Aakash

Medical | IIT-JEE | Foundations

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Previous Year RMO Questions

Solution

1. Solution

    $a^2 + b^2 + c = d^2$

    Let a, b, c are odd $\Rightarrow$ d is odd

    a^2, b^2, $c^2 \rightarrow 4k + 1$

    $a^2 + b^2 + c^2 \rightarrow 4k + 3$

    But d^2 can't be of form $4k + 3$

    Let any one is even without loss of generality lets assume a is even

    $a^2 + b^2 + c^2 \rightarrow$ even $\Rightarrow$ d is even

    $\Rightarrow$ $abcd$ is divisible by 4.

    Let any two numbers be even

    $\Rightarrow$ $abcd$ is divisible by 4

    $\Rightarrow$ $abcd$ must be divisible by 4

    Now, let a, b, $c \in \{3k + 1, 3k + 2\}$

    a^2, b^2, $c^2 \in \{3k + 1\}$

    $a^2 + b^2 + c^2 \in 3m$

    $\Rightarrow$ $abcd$ is divisible by 3

    $\Rightarrow$ Largest positive integer = 12

    For example (a, b, c, d) = (1, 2, 2, 3), (2, 3, 6, 7), …

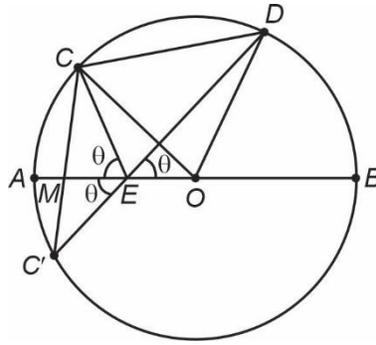   But all should be divided then largest positive integer is 12.

2. Solution

    Let $\angle CEA = \angle DEB = \theta$ (given)

    $\Rightarrow$ $\angle CED = 180° - 2\theta$

    Extend CE so that it meets the circle at point C'

    $\Rightarrow$ $\angle MEC' = \theta$

Therefore, C and C' are mirror image of each other

$\Rightarrow \quad \angle EMC' = 90°$

$\Rightarrow \quad \angle MC'E = 90° - \theta$

Let O be the center of the circle, the angle made by chord CD at the centre is $2(90° - \theta) = 180° - 2\theta$

$\angle CED = \angle COD = 180° - 2\theta$

Therefore $CEOD$ are concyclic

Now in $\triangle COD$

$OD = OC = r$

$CD = 2r\sin(90° - \theta)$

$\sin(90° - \theta) = \dfrac{l}{2r}$

$\Rightarrow \quad 90° - \theta$ is constant as l and r both are constant

$\Rightarrow \quad \theta$ is constant, $\angle CED = 180° - 2\theta$ is also constant

Now as we know that $CEOD$ are concyclic. The circumcircle of $\triangle CED$ will always pass through point O which is a fixed point.

3. Solution

$(S(n))^2 = m$ and $(S(m))^2 = n$

$m < n$

$\Rightarrow \quad$ m and n are perfect square

Claim

"n and m cannot be more than 3 digit number"

If n is 4 digit number

$(S(n))_{max} = 36$

$m_{max} = 1296$

$\Rightarrow \quad m \leq 1296$

$(S(m))_{max} = 19$ when $m = 676$

$\Rightarrow \quad n \leq 361$

But this contradicts the fact that n is 4 digit number

Now lets consider n to be a 3 digit number

$(S(n))_{max}$ = 19 when m = 676

$\Rightarrow$ $m \leq 361$

Now checking

|	10	11	12	**13**	14	15	**16**	17	18	19
m	100	121	144	**169**	196	225	**256**	289	324	361
$n = \left(S(m)\right)^2$	1	4	81	**256**	256	81	**169**	361	81	100
$\left(S(n)\right)^2$	1	16	81	**169**	169	81	**256**	100	81	1

But $m < n$

$\Rightarrow$ m = 169, n = 256

Therefore, only possible pair is (169, 256) $\equiv$ (m, n)

4. Solution

Let the midpoint of PR, QS and O_1O_2 be X, Y and Z respectively foot of perpendiculars from points O_1 and O_2 onto the line $ABCD$ be k_1 and k_2 respectively. Now the line PQ and RS will be parallel as both are perpendicular to the line AD, therefore $PQRS$ is a trapezium let L and M be the intersections of PQ and RS with AD as X and Y are midpoints, so XY will bisect LM by equal intercept theorem.
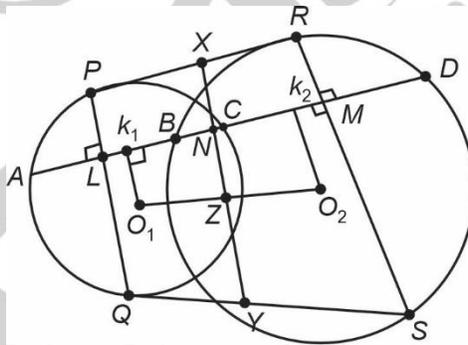
Now $O_1k_1k_2O_2$ will be a trapezium and let line XY meets k_1k_2 at N and O_1O_2 at Z.

To prove

$k_1N = Nk_2$

$LN - k_1L = MN - k_2M$ (as N is the mid-point of LM)

$k_1L = k_2M$

Let $AB = 2\alpha$, $BC = 2\beta$, $CD = 2\gamma$

$AL = \dfrac{AB}{2} = \alpha$

$Ak_1 = \dfrac{AC}{2} = \alpha + \beta$

$k_1L = Ak_1 - AL = \beta$

$DM = \dfrac{CD}{2} = \gamma$

$$Dk_2 = \frac{DB}{2} = \gamma + \beta$$

$k_2M = Dk_2 - DM = \beta$

$k_1L = k_2M = \beta$ (hence proved)

As N is the mid-point of k_1k_2, therefore Z will be mid-point of O_1O_2 by equal intercept theorem. So the points X, Y and Z will be collinear.

5. Solution

$n > k > 1$ and n, $k \in I^+$ and $a_i \in$ positive real numbers

$$\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n} a_i = n$$

By AM-GM inequality

$$\frac{\underbrace{a_i^k + a_i^k + a_i^k + ... + a_i^k}_{k-1} + 1}{k} \geq \left(\underbrace{a_i^k . a_i^k ... a_i^k}_{k-1} . 1\right)^{\frac{1}{k}}$$

$$\Rightarrow \quad (k-1)a_i^k + 1 \geq k\left(a_i\right)^{k-1}$$

Now

$$\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq \sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{ka_i^{k-1}}}$$

$$\leq \sum_{i=1}^{n} \sqrt{a_i}$$

$$\Rightarrow \quad n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \sqrt{a_i}$$

Now applying AM $\leq$ RMS

$$\frac{\sum_{i=1}^{n} \sqrt{a_i}}{n} \leq \sqrt{\frac{\sum_{i=1}^{n} a_i}{n}} = 1$$

$$\Rightarrow \quad \sum_{i=1}^{n} \sqrt{a_i} \leq n$$

$$\Rightarrow \quad \sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq n$$

But given that $\displaystyle\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = n$

From inequality we will get

$$\Rightarrow \quad a_i = 1$$

6. Solution

Consider 4 × 4 grid with unit distance then

Case-I: Only 2 rows are used

All 7 points are in any two rows, then it is easy to find an isosceles right angled triangle as shown, in figure 6.1.

Here $A_1C_1A_3$ is such triangle. $A_1A_3C_3$ is another such pairs of point to form isosceles right angled triangle with dimension $\left(2, 2, 2\sqrt{2}\right)$
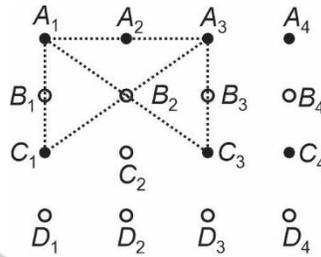

Figure 6.1

Case-II: All four rows are used

By Pigeonhole principle; at least three rows contain 2 points and 7th point can be in the remaining row.

Let's take a symmetric case; figure 6.2

Now, take point at D_4 then $A_1A_4D_4$ makes the required triangle with dimension $\left(4, 4, 4\sqrt{2}\right)$.

Similarly For $D_3 \to B_3C_4D_3$ will form such triangle with dimension $\left(\sqrt{2}, \sqrt{2}, 2\right)$

For $D_2 \to B_2C_1D_2$ forms required triangle with dimension $\left(\sqrt{2}, \sqrt{2}, 2\right)$

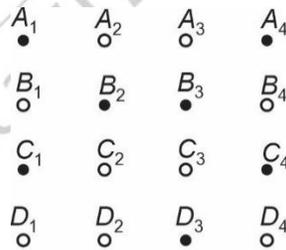For $D_1 \to D_1A_1A_4$ forms required triangle with dimension $\left(4, 4, 4\sqrt{2}\right)$

Figure 6.2

Let's take an asymmetric case of distribution of points as shown in figure 6.3

For point at D_1, $D_1A_1A_4$ forms isosceles right angled triangle with dimension $\left(4, 4, 4\sqrt{2}\right)$

For $D_2 \to D_2C_1B_2$ forms required triangle $\left(\sqrt{2}, \sqrt{2}, 2\right)$

For $D_3 \to D_3B_2A_4$ forms required triangle with dimension $\left(\sqrt{5}, \sqrt{5}, \sqrt{10}\right)$

For $D_4 \to D_4A_4A_1$ forms required triangle with dimension $\left(4, 4, 4\sqrt{2}\right)$

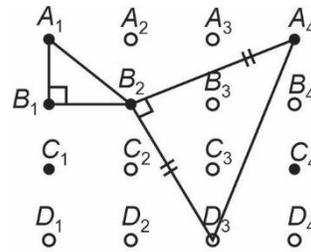Or easily $A_1B_1B_2$ forms such right angled isosceles triangle.

Figure 6.3

Case-III:

When only three rows are used

$\Rightarrow$ Again by Piegonhole principle each row will contain at least two points

$\Rightarrow$ Lets consider such colouring

For $D_3 \to D_3D_1B_1$ forms such triangle.

For $D_4 \to D_2D4B_4$ forms such required triangle.

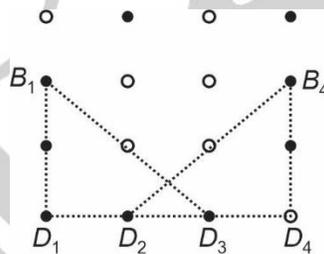Hence, an isosceles right angled triangle must exist in each case.

Figure 6.4

7.   Solution

(a)

Let n be $2k + 1$ for some $k > 1$, $k \in Z$

Then for an arrangement to be nice

$n \nmid \left(a_1 + a_2 + ... + a_n\right)$

But $a_1 + ... + a_n = \dfrac{n(n+1)}{2} = \dfrac{(2k+1)(2k+2)}{2} = (k+1)(2k+1) = (k+1)n$

$\Rightarrow$ $n \mid n(k+1)$ $\Rightarrow$ $n \left| \left(\displaystyle\sum_{i=1}^{n} a_i\right)\right.$

Therefore, no nice rearrangement exists for $n \in$ odd

(b)

Let n be $2k$ for some $k \geq 1$, $k \in z$,

Then lets see through some examples

For $n = 2$

$\Rightarrow$ arrangement of {1, 2} such that $2 \big| \Pi (\text{arrangement})$

$\Rightarrow$ 1 at unit place

$\Rightarrow$ {2, 1} is such rearrangement

For $n = 4$

arrangement of {1, 2, 3, 4}

such that $2 \mid (a_1 + a_2) \Rightarrow \{2, 1\}$

$3 \mid (a_1 + a_2 + a_3) \Rightarrow \{2, 1, 4\}$

And $4 \mid (a_1 + a_2 + a_3 + a_4) \rightarrow$ this clearly satisfies as $4 \mid 10$

$\Rightarrow$ (2, 1, 4, 3) is such rearrangement

$\Rightarrow$ 2, 1, 4, 3, 6, 5 ... will be the nice arrangement

$\{a_i\} \Rightarrow a_i = \begin{cases} i + 1, i \in \text{odd} \\ i - 1, i \in \text{even} \end{cases}$

$\Rightarrow \sum\limits_{i=1}^{k} (a_i)$ should not be divisible by k then

It would lead to nice rearrangement

For even n,

Let, $k = 2m + 1$, where $m \in I^+$

$\sum\limits_{i=1}^{k} a_k = \sum\limits_{i=1}^{2m+1} a_i = (a_1 + a_2 + ... a_{2m} + a_{2m+1})$

$= (2 + 1) + (4 + 3) + ... + (2m + 2m - 1) + (2m + 2)$

$= (1 + 2 + ... + 2m) + 2(m + 1)$

$= \dfrac{(2m)(2m + 1)}{2} + 2(m + 1)$

$= 2m^2 + m + 2m + 2 = (2m^2 + 3m + 1) + 1$

$= (2m + 1)(m + 1) + 1$

Clearly, $(2m + 1) \mid (2m + 1)(m + 1) + 1, m \in I^+ \Rightarrow k \mid \sum\limits_{i=1}^{k} a_i$

Now let $k = 2m$, where $m \in I^+$

$\sum\limits_{i=1}^{k} = \sum\limits_{i=1}^{2m} a_i = (a_1 + a_2) + (a_3 + a_4) + ... + (a_{2m-1} + a_{2m})$

$= 1 + 2 + ... + (2m - 1) + (2m)$

$= \dfrac{(2m)(2m + 1)}{2} = n(2m + 1)$

$2n \mid 2m^2$ but $2m \mid m$ for $n \in I^+$

$\Rightarrow 2m \mid (2m^2 + m) = m(2m + 1)$

$\Rightarrow$ $\{a_i\}$ leads to nice rearrangement.

8.   Solution

For observation

$R(1) = 0$

$R(2) = 0 + 0 = 0$

$R(3) = 0 + 1 + 0 = 1$

$R(4) = 0 + 0 + 1 + 0 = 1$

$R(5) = 0 + 1 + 2 + 1 + 0 = 4$

$R(6) = 0 + 0 + 0 + 2 + 1 + 0 = 3$

$R(7) = 0 + 1 + 1 + 3 + 2 + 1 + 0 = 8$

$R(8) = 0 + 0 + 2 + 0 + 3 + 2 + 1 + 0 = 8$

$R(9) = 0 + 1 + 0 + 1 + 4 + 3 + 2 + 1 + 0 = 12$

So, for R(even) and R(odd) there is a pattern for second half. Now assuming

Case-I:

$n \in$ even

$$R(n) \geq 1 + 2 + ... + \left(\frac{n}{2} - 1\right) = \frac{n(n-2)}{8}$$

$$n - 1 \geq \frac{n(n-2)}{8}$$

$\Rightarrow$ $n^2 - 10n + 8 \leq 0$

$\Rightarrow$ $n \in (0, 9]$

But we can see that for $n = 2, 4, 6, 8$ will not satisfy the given relation.

Case-II:

$n \in$ odd

$$R(n) \geq 0 + 1 + 2 + 3 + ... + \left(\frac{n-1}{2}\right) = \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2} + 1\right)}{2} = \frac{n^2 - 1}{8}$$

$$n - 1 \geq \frac{n^2 - 1}{8}$$

$n^2 - 8n + 7 \leq 0$

$(n - 1)(n - 7) \leq 0$
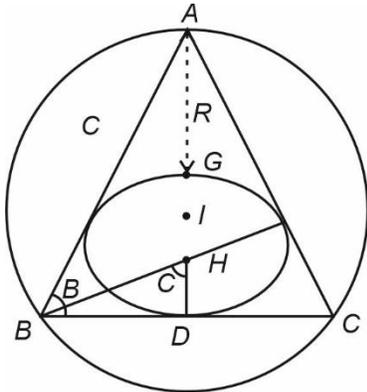
$n \in [1, 7]$

$\Rightarrow$ $n \in 1, 3, 5, 7$.

$\therefore$ Only $n = 1$ and 5 satisfies.

$\therefore$ Positive set of integers $\{1, 5\}$

$\therefore$ Only two integral values satisfies the given statement.

9.   Solution

As $\triangle ABC$ be acute isosceles $\triangle$

$\Rightarrow$ O, G, I and H will lie on same line, where I is Intentre

Let $AD = x$

$\Rightarrow$ $GD = \dfrac{x}{3} = 2GI$ (as GD is diameter)

$\qquad\qquad = 2ID$

$BD = C\sin B$

$HD = \dfrac{C\sin B}{\tan C} = 2R\sin^2 B \cdot \left(AB = AC \Rightarrow \angle B = \angle C\right)$

$OD = R\cos A$,

As $2OD = 23HD$ (Given)

$\Rightarrow$ $2R\cos A = (23)(2R\sin^2 B)$

$\Rightarrow$ $2\cos A = 23(1 - \cos^2 B)$, Also $\angle A = 180° - \angle B - \angle C$

$\Rightarrow$ $\angle A = 180° - 2\angle B$ $\left(\angle B = \angle C\right)$

$\Rightarrow$ $\cos A = \dfrac{23}{25}$, $\sin A = \dfrac{4\sqrt{6}}{25}$

$\cos B = \dfrac{1}{5}$ and $\sin B = \dfrac{2\sqrt{6}}{5} \Rightarrow 5\sin A = 2\sin B$

$\Rightarrow$ $\dfrac{\sin A}{\sin B} = \dfrac{2}{5}$

If $GD = 2r$ then GD will be diameter and G will lie on in circle

$\Rightarrow$ $r = \dfrac{\Delta}{S} = \dfrac{\dfrac{1}{2}xa}{\dfrac{1}{2}(a + b + c)} = \dfrac{2R\sin A.x}{2R\left(\sin A + \sin B + \sin B\right)} = \dfrac{x\sin A}{6\sin A}$

$\qquad = \dfrac{x}{6} = \dfrac{GD}{2}$

$\Rightarrow$ $GD = 2r \Rightarrow G$ lies on in circle of $\triangle ABC$.

10. Solution

Let's assume that for real numbers

a_1, a_2, a_3, a_4 there exists $\left(a_i - a_j\right)^2 > \dfrac{1}{5}$

for $1 \leq i \leq j \leq 4$

without losing the generality

$a_1 \geq a_2 \geq a_3 \geq a_4$, clearly equality doesn't holds otherwise proof will be done

$\Rightarrow$ $a_1 > a_2 > a_3 > a_4$

$\Rightarrow$ $\left(a_1 - a_2\right) > \dfrac{1}{\sqrt{5}}$ ...(i)

     $\left(a_2 - a_3\right) > \dfrac{1}{\sqrt{5}}$ ...(ii)

     $\left(a_3 - a_4\right) > \dfrac{1}{\sqrt{5}}$ ...(iii)

Adding,

$\Rightarrow$ $a_1 > a_4 + \dfrac{3}{\sqrt{5}}$

     $a_2 > a_4 + \dfrac{2}{\sqrt{5}}$

If $a_4 > 0 \Rightarrow a_i > 0 \quad \forall\ i \in \{1, 2, 3, 4\}$

$\Rightarrow$ $a_2^2 > a_4^2 + \dfrac{4}{5} + \dfrac{4}{\sqrt{5}} a_4$

     $a_1^2 > a_4^2 + \dfrac{9}{5} + \dfrac{6}{\sqrt{5}} a_4$

     $a_3^2 > a_4^2 + \dfrac{1}{5} + \dfrac{2}{\sqrt{5}} a_4$

$\Rightarrow$ $\displaystyle\sum_{i=1}^{4} a_i^2 > 4a_4^2 + \dfrac{14}{5} + \dfrac{12}{\sqrt{5}} a_4 > 4\left(a_4 + \dfrac{3}{2\sqrt{5}}\right)^2 + \left(\dfrac{14}{\sqrt{5}} - \dfrac{9}{20}\right) \Rightarrow \displaystyle\sum_{i=1}^{4} a_i^2 > 1$

We reach to contradiction

If $a_1 < 0 \Rightarrow a_i < 0 \quad \forall\ i \in \{1, 2, 3, 4\}$

Let $b_i = -a_i \Rightarrow b_i > 0 \ \forall\ i \in \{1, 2, 3, 4\}$

$\Rightarrow$ from equation (i), (ii) and (iii)

     $b_2 - b_1 > \dfrac{1}{\sqrt{5}}$

$$b_3 - b_2 > \frac{1}{\sqrt{5}}$$
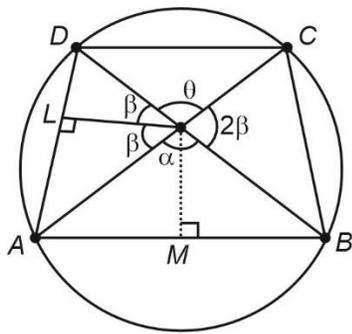
$$b_4 - b_3 > \frac{1}{\sqrt{5}}$$

$\Rightarrow$ $b_1 < b_2 < b_3 < b_4$

Will lead to similar contraction

$\Rightarrow$ $\sum b_i^2 > 1$

$\Rightarrow$ There must exist such a_1, a_2, a_3, $a_4 \in R$ such that $\forall$ i, $j \in \{1, 2, 3, 4\}$ $\left(a_i - a_j\right)^2 \le \frac{1}{5}$

11. Solution

$\because$ $AB \parallel CD$ and $ABCD$ is cyclic

$\Rightarrow$ ABCD is cyclic trapezium

$\Rightarrow$ $AD = CB$ and $AC = BD$

Let r be the radius of circle

$OB \cdot (AB + CD) = r(AB + CD)$

Let $\angle DOC = \theta$, $\angle AOB = \alpha$

$\angle DOL = \angle AOL = \beta$

$\Rightarrow$ $\angle BOC = 2\beta$

$AM = r\sin\dfrac{\alpha}{2}$

$AB = 2r\sin\left(\dfrac{\alpha}{2}\right)$

Similarly, $CD = 2r\sin\left(\dfrac{\theta}{2}\right)$

$AC = 2r\sin\left(\dfrac{2\beta + \theta}{2}\right)$

$OL = r\cos\beta$

LHS = $r\left(2r\sin\dfrac{\alpha}{2} + 2r\sin\dfrac{\theta}{2}\right)$

$$= 2r^2 \left[\sin\frac{\alpha}{2} + \sin\frac{\theta}{2} \right] \qquad\qquad \dots \text{(i)}$$

$$\text{RHS} = r\cos\beta \left[2r\sin\left(\frac{2\beta+\theta}{2}\right) + 2r\sin\left(\frac{2\beta+\theta}{2}\right) \right]$$

$$= 2r^2 \left(2\sin\left(\frac{2\beta+\theta}{2}\right)\cos\beta \right)$$

$$= 2r^2 \left(\sin\left(2\beta+\frac{\theta}{2}\right) + \sin\frac{\theta}{2} \right)$$

Now, $2\beta + 2\beta + \alpha + \theta = 360$

$$2\beta + \frac{\alpha}{2} + \frac{\theta}{2} = 180$$

$$= 2r^2 \left[\sin\left(\frac{\alpha}{2}\right) + \sin\frac{\theta}{2} \right]$$

$\Rightarrow$ LHS = RHS.

12. **Solution**

Lets see via examples

12-chain,

$f(12) = n(S)$, $12 = 2^2 \cdot 3^1$

$S = \{(1, 12), (1, 2, 12), (1, 3, 12), (1, 2, 6, 12), (1, 4, 12), (1, 3, 6, 12) \dots, (1, 6, 12)\}$

$\Rightarrow$ In another way we need to select divisors of n.

Clearly $f(2^k) = f(2^{k-1}) + f(2^{k-2}) + \dots + f(2^1) + f(2^0)$

$= (2^{k-2} + 2^{k-3} + \dots + 1) + 1$

$= (2^{k-1} - 1) + 1 = 2^{k-1}$

Let's proceed with induction

$f(2^k \cdot 3) = (k + 2) \cdot 2^{k-1}$, $k \geq 1$

$\Rightarrow$ Checking the base case

$f(6) = 3 \cdot 2^0 = 3$

$\{(1, 6), (1, 2, 6), (1, 3, 6)\}$

Notice that $(2^k \cdot 3)$ and (2^{k+1}) are proper factor of $(2^{k+1} \cdot 3)$

Also, $f(2^{k+1} \cdot 3^1) = f(2^k \cdot 3) + f(2^{k+1}) + f(2^k \cdot 3)$

$\Rightarrow$ $(k + 2) \cdot 2^{k-1} + 2^k + (k + 2) \cdot 2^{k-1}$

$= (k + 3) \cdot 2^k = [(k + 1) + 1] \cdot 2^{(k-1)+1}$

$\Rightarrow$ Since for $f(k)$, $f(k + 1)$ is also true.

Therefor the proof is done

Alter :

$$f\left(2^m \cdot 3\right) = \sum_k k \cdot \left[{}^{(m-1)}C_{k-1} + {}^{m-1}C_{k-2} \right]$$

Using combinotorics argument for selection of factors when 3 is in divisors factors.

$$\Rightarrow \quad \sum_k k \left({}^{m-1}C_{k-2} \right) = \sum_k \left[(k-1) \, {}^m C_{k-1} + {}^m C_{k-1} \right]$$

$$= \sum_k \left(\frac{m}{k-1} \right) (k-1) \left({}^{m-1}C_{k-2} \right) + \sum_k {}^m C_{k-1}$$

$= m \cdot 2^{m-1} + 2^m$

$= 2^{m-1}(m+2)$

Hence proved.

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