

Chapter 15

Three Dimensional Geometry

In the practical life different objects do not lie in same plane but in space. If we are to locate any object in universe three co-ordinates are required. Hence, three co-ordinate axes namely X , Y & Z intersecting at a point, called origin are chosen in mutually perpendicular directions. In terms of 3 co-ordinates i.e. (X, Y, Z) any point in universe can be uniquely specified.

CO-ORDINATE AXES AND ORIGIN

Let XOX' , YOY' and ZOZ' are three mutually perpendicular lines intersecting at O . ' O ' is called origin of co-ordinate system.

$OX \rightarrow$ Positive direction of x -axis

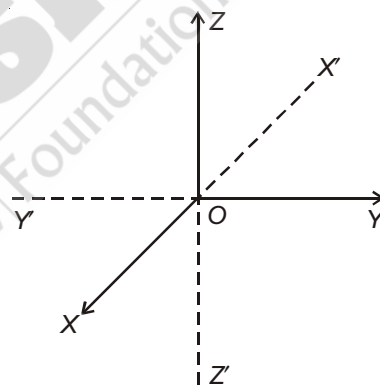
$OX' \rightarrow$ Negative direction of x -axis

$OY \rightarrow$ Positive direction of y -axis

$OY' \rightarrow$ Negative direction of y -axis

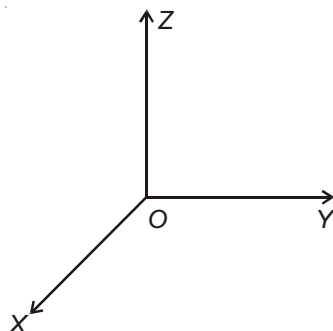
$OZ \rightarrow$ Positive direction of z -axis

$OZ' \rightarrow$ Negative direction of z -axis



COORDINATE PLANES

XOY is called xy -plane which is perpendicular to Z -axis. YOZ is called yz -plane which is perpendicular to X -axis. ZOX is called zx -plane which is perpendicular to Y -axis. These three planes divide the space into 8 octants, namely $XOYZ$, $X'OYZ$, $XOYZ'$, $X'OYZ'$, $XOY'Z$, $X'OY'Z$, $XOY'Z'$, & $X'OY'Z'$.



CO-ORDINATES OF A POINT AND THEIR SIGNS

A point $P(x, y, z)$ is such that

x = Perpendicular distance of point P from yz plane, with proper signs

y = Perpendicular distance of point P from zx plane, with proper signs

z = Perpendicular distance of point P from xy plane, with proper signs

The signs of coordinates would be taken as given below :

Octant → Co-ordinate ↓	OXYZ	OX'YZ	OXY'Z	OXYZ'	OX'Y'Z	OX'YZ'	OXY'Z'	OX'Y'Z'
x	+	−	+	+	−	−	+	−
y	+	+	−	+	−	+	−	−
z	+	+	+	−	+	−	−	−

When a point lies on

Co-ordinates

- | | |
|-------------------|-----------------------|
| (i) x – axis | $(\alpha, 0, 0)$ |
| (ii) y – axis | $(0, \beta, 0)$ |
| (iii) z – axis | $(0, 0, \gamma)$ |
| (iv) XY – plane | $(\alpha, \beta, 0)$ |
| (v) XZ – plane | $(\alpha, 0, \gamma)$ |
| (vi) YZ – plane | $(0, \beta, \gamma)$ |

DISTANCE FORMULA

Distance PQ in cartesian co-ordinates where P is (x_1, y_1, z_1) and Q is (x_2, y_2, z_2)

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \quad \dots (i)$$

In vector form

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\therefore |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad \text{same as } \dots(i)$$

Note : Distance of $P(x_1, y_1, z_1)$ from origin $O(0, 0, 0) = \sqrt{x_1^2 + y_1^2 + z_1^2}$

$$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$|\overrightarrow{OP}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

SECTION FORMULA**I. Internal division**

If point $R(x, y, z)$ divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m : n$ then

$$x = \frac{mx_2 + nx_1}{m + n}$$

$$y = \frac{my_2 + ny_1}{m+n}$$

$$z = \frac{mz_2 + nz_1}{m+n}$$

$$\therefore R = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$

In terms of vector; $\vec{OR} = \frac{m\vec{OQ} + n\vec{OP}}{m+n}$

II. External division

If $R(x, y, z)$ divides PQ externally in $m : n$ then

$$R = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n} \right)$$

In terms of vector; $\vec{OR} = \frac{m\vec{OQ} - n\vec{OP}}{m-n}$

Mid point of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Centroid of the Triangle

If (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) are the vertices of a triangle, then centroid is given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$

Centroid of a Tetrahedron

If (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then the centroid G of tetrahedron is given by

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right)$$

Direction-Cosines of a Line

If α, β, γ are the angles that a given line makes with the positive directions of the co-ordinate axes, then $\cos \alpha, \cos \beta, \cos \gamma$ are called the Direction-cosines (or d.c.'s) of the given line.

Usually the direction cosines of a line are denoted by l, m, n . Thus $[l, m, n]$ denote d.c.'s of a line, where $l^2 + m^2 + n^2 = 1$.

Cor. Direction-cosines of Co-ordinate Axes : The x -axis makes angles of $0^\circ, 90^\circ$ and 90° with axes of X, Y and Z respectively. Hence direction-cosines of X -axis are $[\cos 0^\circ, \cos 90^\circ, \cos 90^\circ]$, i.e., $[1, 0, 0]$.

Similarly d.c.'s of Y -axis are $[0, 1, 0]$ and those of Z -axis are $[0, 0, 1]$.

Position of a point by Radius Vector and Direction-Cosines

Let $P(x, y, z)$ be a point in the space and O the origin. Then length $OP = |\vec{r}|$ is the radius vector of the point P .

Also let $(\cos \alpha, \cos \beta, \cos \gamma)$ be the d.c.'s of OP . Drop PA perpendicular to OX . Then

$$x = OA = OP \cos \alpha = r \cos \alpha,$$

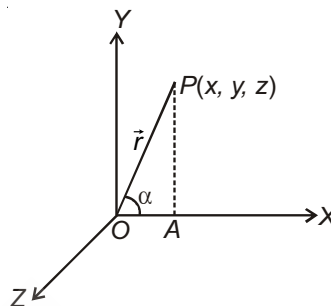
i.e., $x = lr$ if $\cos \alpha = l$.

Similarly $y = mr, z = nr$.

Hence co-ordinates of P are (lr, mr, nr) .

Note : In vector notations $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\text{Since } \cos \alpha = \frac{x}{r}, \cos \beta = \frac{y}{r} \text{ and } \cos \gamma = \frac{z}{r}.$$



Relation between Direction-Cosines

If l, m, n are the direction-cosines of a line then $l^2 + m^2 + n^2 = 1$.

Direction-Ratios : Quantities proportional to direction-cosines of a line are called direction-ratios of that line. Thus if l, m, n are direction-cosines and a, b, c direction-ratios of the line, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

as a, b, c are proportional to l, m, n respectively.

$$\text{Thus } l = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}} \text{ etc.}$$

Hence if a, b, c are direction-ratios of a line, then its direction-cosines are

$$\left(\frac{\pm a}{\sqrt{a^2 + b^2 + c^2}}, \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}}, \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}} \right)$$

Direction-cosines of line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

Let l, m, n be the direction-cosines of PQ ; then projection of PQ on X -axis = $PQ \cdot l$

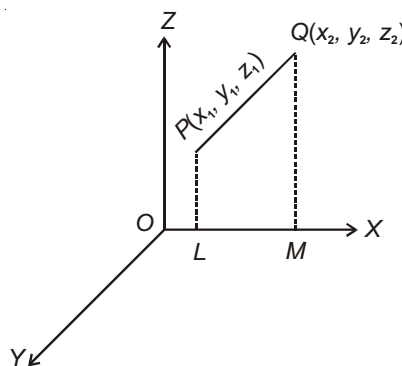
$$PQ \cdot l = x_2 - x_1$$

$$\therefore l = \frac{x_2 - x_1}{PQ}$$

$$\text{Similarly } m = \frac{y_2 - y_1}{PQ}$$

$$\text{and } n = \frac{z_2 - z_1}{PQ}$$

$$\therefore \frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ$$



Therefore direction-cosines of line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

EQUATION OF A STRAIGHT LINE

1. Vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to the vector $\vec{b}$ is

$$\vec{r} = \vec{a} + t\vec{b} \quad (\text{One Point Form})$$

2. **Cartesian Form :**

$$\text{Let } \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \quad \vec{b} = l\hat{i} + m\hat{j} + n\hat{k} \quad \text{and} \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{This gives } \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = t (t \text{ is parameter}).$$

3. Vector equation of a line through the point $A(\vec{a})$ and $B(\vec{b})$ is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ **(Two Points Form)**

4. If $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, then parametric form are

$$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1), \quad z = z_1 + t(z_2 - z_1)$$

i.e., $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$ which is called cartesian form of equation of the line passing through two given points.

ANGLE BETWEEN TWO LINES

The angle θ between the lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 is given by

$$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

Perpendicular and Parallel Lines

Two lines whose direction-cosines are $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$ are perpendicular if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad \text{and parallel if } \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are **coplanar** iff

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Shortest Distance between Two Skew Lines

1. The shortest distance between two non-parallel lines $\vec{r} = \vec{a}_1 + t\vec{b}_1$ and $\vec{r} = \vec{a}_2 + s\vec{b}_2$ is
$$\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}$$

Cor. These lines are coplanar iff $[\vec{a}_2 - \vec{a}_1, \vec{b}_1, \vec{b}_2] = 0$

2. The distance (i.e., the shortest distance) between two parallel lines $\vec{r} = \vec{a}_1 + t\vec{b}$ and $\vec{r} = \vec{a}_2 + s\vec{b}$ is

$$\frac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$$

THE PLANE

- Every equation of the first degree in x, y, z represents a plane i.e., $ax + by + cz + d = 0$
- $\vec{r} \cdot \hat{n} = P$ is the vector equation of a plane called normal form $\hat{n}$ is unit vector perpendicular to the plane and P distance of plane from origin.
- Intercept Form** : $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ represents a plane and intercepts on the three axes of co-ordinates are a, b, c respectively.

- Normal Form** : Let $ON = p$ be the perpendicular from O on the plane ABC . Also let $[l, m, n]$ be the d.c.'s of normal ON .

Take a point $P(x, y, z)$ on the plane.

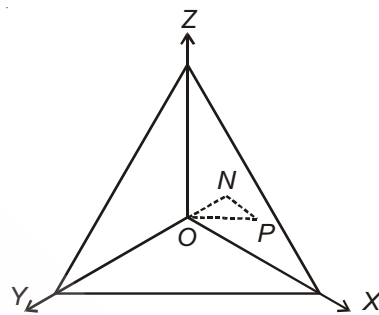
Clearly angle ONP is a right angle because ON is perpendicular to the plane and PN lies in the plane.

Also $ON =$ Projection of OP on a line ON

$$= lx + my + nz$$

But $ON = p$, Perpendicular distance.

Hence $lx + my + nz = p$ is the required equation of the plane.



- One-Point Form** : Vector equation of the plane through the point $A(\vec{a})$ and perpendicular to the direction $\vec{n}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$
- Equation of the plane passing through three non-collinear points (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \quad (\text{Three Point form})$$

Angle between Two Planes

- If θ is the angle between the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|}$$

where $\vec{n}_1$ and $\vec{n}_2$ are vectors along normals to the two planes.

- If $\vec{n}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

and $\vec{n}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- Two planes are perpendicular iff

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad \& \quad \text{Parallel if } \vec{n}_1 \times \vec{n}_2 = 0$$

Angle between a Line and a Plane

$$\sin \theta = \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2} \sqrt{l^2 + m^2 + n^2}},$$

where (a, b, c) are the direction ratios of a line perpendicular to the plane and (l, m, n) are the direction ratios of the line.

This gives the angle between the line and the plane.

1. If line is parallel to the plane, $\theta = 0$ (or the line is perpendicular to the normal AN),

$$\therefore al + bm + cn = 0$$

2. The line will be perpendicular to the plane, if it is parallel to the normal to the plane, i.e., $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$.

Perpendicular Distance of a Point from a Plane

1. **Plane in the normal form :**

Let the equation of the plane be

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p \quad \dots(1)$$

p being the perpendicular distance of the plane from origin

Any plane parallel to (1) is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p' \quad \dots(2)$$

It passes through (x_1, y_1, z_1)

$$\therefore x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma = p' \quad \dots(3)$$

Now perpendicular distance of the point (x_1, y_1, z_1) from (1) = $|p' - p|$

$$= |x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p|$$

2. **Plane given by general equation.** Let the equation of the plane be

$$ax + by + cz + d = 0$$

Changing it into normal form the equation becomes $\frac{ax + by + cz + d}{\pm \sqrt{a^2 + b^2 + c^2}} = 0 \quad \dots(1)$

Perpendicular distance of the (x_1, y_1, z_1) from (1)

$$= \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

PLANE THROUGH THE INTERSECTION OF TWO PLANES

1. Equation of any plane passing through the intersection of the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and

$$\vec{r} \cdot \vec{n}_2 = d_2 \text{ is } \vec{r} \cdot \vec{n}_1 - d_1 + \lambda(\vec{r} \cdot \vec{n}_2 - d_2) = 0$$

2. Equation of any plane through the intersection of the planes

$$P \equiv a_1x + b_1y + c_1z + d_1 = 0$$

$$\text{and } Q \equiv a_2x + b_2y + c_2z + d_2 = 0$$

Then the equation $P + \lambda Q = 0$

$$\text{i.e., } (a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

DISTANCE OF A POINT FROM PLANE

1. **In vector form:** Distance of point $P(\vec{r}_1)$ from plane $\vec{r} \cdot \vec{n} = d$ is $\left| \frac{\vec{r}_1 \cdot \vec{n} - d}{|\vec{n}|} \right|$
2. **In Cartesian form:** Distance of point $P(x_1, y_1, z_1)$ from plane $ax + by + cz + d = 0$ is $\left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$

DISTANCE BETWEEN TWO PARALLEL PLANES

1. Distance between $\vec{r} \cdot \vec{n} = d_1$ and $\vec{r} \cdot \vec{n} = d_2$ is $\left| \frac{d_1 - d_2}{|\vec{n}|} \right|$.
2. Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\left| \frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}} \right|$. Image and foot of perpendicular of point with respect to plane.

Image (α, β, γ) of point (x_1, y_1, z_1) w.r.t. plane $ax + by + cz + d = 0$ is given by

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = \frac{\gamma - z_1}{c} = -2 \left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2} \right)$$

Foot of perpendicular (p, q, r) of point (x_1, y_1, z_1) w.r.t. plane $ax + by + cz + d = 0$ is given by

$$\frac{p - x_1}{a} = \frac{q - y_1}{b} = \frac{r - z_1}{c} = - \left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2} \right)$$

