Chapter 1

Sets, Relations and Functions

SET - "Set is a well-defined collection of distinct and distinguishable objects"

Set-Notations

A set is usually denoted by capital letters A, B, C,... etc. where as its members or elements or objects are denoted by lowercase letters such as a, b, c,....etc.

The greek symbol $\in$ is used to denote the phrase 'belongs to'. Symbol $\in$ is called membership relation.

$x \in A \Rightarrow$ 'x belongs to A' or 'x is an element of A' or 'x is a member of A' or 'x is an object of A'.

$x \notin A \Rightarrow$ 'x does not belong to A'

e.g. $a \in$ set of English alphabet.

$6 \notin$ set of English alphabet.

Representation of a Set

A set can be represented by two methods

1. Roster form or tabular form
2. Set builder form or rule method.

(1) Roster or Tabular Form

Here the elements of set are listed seperated by commas within braces or curly brackets { }. Here order of listing is immaterial and no element is repeated.

For example, the set A of all natural numbers less than 10 is written as

A = {1, 2, 3, 4, 5, 6, 7, 8, 9} or A = {1, 3, 5, 2, 6, 4, 9, 8, 7}

(order is immaterial)

(2) Set-Builder Form

Here we choose a variable (say x), which represents each element of the set satisfying a particular property. Inside the bracket, x is followed by symbol: (or ; or vertical line '|' or oblique line '/' followed by the property or properties, possessed by each element of set. For example, the set A of all even integers less than 10 is written as

$A = \{x : x$ is an even integer less than $10\}$
$= \{x \mid x$ is an even integer less than $10\}$
$= \{x ; x$ is an even integer less than $10\}$
$= \{x / x$ is an even integer less than $10\}$

The symbol following x is read as 'such that'.

The roster form of A is written as

$A = \{0, 2, 4, 6, 8\}$

Standard Notations for Sets of Numbers

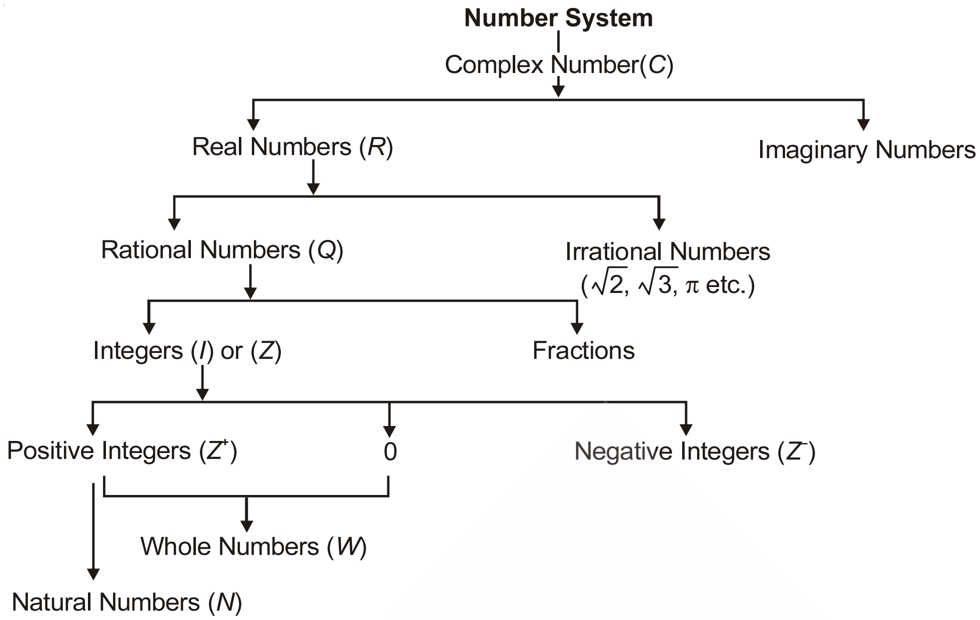
	Set of all	Symbol	*Examples*
1.	Natural number	$\mathbb{N}$	$\mathbb{N} = \{1, 2, 3.......\}$
2.	Integers	$\mathbb{Z}$ or I	$\mathbb{Z}$ or I = $\{.......-3, -2, -1, 0, 1, 2, 3,....\}$
3.	(a) Positive integers	$\mathbb{Z}^+$	$\mathbb{Z}^+ = \{1, 2, 3,......\}$
	(b) Negative integers	$\mathbb{Z}^-$	$\mathbb{Z}^- = \{.....-3, -2, -1\}$
4.	Integers excluding 0	I_0	$I_0 = \{\pm 1, \pm 2, \pm 3........\}$
5.	Even integers	E	$E = \{0, \pm 2, \pm 4........\}$
6.	Odd integers	O	$O = \{\pm 1, \pm 3, \pm 5........\}$
7.	Rational numbers	$\mathbb{Q}$	$\mathbb{Q} = \{x : x = \frac{p}{q}$, p and q are integers, $q \neq 0\}$
8.	Non-zero rational numbers	$\mathbb{Q}_0$	$\mathbb{Q}_0 = \{x : x \in Q, x \neq 0\}$
9.	Positive rational numbers	$\mathbb{Q}^+$	$\mathbb{Q}^+ = \{x : x \in Q, x > 0\}$
10.	Real numbers	$\mathbb{R}$	Here all rational and irrational numbers are included
11.	Non-zero real numbers	$\mathbb{R}_0$	$\mathbb{R}_0 = \{x : x \in R, x \neq 0\}$
12.	Positive real number	$\mathbb{R}^+$	$\mathbb{R}^+ = \{x : x \in R, x > 0\}$
13.	Complex numbers	$\mathbb{C}$	$\mathbb{C} = \{a + ib;\ a, b \in$ R and $i = \sqrt{-1}\}$
14.	Non-zero complex number	$\mathbb{C}_0$	$\mathbb{C}_0 = \{x : x \in C, x \neq 0\}$
15.	Natural numbers less than or equal to k, where k is positive integer	$\mathbb{N}_k$	$\mathbb{N}_k = \{1, 2, 3, 4, ...k\}$
16.	Whole numbers	W	$W = \{0, 1, 2, 3,......\}$

R is a subset of C (R $\subset$ C). Irrational numbers cannot be written in $\frac{p}{q}$ form.

Non-repeating and non-terminating decimals are called irrational.

e.g. $\sqrt{2}, \sqrt[5]{3}, \pi, e, \log_2 10$

NUMBER SYSTEM



Cardinal number of a set :

It is the number of elements in the set. It is also known as order of a set. The cardinal number of set A is denoted by $n(A)$ or $O(A)$.

Types of Sets

1. **Null Set (or Empty Set or Void Set) :** A set which has no element is called null set. It is denoted by ϕ or { }.

   Examples.

   (i) A = Set of odd numbers divisible by 2.

   (ii) B = Set of all omnipresent human beings.

   (iii) C = Set of all negative natural numbers

Remember :

(a) $\{0\}$ or $\{\phi\}$ is not an empty set.

(b) ϕ is called the null set and $\{\phi\}$ not a null set.

(c) ϕ is a subset of every set.

(d) Cardinal number of ϕ is zero.

(e) A set having at least one member is non-empty set.

2. **Singleton Set :** A set having a single element only is called singleton set.

   e.g. $\{\phi\}$, $\{0\}$, $\{2\}$, $\{a\}$ etc. each is singleton set or unit set.

   Examples :

   A = Set of present chief justice of India.

   B = $\{x : x^2 = 1, x > 0\}$

3. **Pair–Set :** A set having two elements only is called pair set.

   e.g. {0, 1}, {± 1}, {x : x is a root of $x^2 - 5x + 6 = 0$}

4. **Set of Sets :** A set S having all its elements as sets is called set of sets or a family of sets or a class of sets.

   e.g. {{1, 2}, {2, 3} {1, 2, 3}} is a set of sets as each member is a set itself.

   {{1, 2}, 7, {1, 7, 4}} is not a set of set as 7 is not a set.

5. **Finite and Infinite Set :** A set having finite number of elements in it is called finite set otherwise it is called infinite set. The number of members in an infinite set are infinite *i.e.* cannot be counted.

   Example :

   (i) I, N, W, Q, R all are infinite sets

   (ii) Set of all IIT in India is a finite set.

   (iii) Set of all supreme court judges in India is a finite set.

   $\boxed{\phi \text{ is a finite set as } n(\phi) = 0}$

6. **Equivalent Sets :** Two given sets A and B are said to be equivalent iff they have same number of elements. Two finite sets A and B are equivalent iff $n(A) = n(B)$.

   e.g. A = {a, b, c, d, e} and B = $\left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$ are equivalent sets as $n(A) = n(B) = 5$

7. **Equal Sets :** Two sets A and B are equal if both have all the elements same *i.e.* A is a subset of B and B is also a subset of A. The order of elements is immaterial.

   $A = B \Leftrightarrow A \subset B$ and $B \subset A$

   e.g. {a, b, c} = {a, c, b}

   $\boxed{A = B \text{ if } x \in A \Rightarrow x \in B \text{ and } x \in B \Rightarrow x \in A}$

   $\boxed{\text{Equal sets are also equivalent but equivalent sets may or may not be equal.}}$

SUBSETS, SUPERSETS, PROPER SUBSETS

A set 'A' is called a subset of set B if every member of set A also belongs to set B. (sign of subset is $\subset$)

$A \subset B \Leftrightarrow (x \in A \Rightarrow x \in B) \Rightarrow$ (A is contained in B or B contains A)

Here set B is called superset of A. $B \supset A$

$A \not\subset B$ is read as 'A is not a subset of B'.

Example :

A = {x : $x \in$ N} and B = {x : $x \in$ Z}

So $A \subset B$ as every natural number is also an integer.

Example :

A = {a, e, i, o, u}; B = {x : x is a letter of English alphabet}

$\therefore \quad A \subset B$

Example :

A = {a, e, i, o, u} and B = {e, o, u, i, a}

$A \subset B$ and $B \subset A \quad \Rightarrow A = B$.

Proper Subset

A set A is said to be a proper subset of a given set B if every element of set A is an element of set B and set B has atleast one extra element which is not an element of A.

i.e., $x \in A \Rightarrow x \in B$ and $\exists$ at least one element $y \in B$ such that $y \notin A$.

Proper subset is denoted by $A \subset B$ (read as "A is a proper subset of B")

$A \not\subset B$ (read as "A is not a proper subset of B")

Example :

A = {a, e, i, o, u}; B = {Set of all letters of English alphabet}.

Clearly $A \subset B$.

Remember :

(i) $A \subset B$ and $A \neq B \Leftrightarrow$ A is proper subset of B

(ii) Every set is a subset as well as superset of itself. $A \subset A$

(iii) No set is a proper subset of itself. $A \not\subset A$

(iv) ϕ is a subset of every set. $\phi \subset A$

(v) ϕ may or may not be a proper subset of any set.

(vi) If $A \subset B$ and $B \subset C \Rightarrow A \subset C$

(vii) If $A \supset B$ and $B \supset A \Leftrightarrow A = B$

$\boxed{N \subset W \subset I \subset Q \subset R \subset C}$

If cardinal number of a finite set is n then number of its possible subsets is 2^n and its number of proper subsets is $(2^n - 1)$.

Comparability of Sets

Two sets A and B are called comparable if any one of the conditions $A \subset B$ or $B \subset A$ or $A = B$, holds good otherwise A and B are called incomparable.

Example :

{*a*, *e*, *i*} and {*a*, *e*, *o*} are incomparable.

{*a*, *e*, *i*} and {*a*, *e*, *i*, *o*, *u*} are comparable.

{*a*, *e*, *i*} and {*e*, *i*, *a*} are comparable.

Power Set

The set of all possible subsets of a given set A is called power set of A and is denoted by P(A) or 2^A.

$P(A) = \{x : x \subset A\}$

$x \in P(A) \Leftrightarrow x \subset A$

$\phi \in P(A)$ and $A \in P(A)$

Example :

A = {1, 2, 3}

P(A) = {ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

n(A) = 3 so n(P(A)) = 2^3 = 8

(i) If A has n elements then its power set P(A) contains 2^n elements. $n(P(A)) = 2^n$.

(ii) ϕ and A both belong to P(A).

(iii) If $A = \phi$ then $P(A) = \{\phi\}$ is a singleton set.

(iv) If $A = \{t\}$ then $P(A) = \{\phi, \{t\}\}$ is a pair set.

(v) If cardinal number of set A is n then total number of subsets of P(A) = 2^{2^n} and proper subsets = $2^{2^n} - 1$.

(vi) If $A \subset B \Rightarrow P(A) \subset P(B)$

Universal Set

Any set which is the superset of all the sets under consideration is called the universal set (Ω or S or U).

Choice of universal set is not unique, but once chosen it is fixed for that discussion.

Example:

Let A = {a, e, i}; B = {i, o, u}; C = {e, f}

then U = {a, e, i, o, u, f}

or U = {a, e, i, p, o, u, f, g}

or U = Set of all English alphabet.

Intervals as Subsets of R

Four types of intervals can be defined on R as given below.

Let $a, b \in R$, such that $a < b$

1. Open Interval

(a, b) or $]a, b[= \{x : a < x < b\}$

= Set of all real numbers between a and b, not including a and b both.

$]a, b[$ or (a, b)

2. Closed Interval

$[a, b] = \{x : a \le x \le b\}$

= Set of all real numbers between a and b as well as including a and b both.

$[a, b]$

3. Open-closed Interval (semi closed or semi open interval)

$(a, b]$ or $]a, b] = \{x : a < x \le b\}$

= Set of all real numbers between a and b, including b and excluding a.

$(a, b]$ or $]a, b]$

4. Closed-open interval (semi closed or semi open interval)

$[a, b)$ or $[a, b[= \{x : a \le x < b\}$

= Set of all real numbers between a and b including a but excluding b.

$[a, b)$ or $[a, b[$

Some More Representations on Number Line

Infinite open interval

$x > a$ i.e. (a, ∞)

$x < b$ i.e. $(-\infty, b)$

Infinite close interval

$x \leq b$ i.e. $(-\infty, b]$

$x \geq a$ i.e. $[a, \infty)$

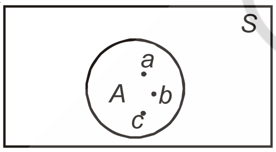
$(0, \infty) = R^+$

$(-\infty, 0) = R^-$

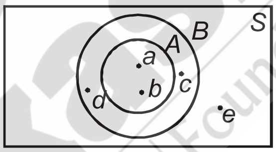
$(-\infty, \infty) = R$

VENN DIAGRAMS

Venn diagrams are introduced by Euler (a Swiss mathematician) and named after John Venn. It is a pictorial representation of sets in which a set is represented by a circle or a closed geometrical figure inside universal set which is shown by a rectangle. Each element of a set is represented by a point within the circle representing that set.



$A = \{a, b, c\}$

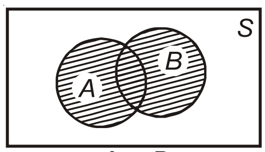


$A \subset B$; $A = \{a, b\}$; $B = \{a, b, c, d\}$; $e \notin A$; $e \notin B$

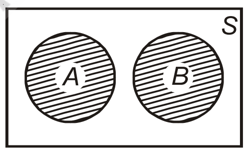
Various Operations on Sets

Union of Sets : $A \cup B$ (read as 'A union B' or 'A cup B' or 'A join B') is a set consisting of all the elements which are either in A or in B or in both.

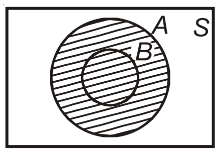
$A \cup B = \{x : x \in A \text{ or } x \in B\}$



$A \cup B$



$A \cup B$



$A \cup B$

$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$
$x \notin A \cup B \Leftrightarrow x \notin A \text{ and } x \notin B$

'$\vee$' denotes 'or'

Example:

$Q \rightarrow$ Set of all rational numbers

$Q' \rightarrow$ Set of all irrational numbers

$R = Q \cup Q'$

Remember :

$A_1 \cup A_2 \cup A_3$ .... $\cup A_n = \bigcup\limits_{i=1}^{n} A_i$

$A_1 \cup A_2 \cup A_3$ .... $= \bigcup\limits_{i=1}^{\infty} A_i$

Union of finite number of finite sets is a finite set

Union of finite sets with an infinite set is an infinite set

$A \cup A = A$, $A \cup \phi = A$, $A \cup S = S$

$S \cup \phi = S$, $\phi \cup \phi = \phi$, If $A \subseteq B$ then $A \cup B = B$

INTERSECTION OF TWO SETS

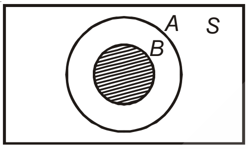
$A \cap B$ (read as 'A intersection B' or 'A cap B' or 'A meet B') is defined as a set containing all the elements common to A and B.

$A \cap B \quad = \{x : x \in A \text{ and } x \in B\}$

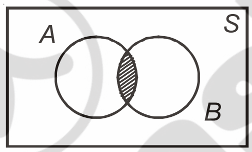
$\qquad\quad = \{x : x \in A \cap x \in B\}$

$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

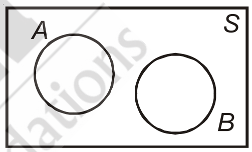
$x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$



$A \cap B$



$A \cap B$



$A \cap B = \phi$

Remember :

$A_1 \cap A_2 \cap A_2$ .... $\cap A_n = \bigcap\limits_{i=1}^{n} A_i = \{x : x \in A_i \ \forall \ i\}$

$A \cap B = AB$

Intersection of finite number of finite sets will be a finite set.

Intersection of finite set with infinite set will be a finite set.

Intersection of two or more infinite sets may or may not be finite.

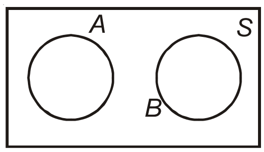
$A \cap A = A$; $A \cap \phi = \phi$; $A \cap S = A$; $S \cap \phi = \phi$

$\phi \cap \phi = \phi$; if $A \supset B$ then $A \cap B = B$; $(A \cup B) \cap A = A$; $(A \cap B) \cup A = A$

Disjoint Sets

Two sets A and B having no element in common are called disjoint or mutually exclusive sets.

i.e. $A \cap B = \phi \Leftrightarrow A$ and B are disjoint.



A and B are disjoint

***Example**:*

Z^+ and Z^- are disjoint sets.

Set of all boys and set of all girls are disjoint sets.

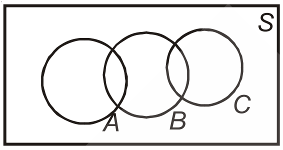
Set of Hindi alphabet and set of English alphabet are disjoint sets.

Set of years of birth of adults and set of years of birth of minors are disjoint

Q and Q' are disjoint

A family of various sets is pairwise disjoint if no two members of this family have a common element.

> ***Note:*** *If A_1, A_2 .... A_n are pairwise disjoint then $A_1 \cap A_2$ .... $\cap A_n = \phi$,*
> *But if $A_1 \cap A_2$ .... $\cap A_n = \phi$, A_1, A_2 .... A_n may not be pairwise disjoint.*



A $\cap B \cap C = \phi$; but A and B, B and C are not disjoint.

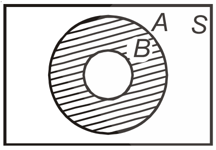
So A, B and C are not pairwise disjoint.

Difference of Two Sets

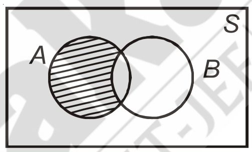
$A - B$ (read as 'A minus B') or (relative complement of B in A) is the set of all those elements of A which are not elements of B.

$A - B = \{x : x \in A \text{ and } x \notin B\}$

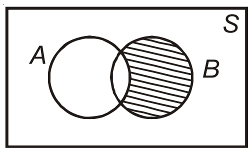
$B - A = \{x : x \in B \text{ and } x \notin A\}$



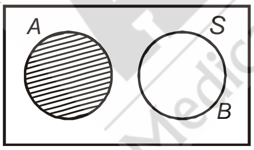
$A - B$ when $B \subset A$



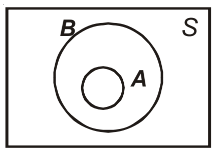
$A - B$



$B - A$



$A - B$, when A and B are disjoint $A - B = A$



$A - B = \phi$ when $A = B$ or $A \subset B$

Remember :

$x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

[Difference of two sets is not commutative]

$A - B \neq B - A$

Delete the elements of set B from set A and the set formed by remaining elements of A is $A - B$.

$(A - B)$, $(B - A)$ and $(A \cap B)$ are disjoint sets.

$A - B \subset A$; $B - A \subset B$, $A - \phi = A$

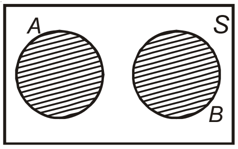
$A - A = \phi$

$A - B = A \sim B = A/B = C_A B$ (complement of B in A)

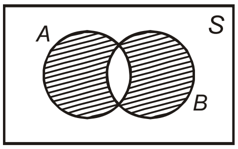
Symmetric Difference of Two Sets

Let A and B be two given sets. Then the symmetric difference of A and B denoted by $A \,\Delta\, B$ or $A \oplus B$, (A direct sum B) is defined as

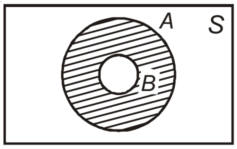
$$A \,\Delta\, B = (A - B) \cup (B - A)$$
$$= (A \cup B) - (A \cap B)$$



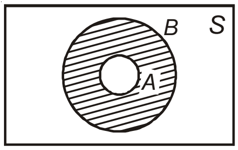
$A \,\Delta\, B = A \cup B$ when A and B are disjoint



$A \,\Delta\, B = (A \cup B) - (A \cap B)$



$A \,\Delta\, B = (A - B)$ when $B \subset A$



$A \,\Delta\, B = (B - A)$ when $A \subset B$

$A \,\Delta\, B = B \,\Delta\, A$; commutative law holds in case of symmetric difference of two sets.

$A \,\Delta\, B = \{x : x \in A \text{ and } x \notin B\} \cup \{x : x \notin A \text{ and } x \in B\}$

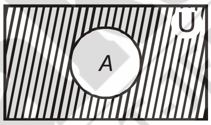
Complement of a Set

If A be a set and U be the universal set such that $A \subset U$ then complement of set A is denoted by A' or A^C or $C(A)$ and is defined as $U - A$

i.e. $A' = A^C = C(A) = U - A = \{x : x \in U \text{ and } x \notin A\}$

if $x \in A \Leftrightarrow x \notin A'$

$x \in A' \Leftrightarrow x \notin A$



$U - A$, or A^c or A' or $C(A)$

$U' = \phi$; $\phi' = U$; $A \cup A' = U$, $A \cap A' = \phi$

ALGEBRA OF SETS

1. **Idempotent Laws :** For any set A, we have
   (a) $A \cup A = A$
   (b) $A \cap A = A$
2. **Identity laws :** For any set A, we have
   (a) $A \cup \phi = A$
   (b) $A \cap \phi = \phi$
   (c) $A \cup U = U$
   (d) $A \cap U = A$
3. **Commutative laws :** For any two sets A and B, we have
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$

4. **Associative laws :** For any three sets A, B and C, we have
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$
5. **Distributive laws :** For any three sets A, B and C, we have
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. **Demorgan's laws :** For any three sets A, B and C, we have
   (a) $(A \cup B)' = A' \cap B'$
   (b) $(A \cap B)' = A' \cup B'$
   (c) $A - (B \cup C) = (A - B) \cap (A - C)$
   (d) $A - (B \cap C) = (A - B) \cup (A - C)$
   $(A')' = A$ (for any set A)
   $P(A) \cap P(B) = P(A \cap B)$
   $P(A) \cup P(B) \subseteq P(A \cup B)$

SOME THEOREMS

For any sets A, B and C, we have

1. $A - B = A \cap B'$
2. $A \cup B = B \Leftrightarrow A \subset B$
3. $A \cap B = A \Leftrightarrow A \subset B$
4. $(A - B) \cup B = A \cup B$
5. $A - (B \cup C) = (A - B) \cap (A - C)$
   $A - (B \cap C) = (A - B) \cup (A - C)$

Some More Operations on Sets.

$A \subset A \cup B$; $A \cap B \subset A$; $(A - B) \cap B = \phi$

$A \subset B \Leftrightarrow B' \subset A'$; $A - B = B' - A'$

$(A \cup B) \cap (A \cup B') = A$; $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

$A - (A - B) = A \cap B$; $A - B = B - A \Leftrightarrow A = B$

$A \cup B = A \cap B \Leftrightarrow A = B$; $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$

Some Basic Results about Cardinal Numbers

If A, B and C are finite sets and U is finite universal set, then we have

1. $n(A') = n(U) - n(A)$
2. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
3. If $n(A \cup B) = n(A) + n(B) \Leftrightarrow A \cap B = \phi$ *i.e.* A and B are disjoint.
4. $n(A \cap B') = n(A) - n(A \cap B)$
5. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
6. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
7. $n(A - B) = n(A) - n(A \cap B)$
8. $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$

9. $n(A \Delta B) = n(A \cup B) - n(A \cap B)$
10. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
11. If $A_1, A_2....A_n$ are pairwise disjoint sets then, $n(A_1 \cup A_2 \cup.......\cup A_n) = n(A_1) + n(A_2) + ...+n(A_n)$
12. Number of elements belonging to exactly two of sets A, B and C
    $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3.\ n(A \cap B \cap C)$
13. Number of elements belonging to atleast two of sets A, B and C
    $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 2.\ n(A \cap B \cap C)$
14. Number of elements belonging to atmost two of sets A, B and C = $n(A \cup B \cup C) - n(A \cap B \cap C)$
15. Number of elements belonging to exactly one of the sets A, B and C
    $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(A \cap C) - 2n(B \cap C) + 3n(A \cap B \cap C)$

Ordered Pair :

It is a pair of objects written in a particular order. Two members are written in a particular order separated by a comma and enclosed in parentheses. Hence in ordered pair (a, b) a is called the first component or the first element or the first co-ordinate and b the second.

Ordered pairs (a, b) and (b, a) are different.

$(a, b) = (c, d)$ iff $a = c$ and $b = d$

i.e., $(1, 3) = (1, 3)$; $(1, 3) \neq (1, 2) \neq (2, 3) \neq (3, 1)$

CARTESIAN PRODUCT

Cartesian product of two sets A × B : Let A and B be given two non-empty sets. Then the cartesian product of A and B written as A × B is defined as $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

It is a set of all ordered pairs such that in each ordered pair first element belongs to set A and second element belongs to set B.

A × B is read as 'A cross B' or 'cartesian product of A and B'

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Thus $(a, b) \in A \times B \Leftrightarrow a \in A$ and $b \in B$.

$B \times A = \{(b, a) : b \in B \text{ and } a \in A\}$

$A \times B \neq B \times A$ (not commutative)

- $n(A \times B) = n(A)\, n(B)$ and $n(P(A \times B)) = 2^{n(A)\, n(B)}$
- $A = \phi$ and $B = \phi \Leftrightarrow A \times B = \phi$
- Cartesian product of n non empty sets $A_1, A_2, ...A_n$ is a set of all n tuples $(a_1, a_2,........a_n)$
  such that each $a_i \in A_i$, $i = 1, 2.........n$.

  $$A_1 \times A_2 \times.......\times A_n = \prod_{i=1}^{n} A_i$$

  $A \times A = A^2$: $R \times R = R^2$ is a set of all points lying in the two dimensional plane

  $R \times R \times R = R^3$ represents set of all points in 3-D space.

  If at least one of A and B is infinite set then A × B is also infinite set,

  provided that other is non-empty set.
- If $n(A \cap B) = k$, then $n(A \times B \cap B \times A) = k^2$

Key Results on Cartesian Product

If A, B, C, D are four sets, then.

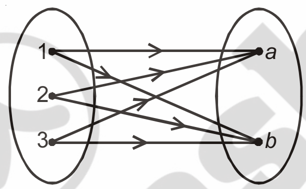
1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
5. If $A \subset B$, then $(A \times C) \subset (B \times C)$
6. If $A \subset B$, then $(A \times B) \cap (B \times A) = A^2$
7. If $A \subset B$, then $A \times A \subset (A \times B) \cap (B \times A)$
8. If $A \subset B$ and $C \subset D$, then $(A \times C) \subset (B \times D)$
9. $A \times B = B \times A \Leftrightarrow A = B$

Pictorial Representation of Cartesian Product of Two Sets :

Arrow diagram :

Let $A = \{1, 2, 3\}$ $B = \{a, b\}$

$A \times B \Rightarrow$

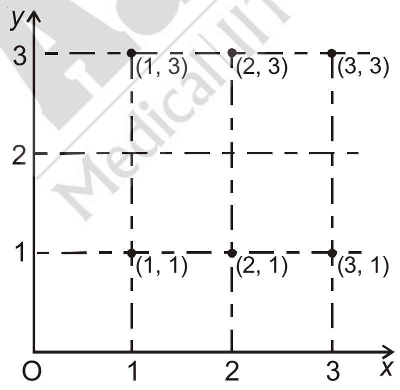


Lattice-Diagram :

Axis OX represents elements of A and perpendicular axis OY represents the elements of set B. Each dot represents an ordered pair of $A \times B$.

Let $A = \{1, 2, 3\}$

$B = \{1, 3\}$



RELATIONS

For any two non-empty sets A and B, every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$ *i.e.* $R \subset A \times B$ defines a relation from A to B.

$a \, R \, b \in A \times B \ \ \forall R$

If $(a, b) \in R$, then $a R b$ is read as 'a is related to b'

If $(a, b) \notin R$, then $a \not R \, b$ is read as 'a is not related to b'

Domain and Range of Relation

Domain of R = Dom(R) = Set of first elements of all the ordered pairs belonging to R.

Range of R = Set of second elements of all the ordered pairs belonging to R.

Co-domain of $R = B$ where R is a relation from A to B.

Range of $R \subset$ Co-domain of R

$\text{Dom}(R) = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$

Range of $R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$

If $R = A \times B$, then Dom(R) = A and Range of $R = B$

Dom $(\phi) = \phi$; Range of $\phi = \phi$

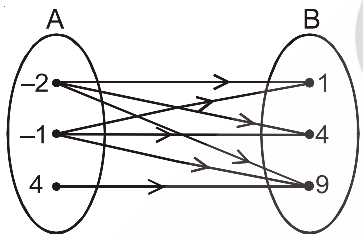
Representation of a Relation

Let $A = \{-2, -1, 4\}$ $B = \{1, 4, 9\}$

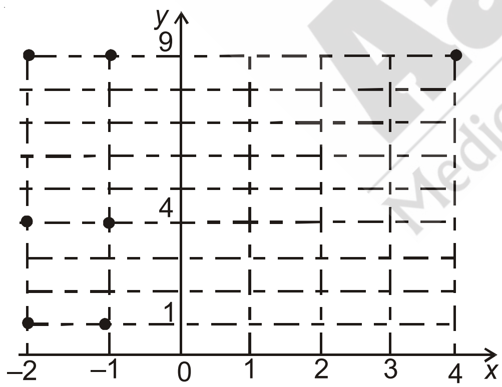
A relation from A to B *i.e.* a R b is defined as a is less than b.

This can be represented in the following ways.

1. **Roster form :** R = {(–2, 1), (–2, 4), (–2, 9), (–1, 1), (–1, 4), (–1, 9), (4, 9)}
2. **Set builder notation :** R = {(a, b): $a \in$ A and $b \in$ B, a is less than b}
3. **Arrow - diagram :**



4. **Lattice-diagram :**



5. **Tabular form :**

R	1	4	9
– 2	1	1	1
– 1	1	1	1
4	0	0	1

Note : *If (a, b) $\in$ R, then we write '1' in the row containing a and column containing b and if (a, b) $\notin$ R, we write '0' in the respective row and column.*

Any subset R of $A \times A$ defines a relation from A to A or in A. This relation is called **Binary relation** or simply relation. If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$

Total number of subsets of $(A \times B) = 2^{pq}$

Hence 2^{pq} different relations are possible from A to B.

Any subset of $A \times A$ is a relation on A.

Inverse Relation

Let $R \subseteq A \times B$ be a relation from A to B.

The inverse relation of R (denoted by R^{-1}) is a relation from B to A defined as $R^{-1} = \{(b, a) : (a, b) \in R\}$

If $(a, b) \in R$, then $(b, a) \in R^{-1}$, $\forall\ a \in A, b \in B$.

Domain of R^{-1} = Range of R

Range of R^{-1} = Domain of R

$(R^{-1})^{-1} = R$

Universal Relation

If A be a set and R is the set $A \times A$, then R is called universal relation in A.

If $R = A \times A$, then R is universal relation in A.

Void Relation

ϕ is called the empty or void relation as $\phi \subset A \times A$

Types of Relations on a Set

1. **Reflexive :**

   Let A be a given non-empty set. Then a relation R in A (i.e. $R \subset A \times A$) is said to be **reflexive** if $(a, a) \in R$, $\forall\ a \in A$

   i.e. $a\ R\ a$, $\forall\ a \in A$

   "is equal to", "is parallel to", are examples of reflexive relations.

2. **Symmetric :**

   A relation R in A (i.e. $R \subset A \times A$) is said to be symmetric

   if $a\ R\ b \Rightarrow b\ R\ a$ $(a, b \in A)$

   i.e. if $(a, b) \in R \Rightarrow (b, a) \in R$, $a, b \in A$

   "is parallel to", "is equal to", are examples of symmetric relations.

3. **Transitive :**

   A relation R in a given non-empty set A (i.e. $R \subset A \times A$) is said to be transitive

   if $a\ R\ b$ and $b\ R\ c \Rightarrow a\ R\ c$ $(a, b, c \in A)$

   i.e. If $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, $a, b, c \in A$

   "is parallel to", "is equal to", "is congruent to" are some examples of the transitive relation.

Equivalence Relation

A relation R on a given non-empty set A is called an equivalence relation if and only if it is Reflexive, Symmetric as well as Transitive.

"is parallel to", "is equal to", "is congruent to" "Identity relation" are some of the equivalence relations.

Every identity relation is an equivalence relation but every equivalence relation need not to be identity relation.

Partition of a Set

If A is a non-empty set, then a partition of A is a collection of non-empty pairwise disjoints subsets of A, such that union of collection of subsets is A.

Example :

If $A_1, A_2, A_3 .... A_n$ are non empty subsets of A, then the set $\{A_1, A_2, A_3 .... A_n\}$ is called partition of A if

(i) $A_1 \cup A_2 \cup A_3 ...... \cup A_n = A$

and (ii) $A_i \cap A_j = \phi \ \forall \ i \neq j \ (i, j = 1, 2, 3 .... n)$

FUNCTIONS

Let *A* and *B* be two given non-empty sets. *A* function *f* from set *A* to set *B* is a rule which associates each element of *A* to a unique element of B. It is denoted by $f : A \to B$

set *A* is called domain of function $'f'$

set *B* is called co-domain of function $'f'$

If element *x* of *A* corresponds to $y (\in B)$ under the function *f*, then we say that *y* is the image of *x* and we write $f(x) = y$.

Note : *(i) Any linear expression represents a function.*

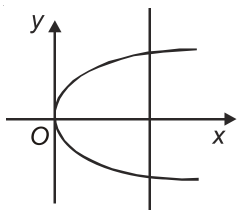
(ii) Range of $'f' \subseteq$ co-domain of $'f'$

(iii) $f : A \to B$ is not a function, if there is atleast one element in A which does not have a f-image in B or if there is an element in A which has more than one f-images in B.

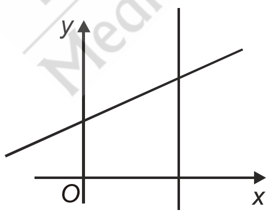
(iv) A function can also be represented as a set of ordered pairs e.g. f = {(1, 2), (2, 3), (3, 4), (4,4)} is a function from {1, 2, 3, 4} to {2, 3, 4}. Clearly f = {(1,2), (1, –1), (2, 2), (3, 3)} is not a function as $1 \to 2$ and $1 \to -1$.

The Graph of a Function

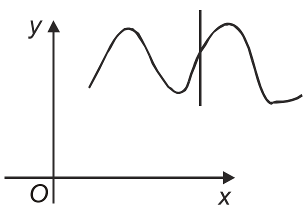
The graph of a function $y = f(x)$ consists of all points $(x, f(x))$ in the Cartesian plane since by definition of a function, there is exactly one value of *y* for each *x*, it follows that no vertical line can intersect the graph of a function of *x* for twice or more.



not a function of *x*



a function of *x*



a function of *x*

Domain of the Function

Domain of the function is set of all those real numbers *x* for which $f(x)$ exists or $f(x)$ is meaningful and finite.

Range of Function

Set of all the images of elements in domain is called the range.

Range = $\{f(x) : x \in \text{domain}\}$

Domain and Range of Standard Functions :

Function	Domain	Range
x^{2n+1}, $n \in W$	$(-\infty, \infty)$	$(-\infty, \infty)$
x^{2n}, $n \in N$	$(-\infty, \infty)$	$[0, \infty)$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\frac{1}{\sqrt{x}}$	$(0, \infty)$	$(0, \infty)$
$\frac{1}{x}$	$R - \{0\}$	$R - \{0\}$
a^x, $a > 0$	$(-\infty, \infty)$	$(0, \infty)$
$\log_a b$	$b > 0$, $a > 0$, $a \neq 1$	$(-\infty, \infty)$
$\sin x$	$(-\infty, \infty)$	$[-1, 1]$
$\cos x$	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	$R - \left\{(2n+1)\frac{\pi}{2}\right\}, n \in I$	$(-\infty, \infty)$
$\cot x$	$R - \{n\pi\}, n \in I$	$(-\infty, \infty)$
$\sec x$	$R - \left\{(2n+1)\frac{\pi}{2}\right\}, n \in I$	$(-\infty, -1] \cup [1, \infty)$
$\operatorname{cosec} x$	$R - \{n\pi), n \in I$	$(-\infty, -1] \cup [1, \infty)$
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Algebraic Operation on Functions

We have the following formulae for domains of functions

(i) dom $(f \pm g)$ = dom $f \cap$ dom g

(ii) dom $(f\,g)$ = dom $f \cap$ dom g

(iii) dom (f/g) = dom $f \cap \{x \in$ dom g: $g(x) \neq 0\}$

(iv) dom $\sqrt{f}$ $\{x \in$ dom f; $f(x) \geq 0\}$

Solution of Inequations :

The knowledge of solutions of inequations is necessary in the problems of domain and range. By the help of following illustration we will know how to solve different type of inequations.

Methods to find the range of a function :

The range of a function can be find out by the following methods.

(i) By using the range of standard function.

(ii) By completing the square.

(iii) By using the concept that for positive numbers we have arithmetic mean $\geq$ geometric mean.

(iv) By using the concept that if $ax^2 + bx + c = 0$ has real roots then $D = b^2 - 4ac \geq 0$.

(v) $-\sqrt{a^2+b^2} \leq a\sin x + b\cos x \leq \sqrt{a^2+b^2}$

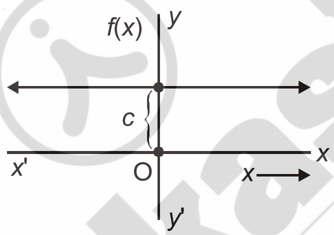
SOME STANDARD FUNCTIONS AND THEIR GRAPHS

Constant Function

A function denoted by $f(x) = c$ (where $c \in R$) is known as constant function

Domain = R

Range = $\{c\}$



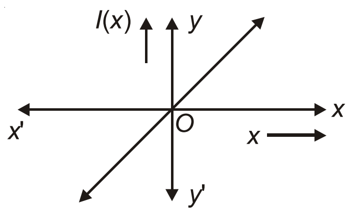
Identity Function $I(x)$

A function which is associated to itself is known as identity function and denoted by $I(x) = x$

Since x can take any value so domain of this function is R, corresponding value of $I(x)$ is also R, so range is R

Domain = R

Range = R



Modulus Function :

This is also known as absolute value function and is denoted by

$f(x) = |x|$

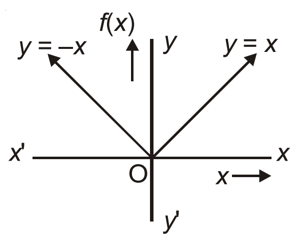
i.e. $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Domain of this function is set of all real numbers because $f(x)$ exists for all $x \in R$ but $|x| \geq 0$ so range is all non-negative real numbers.

Domain = R

Range = $[0, \infty)$

or $R^+ \cup \{0\}$



Properties of modulus function :

(a) $| x |^n = | x^n|$

(b) $| x^n | = x^n$, where n is even and $n \in Z$

(c) $| x y | = | x | | y |$

(d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $(y \neq 0)$

(e) $| | x | - | y | | \leq | x + y | \leq | x | + | y |$

(f) $|f(x)| = a \Rightarrow f(x) = \pm a, a \geq 0$

(g) $|f(x)| \leq a \Rightarrow -a \leq f(x) \leq a, a \geq 0$

(h) $a \leq |f(x)| \leq b \Rightarrow f(x) \in [-b, -a] \cup [a, b]$

(i) The minimum value of $|x - a| + |x - b|$ is $|a - b|$

(j) If $|x - a| + |x - b| = |a - b| \Rightarrow a \leq x \leq b$

(k) $|x + y| = |x| + |y| \Rightarrow xy \geq 0$

$|x + y| = ||x| - |y|| \Rightarrow xy \leq 0$

$|x - y| = |x| + |y| \Rightarrow xy \leq 0$

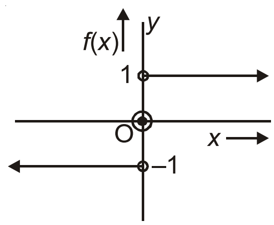
$|x - y| = ||x| - |y|| \Rightarrow xy \geq 0$

Signum Function

The function $f(x)$, defined as $f(x) = \begin{cases} \frac{|x|}{x} \text{ or } \frac{x}{|x|} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$

is called signum function. This signum function may also defined as

$$f(x) = \begin{cases} 1 & ; \quad x > 0 \\ 0 & ; \quad x = 0 \\ -1 & ; \quad x < 0 \end{cases}$$



Domain = R

Range = $\{-1, 0, 1\}$

Greatest Integer Function

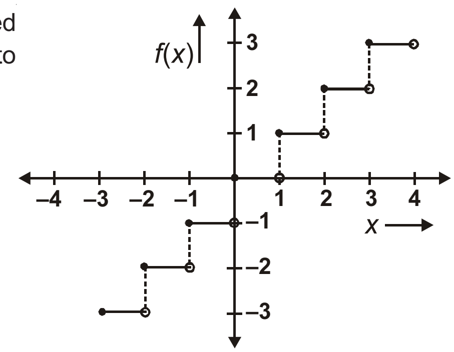
This function is also known as step function or floor function denoted by $f(x) = [x]$. By $[x]$ we mean greatest integer less than or equal to x. If n is an integer and x is any real number between n and $n + 1$

i.e. $n \leq x < n + 1$, then $[x] = n$

Thus $[3.4] = 3$, $[3.99] = 3$

$[-4.99] = -5$, $[-4.001] = -5$

$[0.3] = 0$, $[-0.2] = -1$



Domain of $[x]$ is set of all real numbers because $[x]$ exist $\forall x \in R$

But $[x]$ is always integral number so range is set of all integers Z.

Some Properties of [x] :

(a) $[x + k] = [x] + k$, if $k \in Z$

(b) $[-x] = -[x] - 1$, if $x \notin \mathbb{Z}$

(c) $[x] + [-x] = 0$, $x \in Z$

(d) $[x] + [-x] = -1$, $x \notin Z$

(e) $[x] - [-x] = 2x$, $x \in Z$

(f) $[x] - [-x] = 2[x] + 1$, $x \notin Z$

(g) $x - 1 < [x] \le x$

(h) $[x] \ge n \Rightarrow x \ge n$, $n \in Z$

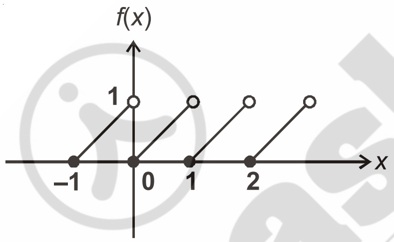
(i) $[x] > n \Rightarrow x \ge n + 1$, $n \in Z$

(j) $[x] \le n \Rightarrow x < n + 1$, $n \in Z$

(k) $[x] < n \Rightarrow x < n$, $n \in Z$

Fractional Part Function :

Function denoted by $f(x) = \{x\}$, known as fractional part function.



Also defined as $f(x) = x - [x]$

If $x \in Z$, then $f(x) = 0$ [*i.e.* $f(2) = 2 - [2] = 0$

If $x \notin Z$, then $f(x)$ lies between 0 to 1.

i.e. $x \notin Z$, $0 < f(x) < 1$ [*i.e.* $f(3.4) = 3.4 - [3.4] = 3.4 - 3 = 0.4$]

Note : *Fractional part function is a periodic function having fundamental period '1'.*

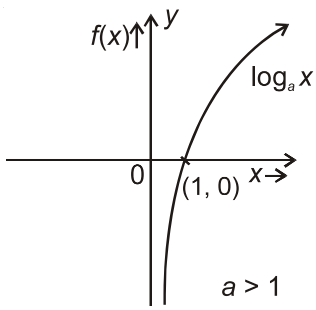
Domain = R

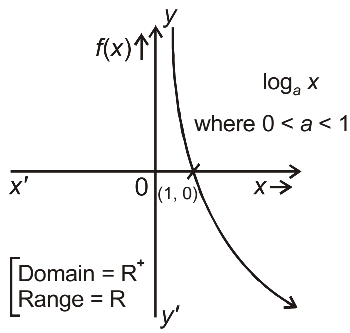
Range [0, 1)

LOGARITHMIC FUNCTION

A function $f : R^+ \to R$ defined as $f(x) = \log_a x$ where $a > 0$ and $a \ne 1$ is known as logarithmic function

Here $f(x)$ exist if $x > 0$ and $0 < a < 1$ or $a > 1$ $(a \ne 1)$





Properties of logarithmic Function

(i) $\log_a (m.n) = \log_a m + \log_a n$

(ii) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n\log_a m,\ m > 0$

(iv) $\log_{a^q} b^p = \dfrac{p}{q}\log_a b$; p and q are not even. Actually $\log_{a^{2q}} b^{2p} = \dfrac{p}{q}\log_{|a|} |b|$

(v) $\log_a b = \dfrac{\log_x b}{\log_x a} = \log_x b . \log_a x$, where $0 < x \neq 1$.

(vi) $\log_a b . \log_b a = 1$

(vii) If $\log_a f(x) = y \quad \Rightarrow \quad f(x) = (a)^y$

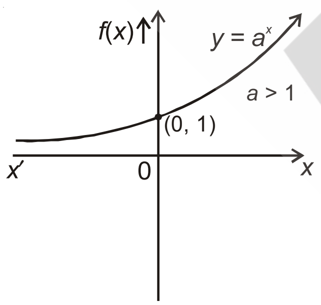
(viii) If $\log_a f(x) \geq \log_a g(x) \Rightarrow \begin{cases} f(x) \geq g(x) \text{ if } a > 1 \\ f(x) \leq g(x) \text{ if } 0 < a < 1 \end{cases}$

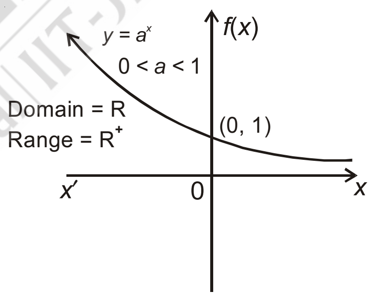
(ix) If $\log_a f(x) \geq y \quad \Rightarrow \begin{cases} f(x) \geq (a)^y \text{ if } a > 1 \\ f(x) \leq (a)^y \text{ if } 0 < a < 1 \end{cases}$

(x) If $\log_a f(x) \leq y \quad \Rightarrow \begin{cases} f(x) \leq (a)^y \text{ if } a > 1 \\ f(x) \geq (a)^y \text{ if } 0 < a < 1 \end{cases}$

Exponential Function

$f(x) = a^x$ is known as exponential function ($a > 0$)





Properties of exponential function

(i) $a^{\log_b x} = x^{\log_b a}$

(ii) $a^{\log_a x} = x$

KINDS OF FUNCTION

Polynomial Function

The function $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$, where $a_0, a_1, a_2, \ldots\ldots a_n \in R$ and $n \in N$ is called a polynomial function of degree n.

Rational Function

A function defined by the quotient of two polynomial function is called rational function.

Example **:**

$\frac{x^2+1}{x^3+x+1}$ is a rational function.

Irrational Function

A function involving one or more radicals of polynomial is called a irrational function

Example :

$x^{\frac{3}{2}}+\sqrt{x}+x^2,\ \frac{x^2+2x+3}{x+\sqrt[3]{x}+5}$ *etc.*

Algebraic Function

An algebraic function is one which consist of a finite number of terms involving power and roots of the variable x and simple operation, addition, subtraction, multiplication and division *i.e.* all rational, and irrational functions are algebraic functions.

Transcendental Function

All function which are not algebric are called transcendental function.

Example :

(a) All trigonometric function *i.e.* sin x, cos x etc.

(b) All exponential function, e^x, a^x *etc.*

(c) Inverse trigonometric function $\sin^{-1} x$, $\cos^{-1} x$, *etc.*

(d) Logarithmic function, log x, *etc.*

> ***Note :*** *A transcendental function is not expressed in a finite number of algebraic terms.*

Explicit Function

A function in which dependent variable (y) is expressed directly in terms of independent variable (say x)

i.e. $y = x^3 + x^2 + 1$, $y = \frac{x^2+3x+5}{x+2}$, *etc.*

Implicit Function

A function in which we can't express dependent variable in terms of independent variable.

Example:

$x^3 + y^3 + 3xy = 0$, note that we can't write y or x in terms of x, or y separately.

Even or Odd Function

(a) Even function : If $f(-x) = f(x)\ \forall\ x \in D_f$ then $f(x)$ is said to be even function.

Example : $f(x) = \cos x$ is a even function [$\because\ f(-x) = \cos(-x) = \cos x = f(x)$]

(b) Odd function : If $f(-x) = -f(x)\ \forall\ x \in D_f$ then $f(x)$ is said to odd function.

Example : $f(x) = x^3 + \tan^3 x$ is a odd function

because $f(-x) = (-x)^3 + [\tan(-x)]^3 = -x^3 - \tan^3 x = -[x^3 + \tan^3 x] = -f(x)$

so $f(-x) = -f(x)$

Note : *(a) Even function is symmetrical about y-axis while odd function is symmetrical about origin (i.e. in opposite quadrant)*

(b) Addition and subtraction of two even function is always even function.

(c) Sum of even ($f(x) \neq 0$) and odd function is neither even nor odd function.

(d) Any function 'f' can be uniquely represented as the sum of an even and an odd function.

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$$

where $\frac{1}{2}[f(x) + f(-x)]$ *is an even and* $\frac{1}{2}[f(x) - f(-x)]$ *is an odd function*

(e) $f(x) = 0$ is the only function which is both odd and even.

Periodic Function

A function 'f' defined on its domain is said to be periodic function if their exists a positive number T such that $f(x + T) = f(x)\ \forall\ x \in D$. Also both $x + T$ and $x - T$ should belong to D.

The least positive value of T, if exists is called, the fundamental period of the function.

Some Standard Functions and their Fundamental Period

Function	Fundamental Period
(i) $\sin^n x$, $\cos^n x$, $\operatorname{cosec}^n x$, $\sec^n x$	(i) π if n is an even and 2π if n is an odd
(ii) $\tan^n x$, $\cot^n x$	(ii) π either n is even or odd
(iii) $\{x\}$	(iii) 1
(iv) Algebraic function *e.g.,* $(x^2, \sqrt{x}, x^3 ...)$	(iv) non periodic

Note : *A constant function is periodic function having no fundamental period.*

Some Special Point about Periodic Function

If period of $f(x)$ is 'T' then

(a) (i) Period of $f(ax)$ and $f(ax + b)$ is $\frac{T}{|a|}$, $a \neq 0$.

(ii) Period of $f\left(\frac{x}{a}\right)$ is $|a|.T$.

(b) If Period of $f(x)$ and $g(x)$ are same say 'T' then period of $f(x) \pm g(x)$ may be

(i) $\frac{T}{2}$ (if $f(x)$ and $g(x)$ both are even).

(ii) T (if $f(x)$ is any function except even).

(c) If period $f(x)$ is T_1 and $g(x)$ is T_2. Then period of $f(x) \pm g(x)$ is given by L.C.M. of T_1 and T_2 (same for $\frac{f(x)}{g(x)}$) or less than that of LCM (T_1, T_2)

Note : *(i)* LCM of $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} = \frac{\text{LCM of } a,c,e}{\text{HCF of } b,d,f}$.

(ii) $\sin\sqrt{x}$ and $\sin x^2$ are not a periodic function because these can't be written in the form of $[f(x + T) = f(x)]$

(iii) L.C.M. of rational with irrational is not possible, e.g., L.C.M. of (π, 2, 2π) is not possible as π, $2\pi \in$ irrational and $2 \in$ rational

Bounded and Unbounded Function

$f(x)$ is said to be bounded above, if there exists a fixed number say M such that $f(x)$ is never greater than M for all value of x. Similarly it's bounded below if there exists a fixed number m (say) that $f(x)$ is never less then m

i.e. $m \le f(x) \le M$ for all value of x.

$f(x)$ is said to be unbounded if one or both of the upper and lower (M and m) bounds of the function are infinite

Example :

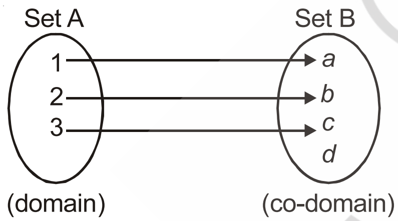
$f(x) = 3 + \sin x$ is a bounded function because maximum and minimum value of $\sin x$ are +1 and –1

So, $2 \le f(x) \le 4$ for all value of x.

TYPES OF MAPPINGS OR FUNCTIONS

One-one Function or Injective Function :

A function is said to be one-one function if different elements in a domain have different images in co-domain.



if $f(x_1) = f(x_2)$ then $x_1 = x_2$
$f(x)$ is one - one function

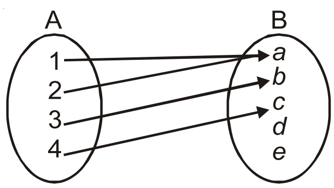
Note : *(i)* ***Example of one-one function :*** *Linear polynomial function (ax + b), x, e^x, log x, are always one-one functions.*

(ii) If $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0 \ \forall \ x \in$ domain *and is continuous, then y = f(x) is said to be one-one function.*

Number of one-one function : If A and B are finite sets having m and n elements respectively, then number of one-one function from A to B = ${}^nP_m$, if $n \ge m$, = 0, if $n < m$.

Many–one Function

A function $f : A \rightarrow B$ is said to be many one if more than one element in set A have same image in Set B.

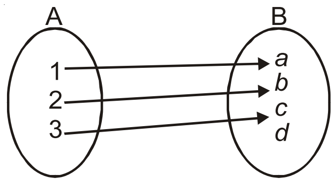


Note : *(i)* *All even function, modulus function, periodic function are always many-one function.*

(ii) *Square function, Trigonometric function are also many–one function in their domain.*

Into Function

A function f: $A \rightarrow B$ is said to be into function if there exist at least one element in set B having no any pre-image in set A.



In fig set B (co-domain) there is no pre-image, for element d, in set A, so function is into function.

Onto Function (Surjective Function)

$f : A \rightarrow B$, said to be onto function if every element in set B has a pre image in set A.

Range of f = co-domain of f.

Example of Onto function :

log x, linear polynomials, are always onto function defined from its domain to R.

One-one onto function or (Bijective function) : If a function is one-one and onto both then it is called one-one onto function or bijections.

If $f : A \rightarrow B$ is bijective then either $f'(x) \geq 0$ or $f'(x) \leq 0$ and range of f = B, and number of elements in A and B are same.

Let A and B both have 'n' elements then number of bijective function from A to B = n!

Example :

If $f : R \rightarrow R$, $f(x) = x^2 + 3x + 2$ then $f(x)$ is many one function.

because $f(x) = x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$

$$f(-2) = \left(-2 + \frac{3}{2}\right)^2 - \frac{1}{4} = 0$$

$$f(-1) = \left(-1 + \frac{3}{2}\right)^2 - \frac{1}{4} = 0$$

So image of –2 and –1 are same

$\therefore$ $f(x)$ is many one.

Example :

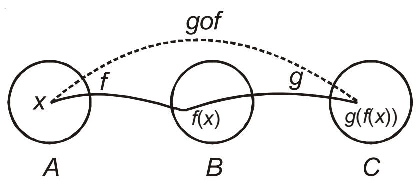
$f(x) = 2x + \sin x$, is one-one because $f'(x) = 2 + \cos x$, minimum value of $\cos x$ is –1.

$\therefore$ $f'(x) > 0$ for all $x \in$ Domain = R

Composition of Function

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ then the composition of g and f is denoted by gof and is defined as $gof : A \rightarrow C$ given by $gof(x) = g(f(x))$

Similarly fog is defined. Note that, gof is defined only if Range $f \subseteq$ dom g and fog is defined only if Range $g \subseteq$ dom f. dom fog = $\{x \in \text{dom } g : g(x) \in \text{dom } f\}$



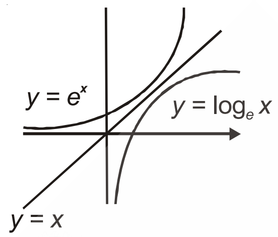
Inverse Function

Two functions f and g are inverse of each other if $f(g(x)) = x$ for $x \in$ dom g and $g(f(x)) = x$ for $x \in$ dom f, *i.e.*, $gof = I_{dom}\ f$ and $fog = I_{dom}\ g$ where $I_{dom}\ f$ is identity function on dom f and $I_{dom}\ g$ is identity function on dom g. We denote g by f^{-1} or f by g^{-1}. To find the inverse of f, write down the equation $y = f(x)$ and then solve x as a function of y. The resulting equation is $x = f^{-1}(y)$.

Note : *For existence of inverse function, function must be one-one onto.*

Graphical explanations:

The graph of f and f^{-1} are related to each other in the following way : If the point (x, y) lies on the graph of f then the point (y, x) lies on the graph of f^{-1} and vice versa. Thus the graph of f^{-1} is the reflection of the graph of f in the line $y = x$ as below (since we know that $y = \log x$ and $y = e^x$ are inverse of each other).



Existence of Inverse Function

A function need not have an inverse. e.g. the function $f(x) = x^2$ has no inverse (where dom $f = R$). To have an inverse, a function must be both *one-one and onto*, i.e. *bijective.*

Few Algebraic Properties of Function

→ If $f(x + y) = f(x) + f(y)$ or $f(ax + by) = af(x) + bf(y)$ then $f(x) = \lambda x$ where λ is a constant

→ If $f(x + y) = f(x).\ f(y)$ then $f(x) = a^{\lambda x}$ where a and λ are constant.

→ If $f(xy) = f(x) + f(y)$, $(x, y > 0)$, then $f(x) = \lambda \log_a x$

→ If $f\left(\frac{x}{y}\right) = f(x) - f(y)$, $(x, y > 0)$, then $f(x) = \lambda \log_a x$

→ If $f(x) \pm f(y) = f\left(\frac{x \pm y}{1 \mp xy}\right)$, then $f(x) = \tan^{-1}(x)$

→ If $f(x) = f'(x)$ then $f(x) = \lambda e^x$, where λ is a constant

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