exists since the tangent function is strictly increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) = (-\infty, \infty)$. Then it is clear that for each $a \in A$, θ_a is contained in $\left[\frac{n\pi}{12}, \frac{(n+1)\pi}{12}\right)$ for some $n \in \{-6, -5, \ldots, 4, 5\}$. By (PP), there must exist some $m \in \{-6, -5, \ldots, 4, 5\}$ and two (distinct) elements of A, say x and y, such that $\left[\frac{n\pi}{12}, \frac{(m+1)\pi}{12}\right)$ contains θ_x and θ_y. Without loss of generality, let us assume that $\theta_x - \theta_y > 0$. Then we must have $\theta_x - \theta_y < \frac{(m+1)\pi}{12} - \frac{m\pi}{12} = \frac{\pi}{12}$. Hence, we have

$$0 < \frac{x-y}{1+xy} = \frac{\tan\theta_x - \tan\theta_y}{1+\tan\theta_x\tan\theta_y} = \tan(\theta_x - \theta_y) < \tan\frac{\pi}{12} = 2-\sqrt{3}$$

As desired. (here, we have used the tangent angle sum formula in the above inequality).

49. Solution

We can label each white square with the number of (distinct) paths that reach the square from the top. As a result, the number of paths to each while square is the sum of the number of paths to the two white squares above it, or equal to the number of paths to the only white square above it.



Thus there are a total of 296 zig-zag paths.

50. Solution

Let N be a number that can be written as a sum of 1990 consecutive integers and as a sum of consecutive positive integers in exactly 1990 ways. The former requirement gives us N = m + (m + 1) + … + (m + 1989) = 995(2m + 1989) for some m. Thus $2 \nmid N$, $5 \mid N$ and 199 | N. The latter requirement tells us that there are exactly 1990 ways to express N as n + (n + 1) + … + (n + k), or equivalently, express 2N as (k + 1) (2n + k). Since N is odd, it follows that one of the factors k + 1 and 2n + k is odd and the other is divisible by 2, but not by 4. Evidently k + 1 < 2n + k. On the other hand, every factorization 2N = ab, 1 < a < b, corresponds to a single pair (n, k), where $n = \frac{b-a+1}{2}$ (which is an integer) and k = a – 1. The number of such factorizations is equal to $\frac{d(2N)}{2-1}$ because a = b is impossible (here d(x) denotes the number of positive divisors of an x $\in$ N). Hence we must have d(2N) = 2·1991 = 3982. Now let $2N = 2 \cdot 5^{e_1} \cdot 199^{e_2} \cdot p_3^{e_3} \ldots p_r^{e_r}$ be a factorization of 2N into prime numbers, where $p_3, \ldots p_r$ are distinct primes other than 2, 5 and 199 and $e_1, \ldots, e_r$ are positive integers. Then $d(2N) = 2(e_1 + 1)(e_2 + 1) \ldots (e_r + 1)$, from which we deduce $(e_1 + 1)(e_2 + 1) \ldots (e_r + 1) = 1991 = 11 \cdot 181$. We thus get $\{e_1, e_2\} = \{10, 180\}$ and $e_3 = \ldots = e_r = 0$. Hence, $N = 5^{10} \cdot 199^{180}$ and $N = 5^{180} \cdot 199^{10}$ are the only possible solutions. There numbers indeed satisfy the desired properties.

51. Solution

For each n let F_n be the number of permutations of {1, 2, …, n} with the required property : call them nice. For n = 1, 2, 3 every permutation is nice, so $F_1 = 1$. $F_2 = 2$, $F_3 = 6$. Take an n > 3 and consider any nice permutation $(a_1, a_2, \ldots, a_n)$ of {1, 2, …, n}. Then n – 1 must be a divisor of the number

$2(a_1 + a_2 + \ldots + a_{n-1}) = 2((1 + 2 + \ldots + n) - a_n) = n(n + 1) - 2a_n = (n + 2)(n - 1) + (2 - 2a_n)$

So, $2a_n - 2$ must be divisible by n – 1, hence equal to 0 or n – 1 or 2n – 2. This means that $a_n = 1$ or $a_n = \frac{n+1}{2}$ or $a_n = n$.

Suppose that $a_n = \frac{(n+1)}{2}$. Since the permutation is nice, taking k = n – 2 we get that n – 2 has to be a divisor of $2(a_1 + a_2 + \ldots + a_{n-2}) = 2((1 + 2 + \ldots + n) - a_n - a_{n-1}) = n(n + 1) - (n + 1) - 2a_{n-1} = (n + 2)(n - 2) + (3 - 2a_{n-1})$.

So $2a_{n-1} - 3$ should be divisible by n – 2, hence equal to 0 or n – 2 or 2n – 4. Obviously 0 and 2n – 4 are excluded because $2a_{n-1} - 3$ is odd. The remaining possibility ($2a_{n-1} - 3 = n - 2$) lead to $a_{n-1} = (n + 1)/2 = a_n$, which also cannot hold. This eliminates $\frac{(n+1)}{2}$ as a possible value of a_n. Consequently $a_n = 1$ or $a_n = n$.

If $a_n = n$ then $(a_1, a_2, \ldots, a_{n-1})$ is a permutation of {1, 2, …, n – 1}. There are F_{n-1} such permutations. Attaching n to any one of them at the end creates a nice permutation of {1, 2, …, n}.

If $a_n = 1$ then $(a_1 - 1, a_2 - 1, \ldots, a_{n-1} - 1)$ is a permutation of {1, 2, …, n – 1.} It is also nice because the number $2((a_1 - 1) + \ldots + (a_k - 1)) = 2(a_1 + \ldots + a_k) - 2k$ is divisible by k, for any $k \le n - 1$. And again, any one of the F_{n-1} nice permutations $(b_1, b_2, \ldots, b_{n-1})$ of {1, 2, …, n – 1} gives rise to a nice permutation of {1, 2, …, n} whose last term is 1, namely $(b_1 + 1, b_2 + 1, \ldots, b_{n-1} + 1)$.

The bijective correspondences established in both cases show that there are F_{n-1} nice permutations of {1, 2, …, n} with the last term 1 and also F_{n-1} nice permutations of {1, 2, …, n} with the last term n. Hence follows the recurrence $F_n = 2F_{n-1}$. With the base value $F_3 = 6$ this gives the outcome formula $F_n = 3\cdot 2^{n-2}$ for $n \geq 3$.

52. Solution

Consider any of the 5 given points. The number of perpendiculars from the chosen point to the lines joining any 2 of the other 4 points is equal to $\binom{4}{2} = 6$. Hence, the total number of perpendiculars is equal to 5 × 6 = 30. Also, since no 2 distinct lines joining any two of the 5 given points are parallel. It follows that there is at most one point of intersection between any 2 distinct perpendiculars. This implies that the maximum number of intersection points between any of the 2 perpendicular is equal to $\binom{30}{2} = 435$.

Firstly, fix any 2 of the 5 given points and consider the line ℓ joining them. As no lines joining the 5 points are perpendicular, we see that the 5 – 2 = 3 perpendiculars to ℓ do not intersect. As there are $\binom{5}{2} = 10$ possibilities for ℓ and hence a total of 10·3 = 30 such perpendiculars, we see that this reduces the maximum number of intersection points between any of the 2 perpendicular to 435 – 30 = 405.

Secondly, let us consider the perpendiculars from a fixed point chosen from the 5 given points to the lines joining any two of the other 4 points. We note that there $\binom{4}{2} = 6$ such lines. Since these perpendiculars only intersect at this fixed point and this point of intersection has been counted $\binom{6}{2} = 15$ times in the counting of the number of intersection points, we see that each intersection point has been counted an extra 15 – 1 = 14 times. Since there are 5 such intersection points, we see that this reduces the maximum number of intersection points between any of the 2 perpendiculars to 405 – 14·5 = 335.

Finally, since no 2 of the lines joining any two points of the 5 given points are coincident, we see that any 3 of the 5 given points determines a non-degenerate triangle and hence a point of intersection, which is the intersection of the altitudes of the triangle. As this point of intersection is counted $\binom{3}{2} = 3$ times in the counting of the number of intersection points. We see that each intersection point has been counted an extra 3 – 1 = 2 maximum number of intersection points between any of the 2 perpendiculars to 335 – 2·10 = 315.

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