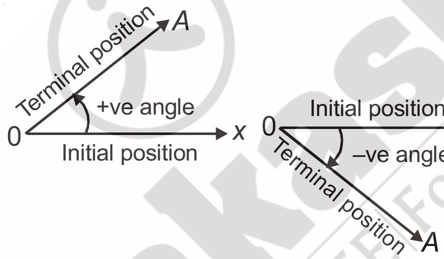
Chapter 8

Trigonometry

ANGLES AND THEIR MEASURES

Angle

A figure traced by rotating a given ray about its end point. The measure of angle is the amount of rotation performed from the initial side to the terminal side. Angle performed by anticlockwise rotation are taken as positive whereas angles formed by clockwise rotation are considered as negative.



Radian or Circular Measure

The angle subtended at the centre of a circle by an arc of the same circle whose length is equal to the radius of the circle is called 1 radian and is denoted by 1^c.

When no unit is mentioned with an angle, it is always understood to be in radian.

Radian measure and real numbers are same.

The ratio of circumference and diameter of a circle is always constant and denoted by Greek letter 'π'.

π is an irrational number, $\pi = \dfrac{\text{Circumference of Circle}}{\text{Diameter of circle}}$

Circumference = $2\pi r = \pi \times$ diameter

$\pi = \dfrac{22}{7}$ (approx) = 3.1415........

Arc-angle relation

Angle = $\dfrac{\text{arc}}{\text{radius}}$; Here angle is always in radian.

$$\theta = \frac{l}{r}$$

Angle subtended by a very small arc is approximately calculated by

$$\theta = \frac{\text{Chord } AB}{r}$$

$\therefore$ Arc AB $\approx$ chord AB for small angle θ

Relation between degree and radian :

$$\pi^c = 180°$$

$$1^c = \left(\frac{180}{\pi}\right)^o$$

$$1° = \left(\frac{\pi}{180}\right)^c$$

Relation between the sides and interior angles of a polygon

(a) Sum of interior angle of polygon of n sides = $(2n - 4) \times 90° = (n - 2)\pi^c$

(b) Each interior angle of a regular polygon of n sides = $\dfrac{(2n-4)90°}{n} = \dfrac{(n-2)}{n}\pi^c$

TRIGONOMETRIC FUNCTIONS (T-RATIOS)

Trigonometric Functions

(i) $\sin\theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{MP}{OP}$

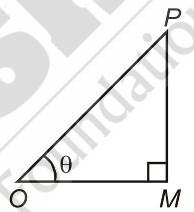
(ii) $\cos\theta = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{OM}{OP}$

(iii) $\tan\theta = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{MP}{OM}$

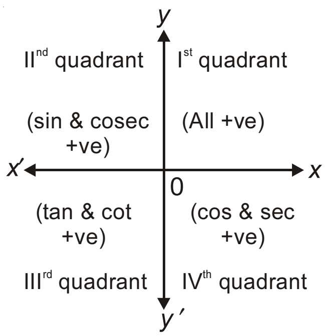
(iv) $\cot\theta = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{OM}{MP}$

(v) $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{OP}{OM}$

(vi) $\text{cosec}\,\theta = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{OP}{MP}$



Signs of T-Ratios

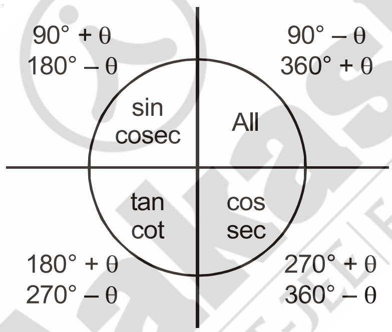


Domain and Range of Trigonometric Functions

Function	Domain	Range
$\sin\theta$	R	$[-1, 1]$
$\cos\theta$	R	$[-1, 1]$
$\tan\theta$	$R \sim \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	R
$\cot\theta$	$R \sim \{n\pi : n \in I\}$	R
$\sec\theta$	$R \sim \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	$(-\infty, -1] \cup [1, \infty)$
$\text{cosec}\,\theta$	$R \sim \{n\pi : n \in I\}$	$(-\infty, -1] \cup [1, \infty)$

Allied Angle

If θ is any angle then, $-\theta$, $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° \pm \theta$ etc. are called as allied angles of θ.



1. To find the sign (+ or –)

   Use the original ratio to find '+ or –' sign (to be affixed) making use of the quadrant rule.

   Thus, ratios shown inside the circle are positive in the corresponding quadrant while other ratios are negative there

2. To find the final ratio

   (a) If π, 2π etc. are present, then there is no change ; *i.e.*, sin remains sin ; cos remains cos etc.

   (b) If $\frac{\pi}{2}, \frac{3\pi}{2}$ are present, then, there is a change as given below :

   sin $\rightleftharpoons$ cos

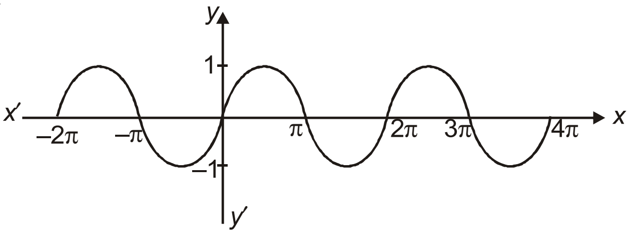
   tan $\rightleftharpoons$ cot

   cosec $\rightleftharpoons$ sec

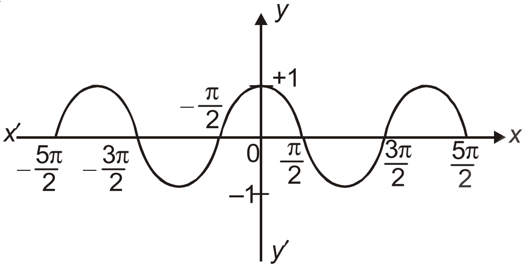
	$-\theta$	$90° - \theta$	$90° + \theta$	$180° - \theta$	$180° + \theta$	$270° - \theta$	$270° + \theta$	$360° - \theta$	$360° + \theta$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

Graphs of standard T-Functions

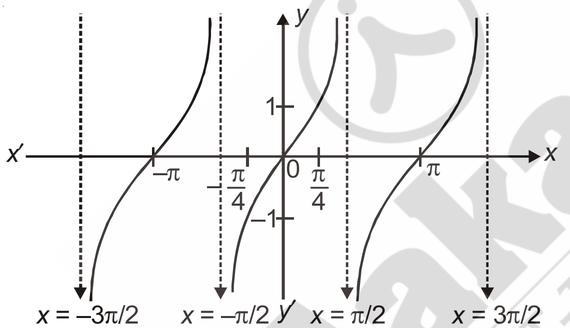
$y = \sin x$



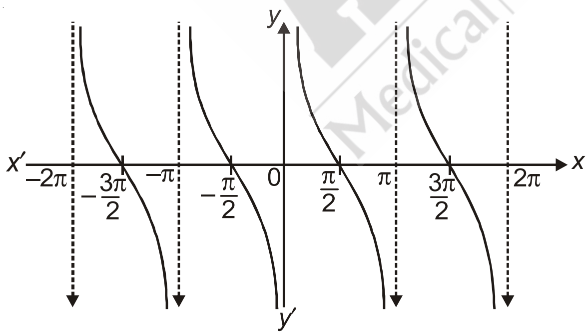
$y = \cos x$



$y = \tan x$



$y = \cot x$



TRIGONOMETRIC IDENTITIES

Fundamental Identities

(i) $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta \Rightarrow \cos^2\theta = 1 - \sin^2\theta$

(ii) $1 + \tan^2\theta = \sec^2\theta \Rightarrow \sec^2\theta - \tan^2\theta = 1$

(iii) $1 + \cot^2\theta = \mathrm{cosec}^2\theta \Rightarrow \mathrm{cosec}^2\theta - \cot^2\theta = 1$

Also note the range within which different trigonometric functions lie

(1) $-1 \le \sin\theta \le 1$; $|\sin\theta| \le 1$

(2) $-1 \le \cos\theta \le 1$; $|\cos\theta| \le 1$

(3) $0 \le \sin^2\theta \le 1$; $0 \le \cos^2\theta \le 1$

(4) $\operatorname{cosec}\theta \le -1$ or $\operatorname{cosec}\theta \ge 1$

(5) $\sec\theta \le -1$ or $\sec\theta \ge 1$

(6) $0 < \cos A < \dfrac{\sin A}{A} < \dfrac{1}{\cos A}$; $0 < A < \dfrac{\pi}{2}$

(7) If θ is very small then $\sin\theta \simeq \theta$

***Note :** Each trigonometric ratio can be expressed in terms of all other T-ratios e.g.,*

$$\sin\theta = \frac{\pm 1}{\sqrt{1+\cot^2\theta}};\ \cos\theta = \frac{\pm\cot\theta}{\sqrt{1+\cot^2\theta}}$$

$$\tan\theta = \frac{1}{\cot\theta};\ \sec\theta = \frac{\pm\sqrt{1+\cot^2\theta}}{\cot\theta}$$

$$\operatorname{cosec}\theta = \pm\sqrt{1+\cot^2\theta}$$

Addition and Subtraction Formulae (Compound Angle)

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
3. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
4. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
5. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
7. $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$
8. $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$
9. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
10. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
11. $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

    or

    $= \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$
12. $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

    or

    $= \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$

13. $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

14. $\tan\left(\dfrac{\pi}{4} + A\right) = \dfrac{1 + \tan A}{1 - \tan A}$

15. $\tan\left(\dfrac{\pi}{4} - A\right) = \dfrac{1 - \tan A}{1 + \tan A}$

Two special series :

1. $\sin(\alpha) + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots\ldots \sin(\alpha+(n-1)\beta)$

$$= \frac{\sin\left[\alpha + (n-1)\left(\frac{\beta}{2}\right)\right].\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

2. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots\ldots + \cos(\alpha + (n-1)\beta)$

$$= \frac{\cos\left[\alpha + (n-1)\left(\frac{\beta}{2}\right)\right].\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Note : *Note that the angles are in A.P.*

TRANSFORMATION FORMULAE

Product into Sum and Difference

1. $2\sin A \cos B = \sin(A + B) + \sin(A - B)$, $A > B$
2. $2\cos A \sin B = \sin(A + B) - \sin(A - B)$, $A > B$
3. $2\cos A \cos B = \cos(A + B) + \cos(A - B)$
4. $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

Sum and Difference into products

1. $\sin C + \sin D = 2\sin\left(\dfrac{C + D}{2}\right)\cos\left(\dfrac{C - D}{2}\right)$

2. $\sin C - \sin D = 2\cos\left(\dfrac{C + D}{2}\right)\sin\left(\dfrac{C - D}{2}\right)$

3. $\cos C + \cos D = 2\cos\left(\dfrac{C + D}{2}\right)\cos\left(\dfrac{C - D}{2}\right)$

4. $\cos C - \cos D = -2\sin\left(\dfrac{C + D}{2}\right)\sin\left(\dfrac{C - D}{2}\right)$

5. $\tan C + \tan D = \dfrac{\sin(C+D)}{\cos C \cos D}$

6. $\tan C - \tan D = \dfrac{\sin(C-D)}{\cos C \cos D}$

7. $\cot C + \cot D = \dfrac{\sin(C+D)}{\sin C \sin D}$

8. $\cot C - \cot D = \dfrac{\sin(D-C)}{\sin C \sin D}$

TRIGONOMETRIC RATIOS OF MULTIPLE AND SUBMULTIPLE ANGLES

T-Ratios of Multiple Angles : (An angle of the form $n\theta$, $n \in I$)

1. $\sin 2A = 2 \sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$

2. $\cos 2A = \cos^2 A - \sin^2 A$
   $= 2\cos^2 A - 1$
   $= 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

   Thus,
   $1 + \cos 2A = 2\cos^2 A$
   $1 - \cos 2A = 2\sin^2 A$

3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$, $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$

4. (i) $\sin 3A = 3\sin A - 4\sin^3 A$

   (ii) $\cos 3A = 4\cos^3 A - 3\cos A$

   (iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$, $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

5. $\cos A \cos 2A \cos 2^2 A \ldots\ldots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

T-Ratios of Submultiple Angles

$\left(\text{An angle of the from } \dfrac{\theta}{n}, n \in I\right)$

1. $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = \dfrac{2\tan\theta/2}{1+\tan^2\theta/2}$

2. $\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2} = \dfrac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

3. $\tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$

4. $\cot\theta = \dfrac{\cot^2\dfrac{\theta}{2}-1}{2\cot\theta/2}$

5. $\cos^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{2}$

6. $\sin^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{2}$

7. $\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

8. $\cot^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{1-\cos\theta}$

9. $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$

10. $\dfrac{1+\cos\theta}{\sin\theta} = \cot\dfrac{\theta}{2}$

T-Ratio of some special angles

(i) $\sin 15° = \cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15° = \sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15° = \cot 75° = 2-\sqrt{3}$

(iv) $\cot 15° = \tan 75° = 2+\sqrt{3}$

(v) $\sin 18° = \cos 72° = \dfrac{\sqrt{5}-1}{4}$

(vi) $\cos 18° = \sin 72° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

(vii) $\sin 36° = \cos 54° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

(viii) $\cos 36° = \sin 54° = \dfrac{\sqrt{5}+1}{4}$

(ix) $\tan 22\tfrac{1}{2}° = \sqrt{2}-1$

(x) $\cot 22\tfrac{1}{2}° = \sqrt{2}+1$

Remember :

(i) $\left|\sin\dfrac{A}{2}+\cos\dfrac{A}{2}\right| = \sqrt{1+\sin A}$

or $\sin\dfrac{A}{2}+\cos\dfrac{A}{2} = \pm\sqrt{1+\sin A}$ *i.e.*, $\begin{cases} +, \text{if } 2n\pi-\dfrac{\pi}{4}\le\dfrac{A}{2}\le 2n\pi+\dfrac{3\pi}{4} \\ -, \text{otherwise}\end{cases}$

(ii) $\left|\sin\frac{A}{2}-\cos\frac{A}{2}\right|=\sqrt{1-\sin A}$

or $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)=\pm\sqrt{1-\sin A}$; *i.e.*, $\begin{cases}+, \text{ if } 2n\pi-\frac{\pi}{4}\le\frac{A}{2}\le 2n\pi+\frac{5\pi}{4}\\ -, \text{ otherwise}\end{cases}$

GREATEST AND LEAST VALUES OF *a*cosθ + *b*sinθ

$S = a\cos\theta + b\sin\theta$

$$= r\left(\frac{a}{r}\cos\theta+\frac{b}{r}\sin\theta\right) \quad : \quad r=\sqrt{a^2+b^2}$$

$$= r\left(\sin(\theta+\alpha)\right) \quad ; \sin\alpha=\frac{a}{r} \quad ; \quad \cos\alpha=\frac{b}{r}$$

Since, $-1\le\sin(\theta+\alpha)\le 1$, therefore, $-r\le S\le r$

CONDITIONAL IDENTITIES

When, three angles *A, B, C* satisfy some given relation, several identities can be established connecting the trigonometric ratios of these angles

In a triangle *ABC*, $A + B + C = \pi$;

$\therefore$ $\sin(A + B) = \sin(\pi - C) = \sin C$

and $\cos(A + B) = \cos(\pi - C) = -\cos C$

Also, $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$;

Hence, $\sin\left(\frac{A+B}{2}\right)=\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cos\frac{C}{2}$

$\cos\left(\frac{A+B}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)=\sin\frac{C}{2}$

Remember :

If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(iii) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iv) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

(vi) $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

(vii) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(viii) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

TRIGONOMETRIC EQUATIONS

An equation involving the trigonometric ratios of an unknown angle is called a trigonometric equation. *e.g.* $\sin x = 0$; $\sin x + \cos x = 1$, etc. The equation is not completely solved unless we obtain an expression for all the angles which satisfy it. Since, all trigonometric ratios are periodic in nature, a trigonometric equation has, in general, an infinite number of solutions.

We note that $\sin \theta = 0$; $\sin \pi = 0$; $\sin(-\pi) = 0$; $\sin 2\pi = 0$ and so on.

$\therefore$ In general, $\sin(n\pi) = 0 \ : \ n \in I$,

where I denotes the set of integers.

Also, $\cos\frac{\pi}{2} = 0$; $\cos\left(-\frac{\pi}{2}\right) = 0$; $\cos\frac{3\pi}{2} = 0$; $\cos\left(-\frac{3\pi}{2}\right) = 0$; $\cos\frac{5\pi}{2} = 0$ and so on.

Thus, in general, $\cos(2n+1)\frac{\pi}{2} = 0 \ : \ n \in I$.

GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

Since trigonometric functions are periodic functions, therefore, solutions of trigonometric equations can be generalised with the help of periodicity of trigonometric functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Thus, a solution generalised by means of periodicity is known as the **General Solution**.

eg., General solution of the equation $\sin \theta = 0$ is $n\pi$. where n is 0 or any positive or negative integer.

General solutions of equation of the forms $\sin \theta = \sin \alpha$, $\cos \theta = \cos \alpha$, $\tan \theta = \tan \alpha$

(i) $\sin \theta = \sin \alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$; $n \in I$; $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos \theta = \cos \alpha \Rightarrow \theta = 2n\pi \pm \alpha$; $n \in I$; $\alpha \in [0, \pi]$

(iii) $\tan \theta = \tan \alpha \Rightarrow \theta = n\pi + \alpha$; $n \in I$; $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

where α is the numerically smallest (positive or negative) angle satisfying the given equation. α is called principle value.

General solution of equations of the forms $\sin \theta = 0$, $\cos \theta = 0$, $\tan \theta = 0$

$\sin \theta = 0 \Rightarrow \theta = n\pi, n \in I$

$\cos \theta = 0 \Rightarrow \theta = (2n + 1)\pi/2, n \in I$

$\tan \theta = 0 \Rightarrow \theta = n\pi, n \in I$

General solution of equations of the form $\sin \theta = 1$

$$\left.\begin{aligned} \sin\theta = 1 &\Rightarrow \theta = (4n+1)\pi/2 \\ \cos\theta = 1 &\Rightarrow \theta = 2n\pi \\ \cos\theta = -1 &\Rightarrow \theta = (2n+1)\pi \end{aligned}\right\}, n \in I$$

$$\left.\begin{aligned} &\text{if } \sin\theta = \sin\alpha \\ &\& \cos\theta = \cos\alpha \end{aligned}\right\} \Rightarrow \theta = 2n\pi + \alpha, \ n \in I$$

General solution of equations of the form $\sin^2\theta = \sin^2\alpha$

$$\left.\begin{aligned}\sin^2\theta &= \sin^2\alpha\\ \cos^2\theta &= \cos^2\alpha\\ \tan^2\theta &= \tan^2\alpha\end{aligned}\right\} \Rightarrow \theta = n\pi \pm \alpha,\ n \in I$$

Solution of an equation of the form $a\cos\theta + b\sin\theta = C$

$a\cos\theta + b\sin\theta = C$

(i) Divide the equation by $\sqrt{a^2+b^2}$

(ii) Let $\dfrac{a}{\sqrt{a^2+b^2}} = \cos\alpha$ and $\dfrac{b}{\sqrt{a^2+b^2}} = \sin\alpha$ (α is least positive angle satisfying the equations)

(iii) Equation become $\cos(\theta-\alpha) = \dfrac{C}{\sqrt{a^2+b^2}} = \cos\beta$

$\therefore$ $\theta - \alpha = 2n\pi \pm \beta$, so $\theta = 2n\pi \pm \beta + \alpha$; $n \in I$

if $|C| > \sqrt{a^2+b^2}$; no real solution exists.

Some Important Precautions

1. While solving a trigonometric equation, squaring the equation at any step should be avoided as far as possible. If squaring is necessary, check the solution for extraneous values.
2. Never cancel terms containing unknown terms on the two sides, which are in product. It may cause loss of the genuine solution.
3. The answer should not contain such values of angles which make any of the terms undefined or infinite.
4. Domain should not change. If it changes, necessary corrections must be made.
5. Check that denominator is not zero at any stage while solving equations.
6. If $\tan\theta$ or $\sec\theta$ is involved in the equation, θ should not be an odd multiple of $\pi/2$.
7. If $\cot\theta$ or $\operatorname{cosec}\theta$ is involved in the equation, θ should not be a multiple of π or 0.
8. Note that $\sqrt{f(\theta)}$ is always positive. For example,

   $\sqrt{\cos^2\theta} = |\cos\theta|$ and not $\pm\cos\theta$.

SOLUTION OF SIMULTANEOUS EQUATIONS

I. Two Equations in One Unknown

II. Two Equations in Two Unknowns

For most general values

$$\tan(A-B) = 1 \Rightarrow A - B = n\pi + \frac{\pi}{4} \qquad \ldots\text{(i)}$$

$$\sec(A+B) = \frac{2}{\sqrt{3}} \Rightarrow A + B = 2m\pi \pm \frac{\pi}{6} \qquad \ldots\text{(ii)}$$

Solving (i) & (ii), we get

$$A = \frac{1}{2}\left[(2m+n)\pi + \frac{\pi}{4} \pm \frac{\pi}{6}\right]$$

$$B = \frac{1}{2}\left[(2m-n)\pi - \frac{\pi}{4} \pm \frac{\pi}{6}\right] m, n \in \text{I}$$

INVERSE FUNCTION

Definition

If a function is one to one and onto from A to B, then function g which associates each element $y \in B$ to one and only one element $x \in A$, such that $y = f(x)$, then g is called the inverse function of f, denoted by $x = g(y)$.

Usually we denote $g = f^{-1}$ {Read as f inverse}

$\therefore \quad x = f^{-1}(y)$.

Inverse Trigonometric Functions

We have seen that the trigonometric functions, sin, cos etc. are all periodic and thus, each of them achieves the same numerical value at an infinite number of points. Thus, the equation $\sin x = \frac{1}{2}$ has an infinite number of solutions, *viz.*, $x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$ etc. If one is to answer the question : "What is the angle whose sine is $\frac{1}{2}$?", there is no unique answer. The difficulty arises as the function $f : R \to R$ defined by $f(x) = \sin x$ is not one to one and thus, does not admit of an inverse. To achieve a unique answer to the above said question, we restrict the domain of $\sin x$ so that the resulting function is invertible. Thus, the function $g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by $g(x) = \sin x$ is one to one and onto and admits of an inverse (denoted by $h = \sin^{-1}$ and read as sine inverse or arc sin) defined as $h : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where $h(y) = x$ if $y = \sin x$. The function $\sin^{-1}$ is the inverse of the sine function when the sine function is viewed in a restricted sense.

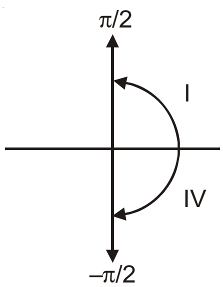
We similarly define the other inverse trigonometric functions.

Important Points

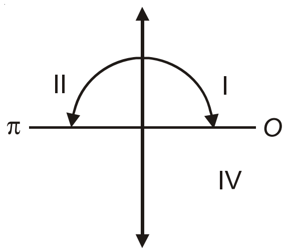
1. $\sin^{-1}x$ is an angle and denotes the smallest numerical angle, whose sine is x.
2. If there are two angles one positive and the other negative having same numerical value. Then we shall take the positive value.

Intervals for Inverse Functions

Here, $\sin^{-1}x$, $\text{cosec}^{-1}x$, $\tan^{-1}x$ belongs to I and IV quadrant.



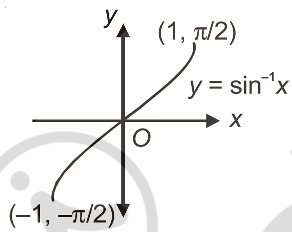
Here, $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ belongs to I and II quadrant.



1. I quadrant is common to all the inverse functions.
2. III quadrant is not used in inverse function.
3. IV quadrant is used in the clockwise direction *i.e.,* $-\pi/2 \le y \le 0$.

DOMAIN, RANGE AND GRAPHS OF INVERSE FUNCTIONS

1. If $\sin y = x$, then $y = \sin^{-1}x$, under certain condition.



$-1 \le \sin y \le 1$; but $\sin y = x$.

$\therefore$ $-1 \le x \le 1$

Again, $\sin y = -1 \Rightarrow y = -\pi/2$

and $\sin y = 1 \Rightarrow y = \pi/2$

Keeping in mind numerically smallest angles or real numbers.

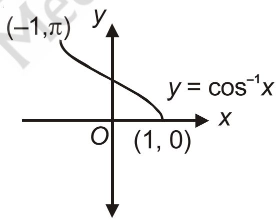
$\therefore$ $-\pi/2 \le y \le \pi/2$

These restrictions on the values of x and y provide us with the domain and range for the function $y = \sin^{-1}x$.

i.e., Domain : $x \in [-1, 1]$

Range : $y \in [-\pi/2, \pi/2]$

2. Let $\cos y = x$ then $y = \cos^{-1}x$, under certain condition $-1 \le \cos y \le 1$.



$\Rightarrow$ $-1 \le x \le 1$

$\cos y = -1 \Rightarrow y = \pi$

$\cos y = 1 \Rightarrow y = 0$

$\therefore$ $0 \le y \le \pi$ {as $\cos x$ is a decreasing function in $[0, \pi]$;

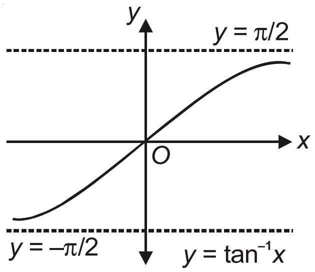
hence $\cos\pi \le \cos y \le \cos 0$}

These restrictions on the values of x and y provide us the domain and range for the function $y = \cos^{-1}x$.

i.e., Domain : $x \in [-1, 1]$

Range : $y \in [0, \pi]$

3. If $\tan y = x$ then $y = \tan^{-1}x$, under certain conditions.



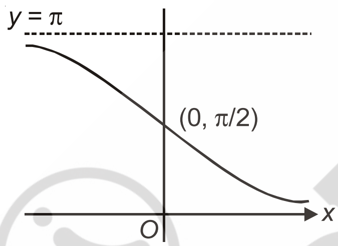
Here, $\tan y \in R \Rightarrow x \in R$

$-\infty < \tan y < \infty \Rightarrow -\pi/2 < y < \pi/2$

Thus, domain $x \in R$

Range $y \in (-\pi/2, \pi/2)$

4. If $\cot y = x$, then $y = \cot^{-1}x$ (under certain conditions)



$\cot y \in R \Rightarrow x \in R$;

$-\infty < \cot y < \infty \Rightarrow 0 < y < \pi$

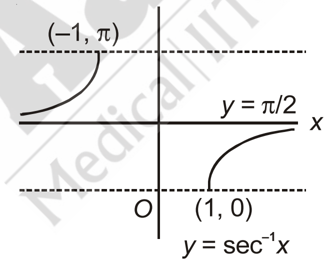
These conditions on x and y make the function, $\cot y = x$ one-one and onto so that the inverse function exists.

i.e., $y = \cot^{-1}x$ is meaningful.

i.e., Domain : $x \in R$

Range : $y \in (0, \pi)$

5. If $\sec y = x$, then $y = \sec^{-1}x$, where $|x| \geq 1$ and $0 \leq y \leq \pi$, $y \neq \pi/2$

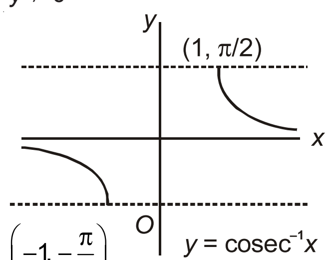


Here, Domain : $x \in R - (-1, 1)$

Range : $y \in [0, \pi] - \{\pi/2\}$

6. If $\operatorname{cosec} y = x$ then $y = \operatorname{cosec}^{-1}x$,

where $|x| \geq 1$ and $-\pi/2 \leq y \leq \pi/2$, $y \neq 0$



$\left(-1, -\frac{\pi}{2}\right)$

Here, domain $\in R - (-1, 1)$

Range $\in [-\pi/2, \pi/2] - \{0\}$

Function	Domain	Codomain = Range
$\sin^{-1}$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	R	$(0, \pi)$
$\sec^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
cosec^{-1}	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Thus, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$; $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ etc.

$\sin\frac{5\pi}{6} = \frac{1}{2}$ but $\left(\frac{5\pi}{6}\right) \neq \sin^{-1}\frac{1}{2}$

$\sin^{-1}\left(\frac{1}{2}\right) = \text{arc}\sin\frac{1}{2} = \frac{\pi}{6}$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property–I

(i) $\sin^{-1}(\sin\theta) = \theta$ only if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos\theta) = \theta$ only if $0 \leq \theta \leq \pi$

(iii) $\tan^{-1}(\tan\theta) = \theta$ only if $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\cot^{-1}(\cot\theta) = \theta$ only if $\theta \in (0, \pi)$

(v) $\sec^{-1}(\sec\theta) = \theta$ only if $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

(vi) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$ only if $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Property-II

(i) $\sin(\sin^{-1}x) = x$: $-1 \leq x \leq 1$

(ii) $\cos(\cos^{-1}x) = x$: $-1 \leq x \leq 1$

(iii) $\tan(\tan^{-1} x) = x$: $x \in R$

(iv) $\cot(\cot^{-1} x) = x$: $x \in R$

(v) $\sec(\sec^{-1} x) = x$: $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\text{cosec}\,(\text{cosec}^{-1} x) = x$: $x \in (-\infty, -1] \cup [1, \infty)$

Property-III

(i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, for all $x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$

(iv) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

Property-IV

(i) $\text{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$ if $-1 \le x < 0$ or $0 < x \le 1$

(ii) $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$ if $-1 \le x < 0$ or $0 < x \le 1$

(iii) $\cot^{-1} x = \tan^{-1}\frac{1}{x}$ if $x > 0$

(iv) $\cot^{-1} x = \pi + \tan^{-1}\left(\frac{1}{x}\right)$ if $x < 0$

(v) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, \text{ for } x > 0 \\ -\pi + \cot^{-1} x, \text{ for } x < 0 \end{cases}$

Property V

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$: $-1 \le x \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$: $x \in R$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$ if $x \le -1$ or $x \ge 1$

Property VI

(i) $$\sin^{-1}x+\sin^{-1}y=\begin{cases}-\pi-\left[\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right] & \text{if} \quad x,y<0, x^2+y^2>1\\ \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) & \text{if} \quad x^2+y^2\le 1 \text{ or } xy<0 \text{ and } x^2+y^2>1\\ \pi-\left[\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right] & \text{if} \quad x,y>0, x^2+y^2>1\end{cases}$$

(ii) $$\sin^{-1}x-\sin^{-1}y=\begin{cases}-\pi-\left[\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\right] & \text{if} \quad x>0, y<0, x^2+y^2>1\\ \sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right) & \text{if} \quad x^2+y^2\le 1 \text{ or } xy>0 \text{ and } x^2+y^2>1\\ \pi-\left[\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\right] & \text{if} \quad x<0, y>0, x^2+y^2>1\end{cases}$$

(iii) $$\tan^{-1}x+\tan^{-1}y=\begin{cases}\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if} \quad x,y>0 \text{ and } xy>1\\ \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if} \quad xy<1\\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if} \quad x,y<0 \text{ and } xy>1\end{cases}$$

(iv) $$\tan^{-1}x-\tan^{-1}y=\begin{cases}\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if} \quad x>0, y<0 \text{ and } xy<-1\\ \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if} \quad xy>-1\\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if} \quad x<0, y>0 \text{ and } xy<-1\end{cases}$$

There is no need to memorize all the formulae given in different books. To arrive at the answer one must keep in mind the range of inverse trigonometric functions. We, however, list some widely used formulae for easy reference. The formulae for inverse cosine (cotangent) can be obtained from those for inverse sine (tangent).

Remark :

If $x_1, x_2, x_3, \ldots, x_n \in R$, then

$$\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{s_1-s_3+s_5\ldots}{1-s_2+s_4-s_6\ldots}\right)$$

where s_k denotes the sum of the product of $x_1, x_2, \ldots, x_n$ taken k at a time.

Property-VII

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$= \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\dfrac{1}{x}\right)$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

$= \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\dfrac{1}{x}\right) = \text{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

$= \cot^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right) = \text{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

***Note :** Transformation of one inverse into another inverse exists iff domain satisfies.*

Property-VIII

(i) $$2\sin^{-1} x = \begin{cases} \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } \dfrac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } -1 \le x \le -\dfrac{1}{\sqrt{2}} \end{cases}$$

(ii) $$3\sin^{-1} x = \begin{cases} \sin^{-1}(3x - 4x^3), & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - \sin^{-1}(3x - 4x^3), & \text{if } \dfrac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x - 4x^3), & \text{if } -1 \le x < -\dfrac{1}{2} \end{cases}$$

(iii) $$2\cos^{-1} x = \begin{cases} \cos^{-1}(2x^2 - 1); & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2 - 1); & \text{if } -1 \le x \le 0 \end{cases}$$

(iv) $$3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3 - 3x), & \text{if } \dfrac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3 - 3x), & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 2\pi + \cos^{-1}(4x^3 - 3x), & \text{if } -1 \le x \le -\dfrac{1}{2} \end{cases}$$

(v) $$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } -1<x\le 1\\ \pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } x>1\\ -\pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } x<-1\end{cases}$$

(vi) $$3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); & \text{if } -\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); & \text{if } x>\dfrac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); & \text{if } x<\dfrac{-1}{\sqrt{3}}\end{cases}$$

(vii) $$2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } -1\le x\le 1\\ \pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x>1\\ -\pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x<-1\end{cases}$$

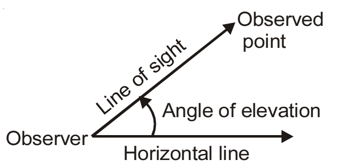
(viii) $$2\tan^{-1}x=\begin{cases}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } 0\le x<\infty\\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } -\infty<x\le 0\end{cases}$$

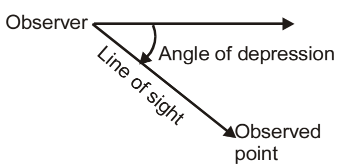
HEIGHT AND DISTANCE

Trigonometry is also used in land surveying and astronomy. Here the distances and heights are measured indirectly, which cannot be measured directly. The convenient parameters are measured and with the help of trigonometry the required parameters are calculated.

ANGLE OF ELEVATION AND DEPRESSION

It is the angle that the line of sight makes with horizontal direction.

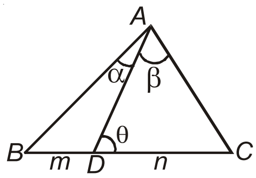




SOME USEFUL RESULTS FROM TRIGONOMETRY

(i) *m* : *n* theorem :

In any triangle ABC if D divides BC internally in the ratio of $m : n$, as shown in figure,



then

$$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$$

or $$(m + n)\cot\theta = n\cot B - m\cot C$$

(ii) In a right angled triangle :

(a) Mid point of hypotenuse is equidistant from the three vertices and it is the circumcentre of triangle.

(b) The vertex containing the right angle is the orthocentre.

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