Corporate Office : *AESL, 3rd Floor, Incuspaze Campus-2, Plot No. 13, Sector-18, Udyog Vihar, Gurugram, Haryana - 122015*

Marks : 102

Regional Mathematical Olympiad

Time : 3 Hrs.

Mock Test

Topic : RMO Full Syllabus Mock Test
Instruction : • **Calculators (in any form) and protractors are not allowed.** • **Rulers and compasses are allowed.** • **Answer all the questions.** • **All questions carry equal marks.** • **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let n, k be positive integers so that $1 < k < n - 1$. Prove that the binomial coefficient $\binom{n}{k}$ is divisible by at least two distinct primes.
2. Let K and L be points on a semicircle with diameter AB. Denote the intersection of AK and BL as T and let N be the point such that N is on segment AB and line TN is perpendicular to AB. If U is the intersection of the perpendicular bisectors of AB and KL and V is a point on KL such that angles UAV and UBV are equal, then prove that NV is perpendicular to KL.
3. Prove that if a, b and c are real numbers, then $(3abc - a^3 - b^3 - c^3)^2 \le (a^2 + b^2 + c^2)^3$.
4. Let $P_1P_2 \ldots P_{100}$ be a cyclic 100-gon and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $\overline{P_{i-2}P_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ for all integers i. Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points $Q_1, Q_2, \ldots, Q_{100}$ are concyclic.
5. In a convex polyhedron with m triangular faces (and possibly faces of other shapes), exactly four edges meet at each vertex. Find the minimum possible value of m.
6. Let a and b be positive integers. Prove that there exist positive integers x and y such that $\binom{x+y}{2} = ax + by$.

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Solution

1. We can assume without loss of generality that $2k \le n$, otherwise we interchange k with $n - k$.

Assume by contradiction that

$$\binom{n}{k} = \frac{n\cdot(n-1)\cdot(n-2)\cdot \ldots \cdot (n-k+1)}{1\cdot 2\cdot \ldots \cdot k} = p^l$$

For some prime p and some integer n. Write every number in the numerator in the form $n - i = p^{\alpha_i} s_i$, with $p \nmid s_i$, where $0 \le i \le k - 1$. **[3]**

First let us observe that we have $s_i \ne s_j$. Indeed, assume by contradiction that $s_i = s_j$ for some $i < j$. Then, as $p^{\alpha_i} s_i = n - i > n - j = p^{\alpha_j} s_j$, we get $\alpha_i \ge 1 + a_j$.

Therefore

$$n \ge p^{\alpha_i} s_i \ge p p^{\alpha_j} s_j = p(n-j) > p(n-k) \ge 2(n-k)$$

Which contradicts $2k \le n$.

This shows that the k terms $s_0, s_1, \ldots, s_{k-1}$ at the top are pairwise distinct. **[3]**

Moreover, as the numerator contains at least two consecutive integers, at least one of those is not divisible by p. Therefore, there exists some j so that $s_j = n - j > n - k \ge k$.

As the elements $s_0, s_1, \ldots, s_{k-1}$ are pairwise distinct and at least one of them is strictly greater than k, we have $s_0 s_1 \cdot \ldots \cdot s_{k-1} > 1 \cdot 2 \cdot \ldots \cdot k$. **[3]**

Moreover, as

$$\frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)}{1\cdot 2\cdot\ldots\cdot k} = p^l$$

We get

$$\prod p^{\alpha_i} s_i = p^l 1\cdot 2\cdot \ldots \cdot k$$

$$\Rightarrow \quad s_1 \cdot \ldots \cdot s_k \mid p^l 1 \cdot 2 \cdot \ldots \cdot k$$

$\Rightarrow \quad s_1 \ldots s_k \mid p'1, 2, \ldots k$ **[3]**

As each s_i is not divisible by p, $s_1 \cdot \ldots \cdot s_k$ is relatively prime with p_l. Therefore

$s_1 \cdot \ldots \cdot s_k \mid 1 \cdot 2 \cdot \ldots \cdot k$

But his contradicts $s_0 s_1 \cdot \ldots \cdot s_{k-1} > 1 \cdot 2 \cdot \ldots \cdot k$

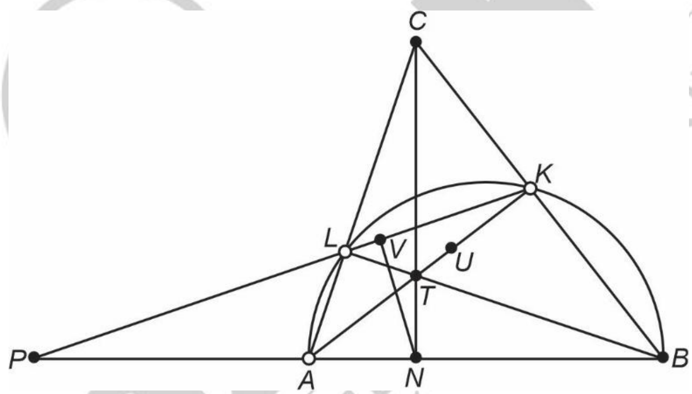
As we got a contradiction, our assumption is wrong, therefore $\binom{n}{k}$ cannot be a power of a prime. **[5]**

2. Denote the intersection of AL and BK by C. Then AK and BL are altitudes in the triangle ABC and therefore T is the orthocentre in this triangle. This implies that C, T, N are collinear.

The condition $\angle UAV = \angle UBV$ means that the quadrilateral $ABUV$ is cyclic.

Let M be the midpoint of AB. We will prove that $MUVN$ is cyclic, which shows that $NV \perp KL$.

If $KL \parallel AB$, then ABC is isosceles, in which case M coincides with N and U coincides with V, so $NV \perp KL$



Otherwise, let P be the intersection of AB and KL. By symmetry, we can assume that A is between P and B. **[3]**

As usual we denote by A, B, C respectively a, b, c the angles respectively the sides in the triangle ABC.

By applying the Menelaus theorem in triangle ABC with transversal $P - L - K$, we obtain

$$\frac{PA}{PB}\frac{KB}{KC}\frac{LC}{LA} = 1 \Leftrightarrow \frac{PA}{c+PA} c\cos(B) b\cos(C) \frac{a\cos(C)}{c\cos(A)} = 1 \Leftrightarrow \frac{PA}{c+PA} = \frac{b\cos(A)}{a\cos(B)}$$ **[3]**

Therefore

$$PA = \frac{bc\cos(A)}{a\cos(B) - b\cos(A)}$$

As $ABUV$ is cyclic, we have

$PU \cdot PV = PA \cdot PB$ **[3]**

To complete the proof, we need to show that

$PU \cdot PV = PM \cdot PN$ **[2]**

We have

$PU \cdot PV = PM \cdot PN \Leftrightarrow$

$PA \cdot PB = PM \cdot PN \Leftrightarrow$

$$PA \cdot (c + PA) = (PA + b\cos(A))\left(PA + \frac{c}{2}\right) \Leftrightarrow$$

$$PA^2 + cPA = PA^2 + bPA\cos(A) + PA\frac{c}{2} + \frac{c}{2}b\cos(A) \Leftrightarrow$$

$$PA\frac{c}{2} = bPA\cos(A) + \frac{c}{2}b\cos(A) \Leftrightarrow$$

$$\frac{bc\cos(A)}{a\cos(B) - b\cos(A)2}\frac{c}{2} = b\frac{bc\cos(A)}{a\cos(B) - b\cos(A)}\cos(A) + \frac{bc\cos(A)}{2} \Leftrightarrow$$

$$\frac{c}{2(a\cos(B) - b\cos(A))} = \frac{2b\cos(A)}{2(a\cos(B) - b\cos(A))} + \frac{a\cos(B) - b\cos(A)}{2(a\cos(B) - b\cos(A))} \Leftrightarrow$$

$C = 2b\cos(A) + a\cos(B) - b\cos(A) \Leftrightarrow$

$C = b\cos(A) + a\cos(B)$ **[6]**

Which is true.

3. As we can see, the expression $3abc - a^3 - b^3 - c^3$ is plausibly the sum of the threefold products each involving some mixture of *a*, *b*, *c*. This structure reminds us of a determinant. The three summands are positive, suggesting each forward extended diagonal has one each of *a*, *b*, *c*. The three cubes are negative, suggesting the back diagonals with repeated values. **[5]**

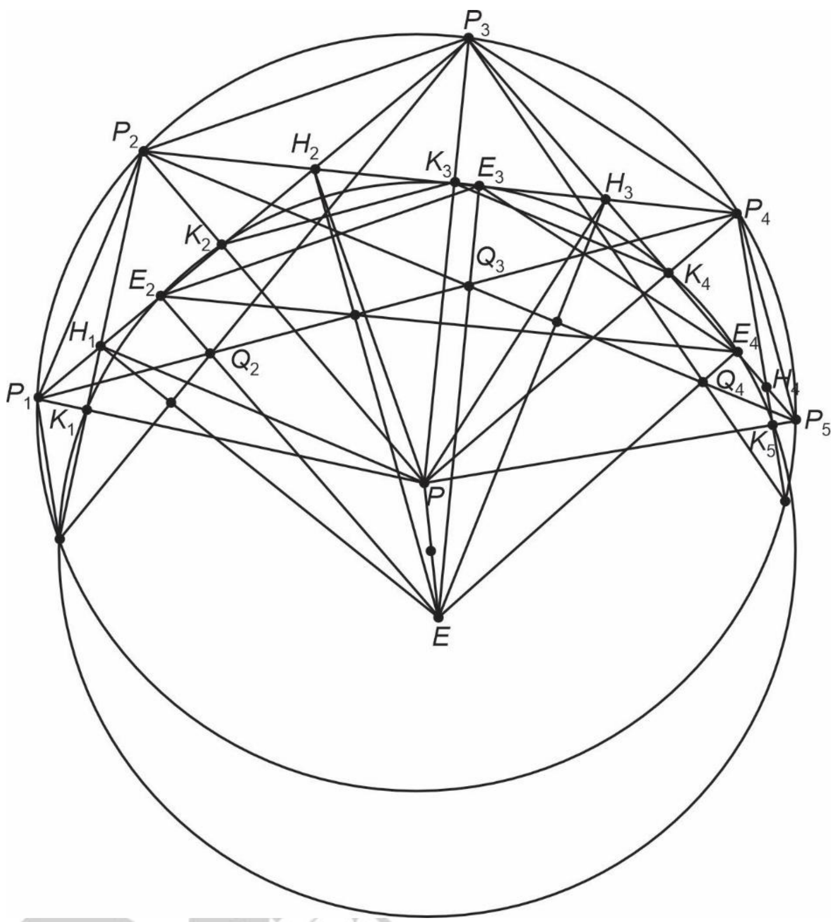
Let $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$.

Then det $A = 3abc - a^3 - b^3 - c^3$. **[5]**

From Hadamard's first theorem, we have

$(detA)^2 \le \prod_{j=1}^{3}\left(\sum_{i=1}^{3} a_{ij}^2\right)$. **[7]**

4. We let $\overline{PP_2}$ and $\overline{P_1P_3}$ intersect (perpendicularly) at point K_2 and define K cyclically.



The points K are concyclic say with circumcircle γ.

Note that $PP_1 \times PK_1 = PP_2 \times PK_2 = \ldots$ so the result follows by inversion at P. **[3]**

Let E_i be the second intersection of line $\overline{P_{i-1}K_iP_{i+1}}$ with γ; then it follows that the perpendiculars to $\overline{P_{i-1}P_{i+1}}$ at E_i all concur at a point E, which is the reflection of P across the centre of γ. **[3]**

We let $H_2 = \overline{P_1P_3} \cap \overline{P_2P_4}$ de note the orthocentre of ΔPP_2P_3 and define H. Cyclically.

We have

$\overline{EH_2} \perp \overline{P_1P_4} \,||\, \overline{K_2K_3}$ and $\overline{PH_2} \perp \overline{E_2E_3} \,||\, \overline{P_2P_3}$ **[3]**

Both parallelisms follow by Reim's theorem through $\angle E_2G_2E_3 = \angle K_2H_2K_3$, so we need to show the perpendicularities. **[2]**

Note that $\overline{H_2P}$ and $\overline{H_3E}$ are respectively circum-diameters of $\Delta H_2K_2K_3$ and $\Delta H_2E_2E_3$. As $\overline{K_2K_3}$ and $\overline{E_2E_3}$ are anti-parallel, it follows $\overline{H_2P}$ and $\overline{H_2E}$ are isogonal and we derive both perpendicularities.

The points E, Q_3, E_3 are collinear. **[2]**

We used the previous claim. The parallelisms imply that

$\frac{E_3H_2}{E_3P_2} = \frac{E_2H_2}{E_2P_3} = \frac{E_4H_3}{E_4P_3} = \frac{E_3H_3}{E_3P_4}$ **[2]**

Now, consider a homothety centered at E_3 sending H_2 to P_2 and H_3 to P_4. Then it should send the orthocentre of ΔEH_2H_3 to Q_3, proving the result.

From all this it follows that $\Delta EQ_2Q_3 \sim \Delta PK_2K_3$ as the opposite sides are all parallel.

Repeating this, we actually find a homothety of 100-gons

$Q_1Q_2Q_3 \ldots \sim K_1K_2K_3 \ldots$

and that concludes the proof. **[2]**

5. Take a polyhedron with m triangular faces and four edges meeting at each vertex. Let F, E and V be the number of faces, edges and vertices, respectively of the polyhedron. **[3]**

For each edges, count the 2 vertices at its endpoints; because each vertex is the endpoint of exactly 4 edges, we count each vertex 2 times in this fashion. **[3]**

Hence, $2E = 4V$. Also, counting the number of edges on each face and summing the F tallies yields a total of a least $3m + 4(F - m)$. **[3]**

Every edge is counted twice in this manner, implying that $2E \geq 3m + 4(F - m)$. **[3]**

By Euler's formula for planar graphs, $F + V - E = 2$. Combined with $2E = 4V$, this equation yields $2E = 4F - 8$.

Thus, $4F - 8 = 2E \geq 3m + 4(F - m)$ **[3]**

Or $m \geq 8$. Equality occurs if and only if every face of the polyhedron is triangular or quadrilateral. A regular octahedron has such faces, implying that $m = 8$ is indeed attainable. **[2]**

6. Denoting $A = 2a + 1$ and $B = 2b + 1$, the equation can be translated into

$$\frac{B-(x+y)}{x} = \frac{(x+y)-A}{y}$$ **[5]**

For $A = B$, any integers with $x + y = A$ satisfy the equation. **[3]**

Now suppose that $A < B$.

Let n be the integer in the interval $[A, B)$ which is divisible by $d = B - A$. **[5]**

Then $n \neq A$ because A is odd and d is even. Now choose

$$x = (B-n)\frac{n}{d},\ y = (n-A)\frac{n}{d}$$

Here $n = x + y$, so the equation is satisfied. **[4]**

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