Chapter 7

Sequences and Series

SEQUENCE, PROGRESSION AND SERIES

A succession of numbers $t_1, t_2, \ldots t_n$ formed according to the some definite rule is called sequence.

"A sequence is a function of natural numbers with codomain as the set of Real numbers or complex numbers"

Domain of sequence	$= N$
if Range of sequence	$\subseteq R \Rightarrow$ Real sequence
if Range of sequence	$\subseteq C \Rightarrow$ Complex sequence

Sequence is called finite or infinite depending upon its having number of terms as finite or infinite respectively.

For example: 2, 3, 5, 7, 11, ... is a sequence of prime numbers. It is an infinite sequence.

A progression is a sequence having its terms in a definite pattern *e.g.*: 1, 4, 9, 16, ... is a progression as each successive term is obtained by squaring the next natural number.

However a sequence may not always have an explicit formula of n^{th} term.

Series is constructed by adding or subtracting the terms of a sequence e.g., 2 + 4 + 6 + 8 + ... is **a series**. The term at n^{th} place is denoted by T_n and is called general term of a sequence or progression or **series**.

ARITHMETIC PROGRESSION (A.P.)

It is sequence in which the difference between any term and its just preceding term remains constant throughout. This constant is called the "common difference" of the A.P. and is denoted by 'd' generally.

A.P. is of the form

$a, (a + d), (a + 2d)$....

where 'a' denotes the first term or initial term

Important Relations :

$a_n - a_{n-1} = d$ = common difference $a_n = n^{th}$ term of A.P. $= \{a + (n - 1)\,d\} = l$
$a'_r = r^{th}$ term of A.P. from the end $= (n - r + 1)^{th}$ term from beginning n = total number of terms *i.e.*, $a'_r = a_{(n-r+1)} = a + (n - r)\,d$

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a'_n = n^{th} term of A.P. from the end $= \{l - (n-1)d\}$
S_n = the sum of first n terms of A.P. $= \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$ $= \frac{n}{2}[2l-(n-1)d]$ $a_n = S_n - S_{n-1}$

Properties of Arithmetic Progression

1. If $a_1, a_2, a_3, \ldots, a_n$ are in A.P., then
   (a) $a_1 \pm k, a_2 \pm k, \ldots, a_n \pm k$ are also in A.P.
   (b) $ka_1, ka_2, \ldots, ka_n$ are also in A.P.
   (c) $\frac{a_1}{k}, \frac{a_2}{k}, \ldots, \frac{a_n}{k}, k \neq 0$ are also in A.P.
2. If $a_1, a_2, a_3, \ldots$, and $b_1, b_2, b_3, \ldots$ are two A.Ps., then
   (a) $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$, are also in A.P.
   (b) $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$, are also in A.P.
3. If $a_1, a_2, a_3, \ldots a_n$, are in A.P., then
   (a) $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots = 2a_1 + (n-1)d$
   (b) $a_r = \frac{a_{r-k} + a_{r+k}}{2}, 0 \le k \le n-r$
4. If n^{th} term of a sequence is a linear expression in n then the sequence is an A.P.
5. If the sum of first n terms of a sequence is a quadratic expression in n, then the sequence is an A.P.
6. Three numbers a, b, c are in A.P. if and only if $b - a = c - b$, i.e., if and only if $a + c = 2b$.
7. Any three numbers in an A.P. can be taken as $a - d$, a, $a + d$. Any four numbers in an A.P. can be taken as $a - 3d$, $a - d$, $a + d$, $a + 3d$. Similarly 5 numbers in A.P. can be taken as $a - 2d$, $a - d$, a, $a + d$, $a + 2d$.

GEOMETRIC PROGRESSION (G.P.)

A sequence (finite or infinite) of non-zero numbers in which every term, except the first one, bears a constant ratio with its preceding term, is called a geometric progression.

The constant ratio, also called the common ratio of the G.P. is usually denoted by r.

For example, in the sequence, 1, 2, 4, 8, ...

$\frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \ldots = 2$, which is a constant.

Thus, the sequence is a G.P. whose first term is 1 and the common ratio is 2.

Important Relations

For a G.P.; $a, ar, ar^2, \ldots ar^{n-1}$

$a_n = n^{\text{th}}$ term of G.P.$= ar^{n-1} = l$ (last term) where a = first term $\neq 0$ $r = \dfrac{a_n}{a_{n-1}}$ $(r \neq 0)$ $a'_n = n^{\text{th}}$ term from end $= \dfrac{l}{r^{n-1}}$ $a'_r = r^{\text{th}}$ term from end of a G.P. having n terms $= a_{(n-r+1)}$ term from beginning. $= ar^{(n-r)}$ S_n = Sum of n terms from beginning $= \dfrac{a(r^n - 1)}{r-1} = \dfrac{lr - a}{r-1}$ when $r \neq 1$ $= na$ when $r = 1$
S_∞ = Sum of infinite G.P. $= \dfrac{a}{1-r}$; where $\lvert r \rvert < 1$

Properties of Geometrical Progression

1. $a_1, a_2, a_3, \ldots$ are in G.P. then
   $a_1k, a_2k, a_3k, \ldots$ and $a_1/k, a_2/k, a_3/k \ldots$ are also in G.P. $(k \neq 0)$
2. If $a_1, a_2, a_3, \ldots$ are in G.P. Then $1/a_1, 1/a_2, 1/a_3, \ldots$ are also in G.P.
3. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $a_1/b_1, a_2/b_2, a_3/b_3, \ldots$ are also in G.P.
4. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are not in G.P.
5. If $a_1, a_2, a_3, \ldots$ are in G.P. $(a_i > 0, \forall i)$, then $\log a_1, \log a_2, \log a_3, \ldots$ are in A.P. In this case the converse also holds good.
6. If $a_1, a_2, a_3, \ldots a_n$ are in G.P. , then
   (a) $a_1a_n = a_2a_{n-1} = a_3a_{n-2} = \ldots = a^2 r^{n-1}$
   (b) $a_r = \sqrt{a_{r-k}a_{r+k}}, \quad 0 \le k \le n-r$
7. If $a_1, a_2, a_3, a_4, \ldots, a_{n-1}, a_n$ are in G.P.

   then $\dfrac{a_2}{a_1} = \dfrac{a_3}{a_2} = \dfrac{a_4}{a_3} = \ldots = \dfrac{a_n}{a_{n-1}}$

   $\Rightarrow \quad a_2^2 = a_3a_1, a_3^2 = a_2a_4, \ldots$
   also $a_2 = a_1r, \ a_3 = a_1r^2, \ a_4 = a_1r^3, \ldots, a_n = a_1r^{n-1}$
   where r is the common ratio.

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8. Three numbers in G.P. can be taken as $\frac{a}{r}, a, ar$; Five numbers in G.P. can be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ etc.

In general: ($2m$ + 1) numbers in G.P. can be written as $(m \in N)$

$$\frac{a}{r^m}, \frac{a}{r^{m-1}}, ..., \frac{a}{r}, a\ ar, ..., ar^{m-1}, ar^m$$

9. Four numbers in G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$; Six numbers in G.P. can be taken as

$\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$; etc.

In general: ($2m$) numbers in G.P. can be written as $(m \in N)$

$$\frac{a}{r^{2m-1}}, \frac{a}{r^{2m-3}}, ..., \frac{a}{r^3}, \frac{a}{r}\ ar, ar^3, ..., ar^{2m-3}, ar^{2m-1}$$

RECOGNIZATION OF A.P. & G.P.

If a, b, c are three successive terms of a sequence

If $\frac{a-b}{b-c} = \frac{a}{a} = 1$, then a, b, c are in A.P.

If $\frac{a-b}{b-c} = \frac{a}{b}$, then a, b, c are in G.P.

MEANS

Arithmetic Mean

If three terms 'a, b, c' are in A.P., then middle term 'b' is called A.M. between the other two i.e., a and c

$$b = \frac{a+c}{2}$$

i.e., A.M. of two numbers x_1 and x_2 is $\frac{x_1 + x_2}{2}$

A.M. of n positive numbers = $\frac{a_1 + a_2 + a_3 .... + a_n}{n}$

Insertion of *n* Arithmetic means between two numbers

Let A_1, A_2, ... A_n are n A.M. between a and b then

a, A_1, A_2, ..., A_n, b form an A.P.

b is $(n + 2)^{th}$ term

$$\therefore \quad b = a + (n+1)d \Rightarrow d = \frac{(b-a)}{n+1}$$

$$\boxed{A_1 + A_2 + ... + A_n = \frac{n}{2}(a+b)}$$

$$A_r = a + \frac{r(b-a)}{n+1}$$

Geometric Mean

If three terms a, b, c are in G.P., then b is called G.M. of a and c such that

$$\boxed{b=\sqrt{ac}}$$

G.M. of n numbers = $\sqrt[n]{a_1.a_2.a_3....a_n}$

Insertion of *n* G.M. between two numbers (*a* and *b*)

Here, a, G_1, G_2, ... , G_n, b will be in G.P.

So, b = $(n + 2)^{th}$ term of G.P.

Hence, $b = a\,.\,r^{n+1} \Rightarrow r=\left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

$$G_k = ar^k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$$

Harmonic Progression :

(i) **n^{th} term :** A sequence $\{a_n\}$ is called a harmonic sequence or harmonic progression (H.P) if the sequence made up of reciprocals of the terms of the sequence $\{a_n\}$ is an A.P.

As, $\left\{\dfrac{1}{a_n}\right\}$ is an A.P., we have (Assume $a_n \neq 0$)

$$\frac{1}{a_n}=\frac{1}{a_1}+(n-1)d$$

$$d=\frac{1}{a_2}-\frac{1}{a_1}$$

(ii) **Harmonic Mean(s)**

If between 2 numbers a, b we have number H such that a H b are in H.P then H is the harmonic mean of a, b.

As, a, H, b are in H.P

$\dfrac{1}{a},\dfrac{1}{H},\dfrac{1}{b}$ are in A.P.

$$\Rightarrow \quad \frac{2}{H}=\frac{1}{a}+\frac{1}{b}$$

$$\Rightarrow \quad \frac{2}{H}=\frac{a+b}{ab}$$

$$\Rightarrow \quad H=\frac{2ab}{a+b}$$

(iii) ***n* Harmonic Means :** Suppose now we insert n Harmonic means between a and b denoted by H_1, H_2, H_3 ...H_n.

Then a, H_1, H_2, H_3, ...H_n, b will be an H.P.

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} \cdots \frac{1}{H_n}, \frac{1}{b}$ will be an A.P.

$\frac{1}{b} = (n+2)^{\text{th}}$ term of A.P. $\frac{1}{b} = \frac{1}{a} + (n+1)d$

$\therefore \quad d = \frac{(a-b)}{ab(n+1)}$

$\frac{1}{H_k}$ = Reciprocal of k^{th} Harmonic mean

= $(k + 1)^{\text{th}}$ term of corresponding A.P.

$= \frac{1}{a} + \frac{k\left(\frac{1}{b} - \frac{1}{a}\right)}{n+1}$

$= \frac{(n-k+1)\frac{1}{a} + \frac{k}{b}}{n+1}$

Thus, $H_k = \frac{ab(n+1)}{ka + (n-k+1)b}$

Relations between A.M., G.M. and H.M

For two real positive numbers a and b

$A = \frac{a+b}{2}, \ G = \sqrt{ab}$

$A > G$ if $a \neq b$ ...(i)

$A = G$ if $a = b$ ...(ii)

So combining (i) & (ii), we get

A $\geq$ G, the equality holds when $a = b$

Let a, b be positive real numbers

Then $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$

$a + b - 2\sqrt{ab} \geq 0$

$\frac{a+b}{2} \geq \sqrt{ab}$

A.M. $\geq$ G.M. ...(iii)

Replacing a by $\frac{1}{a}$ and b by $\frac{1}{b}$, we get

$\frac{\frac{1}{a} + \frac{1}{b}}{2} \geq \frac{1}{\sqrt{ab}}$

$\sqrt{ab} \geq \frac{2}{\left(\frac{1}{a} + \frac{1}{b}\right)}$

G. M. ≥ H.M ...(iv)

From (iii) and (iv), A.M. ≥ G.M. ≥ H.M.

equality holds if and only if $a = b$

If a_1, a_2, a_3... a_n are n positive numbers, then

Above discussion leads to the result that, $\dfrac{a_1 + a_2 + a_3 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 .... a_n}$

Another Important Property to Remember

(i) $\dfrac{a^m + b^m + c^m + ...k^m}{n} \geq \left(\dfrac{a+b+c+........k}{n}\right)^m$ {When m is any rational number $m.\notin$ (0, 1) $m \neq 0$, $m \neq 1$}

(ii) $\dfrac{a^m + b^m + c^m + ...k^m}{n} \leq \left(\dfrac{a+b+c+...k}{n}\right)^m$ {if $m \in (0, 1)$}

SPECIAL SERIES

Sigma (Σ) notation

Σ indicates sum *i.e.*, $\sum_{i=1}^{n} i = \sum n = 1 + 2 + 3 + ... + n$

(i) $\sum_{i=1}^{n} \dfrac{i+1}{i+2} = \dfrac{1+1}{1+2} + \dfrac{2+1}{2+2} + \dfrac{3+1}{3+2} + ... + \dfrac{n+1}{n+2}$

(ii) $\sum_{i=1}^{m} a = a + a + ... + a$ (m times)

$= am$, where a is constant

(iii) $\sum_{i=1}^{m} ai = a\sum_{i=1}^{m} i = a(1 + 2 + ... + m)$

(iv) $\sum_{i=1}^{m} (i^3 - 2i^2 + i) = \sum_{i=1}^{m} i^3 - 2\sum_{i=1}^{m} i^2 + \sum_{i=1}^{m} i$

Important Results

(i) Sum of first n natural numbers

$\sum n = 1 + 2 + ... + n$

$= \dfrac{n(n+1)}{2}$

(ii) Sum of squares of first n natural numbers

$\sum n^2 = 1^2 + 2^2 + ... + n^2$

$= \dfrac{n(n+1)(2n+1)}{6}$

(iii) Sum of cubes of first n natural numbers

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 = (\Sigma n)^2$$

(iv) Sum of n terms of a sequence $T_n = an^3 + bn^2 + cn + d$

$$\boxed{S_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn}$$

ARITHMETICO-GEOMETRIC SERIES (A.G.S.)

n^{th} term of A.G.. S. = (n^{th} term of an A.P.) × (n^{th} term of a G.P.)

If a, $(a + d)$, $(a + 2d)$ + ... be an A.P. &

b, br, br^2 + ... be a G.P. then

$ab + (a + d)\, br + (a + 2d)br^2 + \ldots$ is the corresponding A.G.S.

$$\boxed{T_n \text{ of A.G.S.} = (T_n \text{ of A.P.}) \times (T_n \text{ of G.P.})}$$

Sum of finite A-G series

$$\Rightarrow \quad S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{(1-r)}$$

For infinite A.G. series

$$\therefore \quad S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} (|r| < 1).$$

METHOD OF DIFFERENCE

Let $t_1, t_2, \ldots$ be the terms of the sequence $\{t_n\}$

$$\begin{array}{ccccccccccc} t_1 & & t_2 & & t_3 & & t_4 & & t_5 & & t_6 \\ & \Delta t_1 & & \Delta t_2 & & \Delta t_3 & & \Delta t_4 & & \Delta t_5 & & \text{1st order difference} \\ & & \Delta^2 t_1 & & \Delta^2 t_2 & & \Delta^2 t_3 & & \Delta^2 t_4 & & & \text{2nd order difference} \end{array}$$

and so on.

$\Rightarrow$ First difference in A.P. ($T_n = an^2 + bn + c$)

$\Rightarrow$ First difference in G.P. ($T_n = ar^{n-1} + b$)

$\Rightarrow$ Second difference in A.P. ($T_n = an^3 + bn^2 + cn + d$)

$\Rightarrow$ Second difference in G.P. ($T_n = ar^{n-1} + bn + c$)

Telescopic Sum and Difference

Many sums and products in algebra (and trigonometry) are elegantly evaluated by transforming the n^{th} term, t_n, by clever manipulation of algebraic (trigonometric) identities, to the telescopic form $t_n = F(k + 1) - F(k)$.

The sum, $\sum_{k=1}^{n} t_k$, telescopes to $t_1 + t_2 + \ldots + t_n$

$= F(2) - F(1) + F(3) - F(2) + \ldots + F(n + 1) - F(n)$

$= F(n + 1) - F(1)$

[Note that the terms cancel in pairs and we are left with just two terms].

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The Irrational Number ‘*e*’

Definition : $e = \lim\limits_{n\to\infty}\left[1+\frac{1}{n}\right]^n = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\infty \approx 2.7182818284$

Properties of *e* :

(1) $2 < e < 3$

(2) e is the base of natural logarithm

Exponential Series :

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!} = \dots\infty$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Note that 0! = 1

and $\frac{1}{k!}$ = 0 where k is negative integer.

Exponential Function (a^x, a > 0)

$$a^x = e^{\log_e a^x} = e^{x\log_e a} = e^{x\alpha}\ [\alpha = \log_e a]$$

$$a^x = 1+\frac{x\alpha}{1!}+\frac{x^2\alpha^2}{2!}+\frac{x^3\alpha^3}{3!}+\dots\infty$$

$$\therefore\quad a^x = 1+\frac{x(\log_e a)}{1!}+\frac{x^2(\log_e a)^2}{2!}+\frac{x^3(\log_e a)^3}{3!}+\dots\dots\infty$$

Important Deductions from Exponential Series

(i) $e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\dots+\frac{(-1)^n}{n!}x^n+\dots\infty = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$

(ii) $\frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\infty = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

(iii) $\frac{e^x-e^{-x}}{2} = \frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$

(iv) $\frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots\infty = \sum_{n=0}^{\infty}\frac{1}{(2n)!}$

(v) $\frac{e-e^{-1}}{2} = 1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\dots\infty = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}$

(vi) $\frac{e^{ix}+e^{-ix}}{2} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\infty = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

(vii) $\frac{e^{ix}-e^{-ix}}{2i} = \frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\infty = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$

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Important Results to be Remembered

1. $\sum_{n=0}^{\infty}\frac{1}{n!}=\sum_{n=0}^{\infty}\frac{n}{n!}=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}=\sum_{n=k}^{\infty}\frac{1}{(n-k)!}=e$

2. $\sum_{n=1}^{\infty}\frac{1}{n!}=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...\infty=e-1$

3. $\sum_{n=2}^{\infty}\frac{1}{n!}=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...\infty=e-2$

4. $\sum_{n=0}^{\infty}\frac{1}{(n+1)!}=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+......\infty=e-1$

5. $\sum_{n=0}^{\infty}\frac{1}{(n+2)!}=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty=e-2$

6. $\sum_{n=1}^{\infty}\frac{1}{(n+1)!}=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty=e-2$

7. $\sum_{n=0}^{\infty}\frac{1}{(2n)!}=1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+......\infty=\frac{e+e^{-1}}{2}=\sum_{n=1}^{\infty}\frac{1}{(2n-2)!}$

8. $\sum_{n=0}^{\infty}\frac{n^2}{n!}=2e=\sum_{n=1}^{\infty}\frac{n^2}{n!}$

9. $\sum_{n=0}^{\infty}\frac{n^3}{n!}=5e=\sum_{n=1}^{\infty}\frac{n^3}{n!}$

10. $\sum_{n=0}^{\infty}\frac{n^4}{n!}=15e=\sum_{n=1}^{\infty}\frac{n^4}{n!}$

Logarithmic Series

(i) $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.....\infty;\ -1<x\le 1$

(ii) $\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}....\infty;\ -1\le x<1$

(iii) $\log_e(1+x)+\log_e(1-x)=\log_e(1-x^2)=-2\left[\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}...\infty\right];\ [-1<x<1]$

(iv) $\log_e(1+x)-\log_e(1-x)=\log_e\left[\frac{1+x}{1-x}\right]=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+...\infty\right];\ -1<x<1$

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