Aakash

Medical | IIT-JEE | Foundations

Answers & Solutions

for

RMO 2025-26

Our Problem *Solvers* shine bright in **JEE (Advanced) 2025**

HOME OF PROBLEM SOLVERS

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **No marks will be awarded for stating an answer without justification.**
- **Answer all the questions.**
- **All questions carry equal marks.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. (a) Let $n \geq 3$ be an integer. Find a configuration of n lines in the plane which has exactly

   (i) $n - 1$ distinct points of intersection;

   (ii) n distinct points of intersection;

   (b) Give configurations of n lines that have exactly $n + 1$ distinct points of intersection for

   (i) $n = 8$ and

   (ii) $n = 9$.

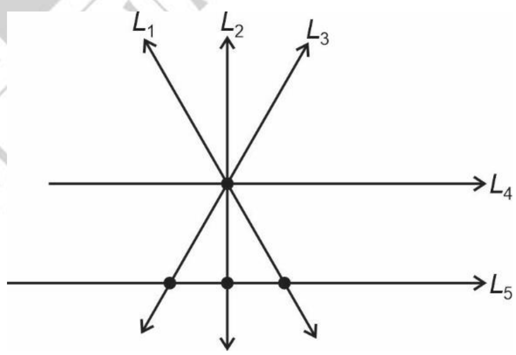
Sol. (a) (i)

There are two possible cases with $(n - 1)$ distinct points of intersection with n lines in a plane as follows

Case-I :

$(n - 1)$ concurrent non coincident line and the n^{th} line is parallel to any one of the concurrent lines.

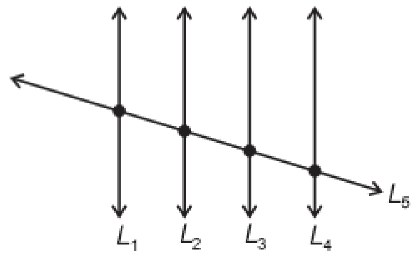
Example for $n = 5$



Case-II :

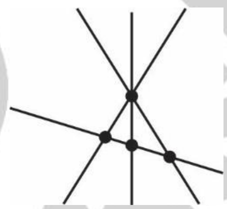
Out of n lines $(n - 1)$ lines are parallel to each other and non coincident and the n^{th} line is not parallel to $(n - 1)$ set of parallel lines.

For example $n = 5$



(ii)

For n distinct point of intersection made by n distinct lines in a plane $(n - 1)$ lines must be concurrent and the n^{th} should not be parallel to any of the $(n - 1)$ concurrent lines. For example, $n = 4$

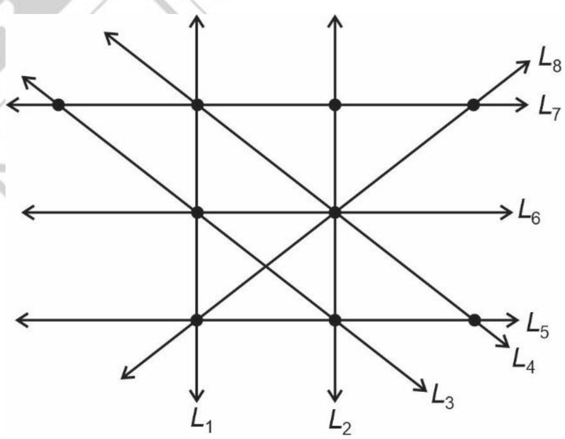


(b)

For n lines in a plane, total number of possible point of intersection is ${}^nC_2$. To make number of point of intersection less, we have to take set of parallel lines and concurrent lines.

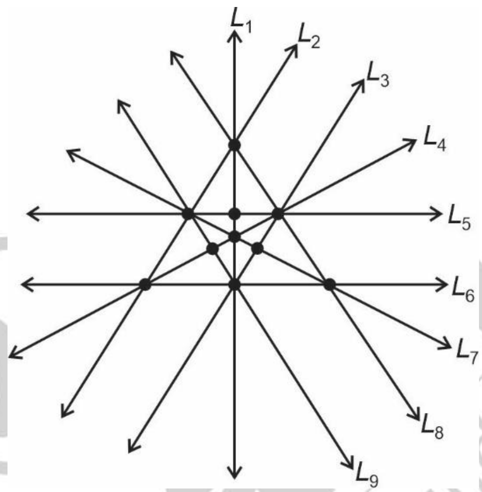
(i)

For $n = 8$, if we take 3 set of parallel lines where exactly 3 lines are parallel to each other exactly 2 lines are parallel to each other and other two lines are parallel to each other and 8^{th} line has 3 points of concurrency as follows :



(ii)

For n = 9, there are 3 set of parallel lines, where exactly 2 lines are parallel to each other, other 2 lines are parallel to each other, other 2 lines are parallel to each other. Remaining 3 lines are concurrent with these parallel lines as follows :



2. Let a, b, c be distinct nonzero real numbers satisfying

$$a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}$$

Determine the value of $|a^2b+b^2c+c^2a|$.

Sol. $(6\sqrt{2})$

$$a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}=k$$

$$\begin{array}{c|c|c|}
a-b=\dfrac{2}{c}-\dfrac{2}{b} & b-c=\dfrac{2}{a}-\dfrac{2}{c} & a-c=\dfrac{2}{a}-\dfrac{2}{b} \\
a-b=\dfrac{2(b-c)}{bc} & b-c=\dfrac{2(c-a)}{ac} & a-c=\dfrac{2(b-a)}{ab} \\
abc-b^2c=2(b-c) & abc-c^2a=2(c-a) & a^2b-abc=2(b-a) \\
 & & abc-a^2b=2(a-b)
\end{array}$$

3abc – (a^2b + b^2c + ca) = 0

$a^2b + b^2c + c^2a = 3abc$

$$(a-b)=\frac{2(b-c)}{bc}$$

$$(b-c)=\frac{2(c-a)}{ac}$$

$$(c-a)=\frac{2(a-b)}{ab}$$

$$(a-b)(b-c)(c-a)$$
$$=\frac{8(a-b)(b-c)(c-a)}{a^2b^2c^2}$$

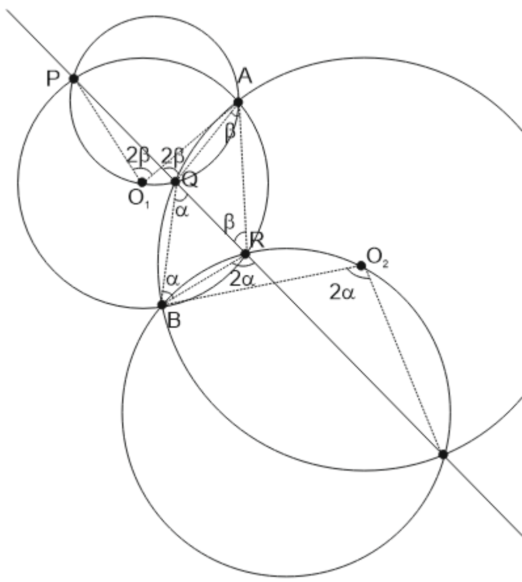
$$(abc)^2=8$$

$$abc=\pm 2\sqrt{2}$$

$$\left|a^2b+b^2c+c^2a\right|=3\times 2\sqrt{2}=6\sqrt{2}$$

3. Let Ω and Γ be circles centred at O_1, O_2 respectively. Suppose that they interest in distinct points A, B. Suppose O_1 is outside Γ and O_2 is outside Ω. Let l be a line not passing through A and B that intersects Ω at P, R and Γ at Q, S so that P, Q, R, S lie on the line in this order. Furthermore, the points O_1, B lie on one side of the l and the points O_2, A lie on the other side of l. Given that the points A, P, Q, O_1 are concyclic and B, R, S, O_2 are concyclic as well, prove that $AQ = BR$.

Sol.



Let $\angle SQB=\alpha$

$\Rightarrow \angle SO_2B = 2\alpha$

Now $\angle BRS = \angle SO_2B = 2\alpha$[as B, R, S, O_2 are concyclic]

Now, In ΔBRS

$\angle BRS = \angle SQB + \angle QBR$ [exterior angle theorem]

$\Rightarrow \angle QBR = 2\alpha - \alpha$

$\angle QBR = \alpha$

$\therefore$ In ΔBRS

$\angle QBR = \angle BQR = \alpha$

$\Rightarrow QR = BR$ … (i)

Now let $\angle PRA = \beta$

$\Rightarrow \angle PO_1 A = 2\beta$

Now $\angle PQA = \angle PO_1A = 2\beta$ [as P, O_1, Q, A are concyclic]

Now In ΔQAR

$\angle PQA = \angle PRA + \angle QAR$ [exterior angle theorem]

$\Rightarrow \angle QAR = \beta$

$\therefore$ In ΔQAR

$\angle QAR = \angle QRA = \beta$

$\Rightarrow AQ = QR$ … (ii)

From (i) and (ii)

$AQ = BR$

Hence proved

4. Prove that there do not exist positive rational numbers x and y such that $x + y + \frac{1}{x} + \frac{1}{y} = 2025$

Sol. $\left(x+\frac{1}{x}\right)+\left(y+\frac{1}{y}\right)=2025$

Let $x=\frac{a}{b}, y=\frac{c}{d}$ $\begin{bmatrix}\gcd(a,b)=1\\ \gcd(c,d)=1\end{bmatrix}$

$\Rightarrow \frac{a^2+b^2}{ab}+\frac{c^2+d^2}{cd}=2025$

$\Rightarrow cd\left(a^2+b^2\right)+ab\left(c^2+d^2\right)=2025abcd$

lemma 1: If $gcd\ (x, y) = 1$

$\Rightarrow gcd(xy, x^2 + y^2) = 1$

Let $gcd\ (xy, x^2 + y^2) = d$

If $d > 1$, let p be a prime such that $p \mid d$

$\Rightarrow p \mid xy, p \mid x^2 + y^2$

$\Rightarrow p \mid x$ or $p \mid y$ as gcd $(x, y) = 1$

If $p \mid x$

$\Rightarrow P \mid x^2 + y^2$

$\Rightarrow \quad P \mid y^2$ again contradiction as $gcd\,(x, y) = 1 \Rightarrow d = 1$

$\Rightarrow \quad cd(a^2 + b^2) \equiv 0 \text{ mod } (ab)$

$\Rightarrow \quad cd \equiv 0 \text{ mod } (ab)$ due to lemma 1

$\Rightarrow \quad ab \mid cd$, similarly $cd \mid ab$ by taking cd

$cd \mid \left(c^2 + d^2\right)ab$

$\Rightarrow \quad |ab| = |cd|$

Since $x, y \in R^+$

$\Rightarrow \quad ab = cd$

Let $a = mp$

$b = nq$

$c = mq,\ d = \cap p$

such that $ab = cd$ & $gcd\,(a, b) = 1$, $gcd(c, d) = 1$

for m, n, p, q relatively coprimes.

$\Rightarrow (a^2 + b^2) + (c^2 + d^2) = (m^2 + n^2)\,(p^2 + q^2)$

$= 2025\,mnpq$

$\Rightarrow (p^2 + q^2) | 2025\,mnpq$

Due to lemma 1

$(p^2 + q^2) | 2025\,mn$

Since p, q, m, n are relatively prime $\Rightarrow gcd\,(p^2 + q^2, mn) = 1$

$\Rightarrow (p^2 + q^2) | 2025 = 3^4.5^2$

and $(m^2 + n^2) | 2025 = 3^4.5^2$

lemma 2 : If $3 | (x^2 + y^2)$

$\Rightarrow 3|x$ & $3|y$

If $gcd\,(x, y) = 1$

Then $3 \nmid (x^2 + y^2)$

Due to lemma 2:

$p^2 + q^2 | 25$ & $m^2 + n^2 | 25$

$\Rightarrow p^2 + q^2 = 5$ or 25, $\left(m^2 + n^2\right) = 5$ or 25

$\Rightarrow \left(p^2 + q^2\right) \le 25,\ m^2 + n^2 \le 25$

$\Rightarrow (p^2 + q^2)(m^2 + n^2) \le 625$

but $2025mnpq \ge 2025$

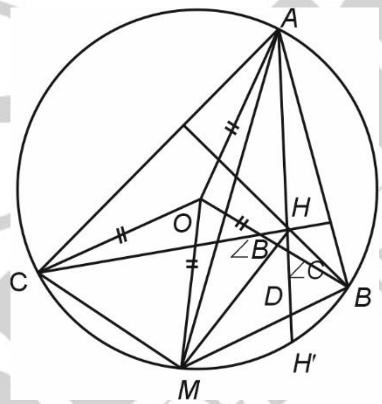
$\Rightarrow$ no such *p, q, m, n* exist

$\Rightarrow$ no such *a, b, c, d* exists

$\Rightarrow$ no such rational numbers *x, y* exist.

5. Let *ABC* be an acute-angled triangle with $AB < AC$, orthocentre *H* and circumcircle Ω. Let *M* be the midpoint of minor arc *BC* of Ω. Suppose that *MH* is equal to the radius of Ω. Prove that $\angle BAC = 60°$.

Sol.



Since, mid point *M* of arc *BC* not containing *A*, chords $MC = MB$ and $OM \perp BC$

$\Rightarrow \quad \angle CHB = 180° - \angle CAB$

$\angle CMB = 180° - \angle CAB$

Let *H′* be image of *H* about the line *BC*

$\Rightarrow \quad HH' \perp BC$

$\Rightarrow \quad BC \perp HH'$ and $BC \perp OM$

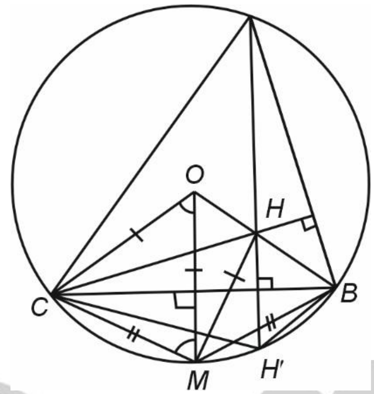
$\Rightarrow \quad OM \parallel HH'$

Since *BC* bisect $HH' \Rightarrow BC$ bisects *OM*

$\Rightarrow \quad$ *M* is image of *O* about $BC \Rightarrow \angle COM$

$= \quad \angle CMO$

$\Rightarrow \quad CO = CM$



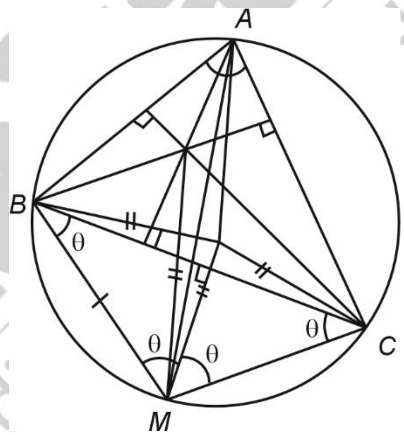
Quadrilateral $MH'HO$ is isosceles trapezium as $BC \perp HH'$ and $OM \perp BC$ and diagonals are equal

$\Rightarrow$ $HH' \parallel OM$ and $OH' = R = MH$

$\Rightarrow$ BC is bisecting both OM and HH'

$\Rightarrow$ $CM = CO$

Now, chasing angles, we get



$A + 2\theta = 180°$

$\Rightarrow$ also, $\angle BOC = 2A$

Since $CO = CM \Rightarrow \theta = 60°$

$\Rightarrow$ $\angle BAC = 60°$

6. Let $p(x)$ be a nonconstant polynomial with integer coefficients, and let $n \geq 2$ be an integer such that no term of the sequence p(0), p(p)(0)), $p(p(p(0)))$, … is divisible by n. Show that there exist integers a, b such that $0 \leq a < b \leq n - 1$ and n divides $p(b) - p(a)$.

Sol. For the contradiction, assume that

$$p(b) \not\equiv P(a) \bmod n \quad \forall\ 0 \leq a < b \leq n - 1$$

$\Rightarrow$ For the $S = \{0, 1, \ldots, n - 1\}$

$P(e_1) \not\equiv P(e_2) \bmod (n)$ $\forall\, e_1, e_2 \in S, e_1 \neq e_2$

$\Rightarrow$ The set $\{P(0), P(1), \ldots P(n + 1)\}$ for the residue system mod n.

Since for polynomial

$i \equiv j \bmod (n)$

$\Rightarrow$ $P(i) \not\equiv P(i)$ mod (n)

$\Rightarrow$ The set $\{P(0), P(1), \ldots P(n - 1)\}$ and residue set $\{0, 1, \ldots, n - 1\}$ will be a one-one, onto function

$\Rightarrow$ Bijection function

Defined $q^k(0) = q^{k-1}(0)$, $q^2(0) = q(q(0))$

Such that $q(x) = P(x)$ mod (n)

Such that $g : \{0, 1, \ldots, n - 1\} \rightarrow \{0, 1, \ldots, n - 1\}$

$\Rightarrow$ Since $n \nmid P(0), p(p(0) \ldots \underset{\cdots}{p\ \ p}\ (p)\ (0) \ldots)$

Now if none of $q^k(0) \neq 0$, then there will be n piegons but $(n - 1)$ holes

$\Rightarrow$ $q^l(0) = q^m(0)$ for some $l, m \in (1, 2, \ldots, n - 1\}$

$\Rightarrow$ since $q(x)$ is one-one

$\Rightarrow$ $q^{l-1}(0) = q^{m-1}(0)$

$\Rightarrow$ Similarly

$q^{l-l} = q^{m-1}(0)$, for $l < m$

$\Rightarrow$ $q^0(0) = q^{m-1}(0)$ and that is contradiction

$\Rightarrow$ $q^k(0) = 0$ and that means

$P^k(0) = P(P \ldots k \text{ times})\ (0) = 0$

And that is contradiction

$\Rightarrow$ $P(a) \equiv P(b)$ mod (n)

$\Rightarrow$ $n \mid P(a) - P(b)$

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