

Chapter 14

Vector Algebra

Vector algebra deals with addition / subtraction / product of vector quantities. Application of vectors to many geometrical problems cuts short the procedure. Hence geometrical significance of vectors should be well understood.

VECTORS AND SCALARS

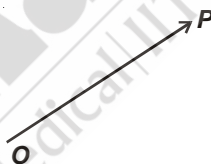
The physical quantities (we deal with) are generally of two types:

Scalar Quantity: A quantity which has magnitude but no sense of direction is called Scalar quantity (or scalar), e.g., mass, volume, density, speed etc.

Vector Quantity: A quantity which has magnitude as well as a sense of direction in space is called a vector quantity, e.g., velocity, force, displacement etc.

NOTATION AND REPRESENTATION OF VECTORS

Vectors are represented by $\vec{a}, \vec{b}, \vec{c}$ and their magnitude (modulus) is represented by a, b, c , or $|\vec{a}|, |\vec{b}|, |\vec{c}|, \dots$.
The vectors are represented by directed line segments.



For example, line segment $\overrightarrow{OP}$ represents a vector with magnitude OP (length of line segment), arrow denotes its direction. O is initial point and P is terminal point.

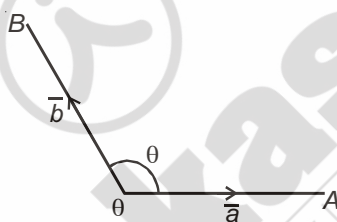
SOME SPECIAL VECTORS

1. **Null vectors:** A vector with zero magnitude and indeterminate direction, denoted by $\vec{0}$ is called null vector or zero vector.
2. **Unit vector:** A vector with unit magnitude (one unit), denoted by $\hat{a}$ where $|\hat{a}| = 1$ unit is called unit vector.
3. **Equal vectors:** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have same sense of direction and $|\vec{a}| = |\vec{b}|$, denoted by $\vec{a} = \vec{b}$.
4. **Like and unlike vectors:** Vectors having same sense of directions are called Like vectors and opposite sense of directions are called Unlike vectors.

5. **Negative of a vector:** Negative of a vector $\vec{a}$, denoted by $-\vec{a}$, is a vector whose magnitude is $|\vec{a}|$ and direction is opposite of $\vec{a}$.
6. **Collinear vectors:** Vectors whose lines of action act along a line or parallel to a given line are called collinear vectors.
7. **Parallel vectors:** Vectors having same line of action or are parallel to a fixed straight lines are called parallel vectors.
8. **Coplanar vectors:** The vectors which lie in the same plane. At least three coplanar unequal vectors are required to make the sum zero and at least four if non-coplanar.
9. **Free vectors:** A vector not restricted to pass through a fixed point is called free vectors.
10. **Localized vectors:** A vector restricted to pass through a fixed point is called localized vectors.
11. **Co-initial vectors:** Vectors having same initial point are called co-initial vectors.
12. **Position vectors:** Let O be fixed point in space, then vector $\vec{OP}$ (P is any point in space) is called position vector of P w.r.t. O . If A and B are any two points in space then

$$\vec{AB} = \text{p.v. of } B - \text{p.v. of } A = \vec{OB} - \vec{OA}.$$

ANGLE BETWEEN TWO VECTORS



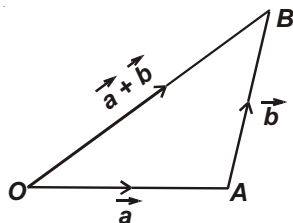
The angle between two vectors $\vec{a}$ and $\vec{b}$ represented by $\vec{OA}$ and $\vec{OB}$ is $\angle AOB = \theta$

$$0 \leq \theta \leq \pi$$

If $\theta = \frac{\pi}{2}$, then vectors are called orthogonal or perpendicular vectors

if $\theta = 0$ or π then vectors are called parallel or coincident vectors.

ADDITION OF VECTORS



Let $\vec{a}$ and $\vec{b}$ be two vectors. Draw $\vec{OA}$ representing vector $\vec{a}$. Taking terminal point of $\vec{a}$ as initial point of vector $\vec{b}$, draw $\vec{AB}$ representing vector $\vec{b}$. Then vector $\vec{OB}$ is called sum of vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} + \vec{b}$ (triangle law of addition)

Triangle Law of Addition

Three vectors are in equilibrium if represented by, three sides of a closed triangle taken in order, (in magnitude and direction).

Converse of triangle law is also true.

Law of polygons : If several vectors, when added, form a closed polygon, their resultant is zero.

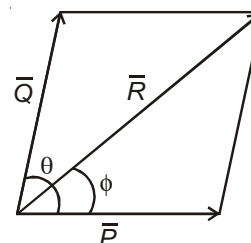
Parallelogram law of addition – If two vectors are represented in magnitude and direction, by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction, by the co-initial diagonal of that parallelogram.

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta} \quad \text{and} \quad \tan \phi = \frac{Q \sin \theta}{P + Q \cos \theta}$$

If $\theta = 0^\circ$ then $R = P + Q$ (maximum)

$\theta = \pi$ then $R = |P - Q|$ (minimum)

If $\theta = \frac{\pi}{2}$ then $R = \sqrt{P^2 + Q^2}$ and $\tan \phi = \frac{Q}{P}$



Properties of Vector Addition

1. Vector addition is commutative.

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

2. Vector addition is associative

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

3. $\vec{0}$ or null vector is additive identity.

$$\vec{A} + \vec{0} = \vec{A} = \vec{0} + \vec{A}$$

4. Additive inverse of $\vec{a}$ is $(-\vec{a})$, Since

$$\vec{a} + (-\vec{a}) = \vec{0}$$

SUBTRACTION OF TWO VECTORS

The subtraction of two vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

i.e. the vector to be subtracted is reversed and added to other vector.

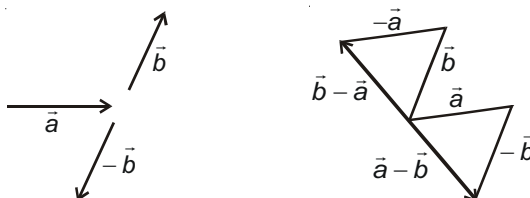
$$\vec{a} - \vec{b} \neq \vec{b} - \vec{a} \quad \text{not commutative}$$

$$\vec{a} - \vec{b} = -(\vec{b} - \vec{a}) \quad \text{i.e. directions are opposite but magnitudes are same}$$

For non-zero vectors $\vec{a}$ & $\vec{b}$

$$\vec{a} + \vec{b} \neq \vec{a} - \vec{b}$$

$|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ means $\vec{a} \perp \vec{b}$ i.e. vectors $\vec{a}$ and $\vec{b}$ are perpendicular in this case.



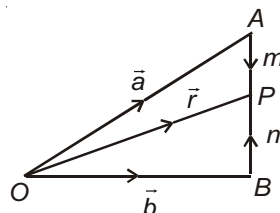
SECTION FORMULAE

Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$ then p.v. of point P which divides AB internally in the ratio $m : n$ given by

$$\vec{OP} = \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n} \quad (m \neq -n)$$

If P divides AB externally in the ratio $m : n$ then

$$\vec{OP} = \frac{n\vec{a} - m\vec{b}}{n-m}$$



VECTOR RESOLUTION OF A VECTOR (COMPONENTS OF A VECTOR)

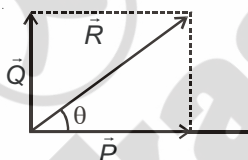
A vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ has its x , y and z components as a , b & c respectively.

The vector resolution can be orthogonal or non-orthogonal. Orthogonal resolution means the components of the vector are mutually perpendicular otherwise non-orthogonal. Orthogonal components of vector $\vec{R}$ are as shown.

$$P = R \cos \theta$$

$$\vec{R} = \vec{P} + \vec{Q}$$

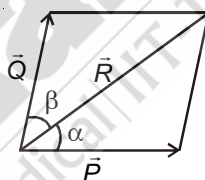
$$Q = R \sin \theta$$



Non-orthogonal components of $\vec{R}$ are as shown

$$P = \frac{R \sin \beta}{\sin(\alpha + \beta)}$$

$$Q = \frac{R \sin \alpha}{\sin(\alpha + \beta)}$$



LINEAR COMBINATION OF VECTORS

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, then any vector $\vec{r}$ in the plane of $\vec{a}$ and $\vec{b}$ is uniquely expressed as

$\vec{r} = x\vec{a} + y\vec{b}$, where x and y are scalars. (Similar linear combination (unique) exists for three non-coplanar vectors, $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$).

LINEARLY DEPENDENT AND INDEPENDENT VECTORS

Vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are said to be linearly dependent if for scalars $x_1, x_2, \dots, x_n$ (not all zero),

$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{0}$. If $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0} \Rightarrow x_1 = x_2 = \dots = x_n = 0$ then vectors are called linearly independent.

Note: 1. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then the vectors are linearly independent.

2. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are linearly dependent then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

3. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanars, then vectors $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$ and $x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$ are coplanar iff.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

4. Two non-zero, non-collinear vectors, or any three non-coplanar vectors are linearly independent.

5. Two collinear or any three coplanar or any four vectors in 3-D are linearly dependent.

6. Three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear iff there exist scalars x , y and z (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ where $x + y + z = 0$.

7. Four points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are said to be coplanar iff there exist scalars x , y , z and w (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$ where $x + y + z + w = 0$

Multiplication of a Vector by a Scalar

If m is a scalar and $\vec{a}$ is a vector, then $m\vec{a}$ (scalar multiple) is a vector whose magnitude is $|m| |\vec{a}|$ and direction is same as of $\vec{a}$ (if m +ve) and opposite that of $\vec{a}$ if m is -ve.

Properties

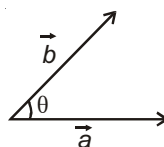
1. $m(\vec{a} \pm \vec{b}) = m\vec{a} \pm m\vec{b}$
2. $(mn)\vec{a} = m(n\vec{a}) = n(m\vec{a})$
3. $(m \pm n)\vec{a} = m\vec{a} \pm n\vec{a}$

Scalar (dot) Product of Two Vectors

The scalar product of vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, θ is angle between two vectors.

Properties :

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $\vec{a} \cdot (\vec{b} \pm \vec{c}) = \vec{a} \cdot \vec{b} \pm \vec{a} \cdot \vec{c}$
3. $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$



4. If $\theta = 0 \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$ (like vectors)
5. If $\theta = \pi \Rightarrow \vec{a} \cdot \vec{b} = -|\vec{a}| |\vec{b}|$ (unlike vectors)
6. If $\hat{a}$ and $\hat{b}$ are unit vectors then $\hat{a} \cdot \hat{b} = \cos \theta$ (where θ is angle between them).
7. $\vec{a} \cdot \vec{a} = |\vec{a}|^2 \Rightarrow |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$
8. If $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$ but $\vec{a} \cdot \vec{b} = 0 \Rightarrow$ either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.
9. If $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along the rectangular coordinate axes OX, OY and OZ then $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$,
 $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.
10. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\text{and } \cos \theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

11. Components of a vector $\vec{r}$ in the direction of vector $\vec{a}$ is $\left(\frac{\vec{r} \cdot \vec{a}}{|\vec{a}|^2} \right) \vec{a}$ and $\perp$ to $\vec{a}$ is $\vec{r} - \left(\frac{\vec{r} \cdot \vec{a}}{|\vec{a}|^2} \right) \vec{a}$.

12. Work done by a force $\vec{F}$ in displacing a particle from A to B ($\vec{AB} = \vec{d}$) is

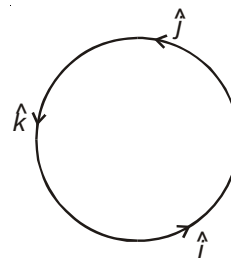
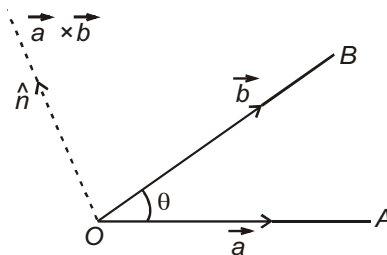
$$W = \vec{F} \cdot \vec{AB} = \vec{F} \cdot \vec{d}$$

VECTOR (CROSS) PRODUCT OF TWO VECTORS

Vector product of two vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ whose direction is that of unit vector $\hat{n}$ which is $\perp$ to both $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}, \vec{b}, \hat{n}$ form a right handed triad (right handed screw system).

Properties

1. In general, $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$. In fact $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
2. For scalar m , $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$.
3. $\vec{a} \times (\vec{b} \pm \vec{c}) = \vec{a} \times \vec{b} \pm \vec{a} \times \vec{c}$
4. If $\vec{a} \parallel \vec{b}$ then $\theta = 0$ or $\pi \Rightarrow \vec{a} \times \vec{b} = \vec{0}$ (but $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$). In particular $\vec{a} \times \vec{a} = \vec{0}$.
5. If $\vec{a} \perp \vec{b}$ then $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \hat{n}$ (or $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}|$)
6. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$ and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ and
 $\hat{k} \times \hat{i} = \hat{j}$ (use cyclic system)
7. Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is given by $\pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$



8. If θ is angle between $\vec{a}$ and $\vec{b}$ then $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$

9. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

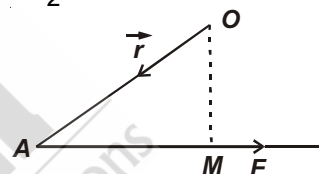
10. (i) If $\vec{a}$ and $\vec{b}$ are adjacent sides of a parallelogram, then

$$\text{Area of parallelogram} = |\vec{a} \times \vec{b}|$$

(ii) If diagonals of parallelogram are $\vec{a}$ and $\vec{b}$, then area of parallelogram = $\frac{1}{2} |\vec{a} \times \vec{b}|$

11. Vector moment of a force about a point :

The vector moment or torque $\vec{M}$ of a force $\vec{F}$ acting at A about the point O is given by $\vec{M} = \vec{r} \times \vec{F}$



SCALAR TRIPLE PRODUCT

Scalar triple product: If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors then $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a} \vec{b} \vec{c}]$ i.e. $\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a} \vec{b} \vec{c}]$.

Properties

1. If $\vec{a}, \vec{b}$ and $\vec{c}$ are adjacent sides of a parallelepiped then volume = $[\vec{a} \vec{b} \vec{c}]$
2. $[\vec{a} \vec{b} \vec{c}] = [\vec{b} \vec{c} \vec{a}] = [\vec{c} \vec{a} \vec{b}]$ (cyclic permutations of $\vec{a}, \vec{b}$ and $\vec{c}$ makes no change in value) but $[\vec{a} \vec{b} \vec{c}] = -[\vec{a} \vec{c} \vec{b}]$ etc.

i.e., Dot and cross can be interchanged, keeping the same cyclic order i.e. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

3. For scalar m , $[m\vec{a} \vec{b} \vec{c}] = [\vec{a} m\vec{b} \vec{c}] = [\vec{a} \vec{b} m\vec{c}] = m[\vec{a} \vec{b} \vec{c}]$
4. The value of scalar triple product is zero if any two vectors are equal or parallel

5. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $[\vec{a} \vec{b} \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

6. Condition of coplanarity: Three non-collinear, non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are called coplanar iff $[\vec{a} \vec{b} \vec{c}] = 0$

Vector Triple Product

Vector triple product: If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors, their vector triple product is defined by

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \text{ and } (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

Note: $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

APPLICATION OF VECTORS TO GEOMETRY

1. Vector Equation of Straight Line

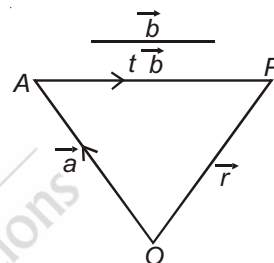
Case I : Vector equation of a straight line passing through a point $\vec{a}$ and parallel to vector $\vec{b}$. Let O be the origin and $\vec{OA} = \vec{a}$. Let P be any point on the line with $\vec{OP} = \vec{r}$.

Since $\vec{AP} \parallel \vec{b}$, for some scalar t ,

$$\vec{AP} = t\vec{b}$$

$$\Rightarrow \vec{r} - \vec{a} = t\vec{b}$$

$$\Rightarrow \boxed{\vec{r} = \vec{a} + t\vec{b}} \text{ which is (required equation) of a straight line passing through } \vec{a} \text{ and parallel to } \vec{b}.$$



Case II : Vector equation of a straight line passing through two points $\vec{a}$ and $\vec{b}$ is given by

$$\boxed{\vec{r} = \vec{a} + t(\vec{b} - \vec{a})}$$

2. Vector Equation of a Plane

Case I : Vector equation of a plane passing through the point $\vec{a}$ and parallel to two given vectors $\vec{b}$ and $\vec{c}$.

Let O be the origin, $\vec{OA} = \vec{a}$. Let AB and AC be the lines on the plane $\parallel$ to vectors $\vec{b}$ and $\vec{c}$. Let $P(\vec{r})$ be any point on the plane, draw $PL \parallel AC$, ($AL \parallel AB$).

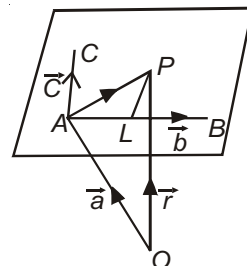
Now $\vec{AL} = s\vec{b}$ and $\vec{LP} = t\vec{c}$ (s, t are scalars)

$$\text{Now } \vec{AP} = \vec{AL} + \vec{LP} = s\vec{b} + t\vec{c}$$

$$\text{and } \vec{OP} = \vec{OA} + \vec{AP} = \vec{a} + s\vec{b} + t\vec{c}$$

Required equation of the plane is $\boxed{\vec{r} = \vec{a} + s\vec{b} + t\vec{c}}$ which can be written as

$$\boxed{[\vec{r} \vec{b} \vec{c}] = [\vec{a} \vec{b} \vec{c}]}$$



Case II : Vector equation of a plane passing through the point

$\vec{a}$, $\vec{b}$, $\vec{c}$ is given by

$$\vec{r} = (1-s-t)\vec{a} + s\vec{b} + t\vec{c} \quad (\text{where } s, t \text{ are scalars}) \text{ or equivalently}$$

$$\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}) = [\vec{a} \ \vec{b} \ \vec{c}]$$

Case III: Vector equation of plane passing through a point $\vec{a}$ and perpendicular to vector $\vec{n}$:

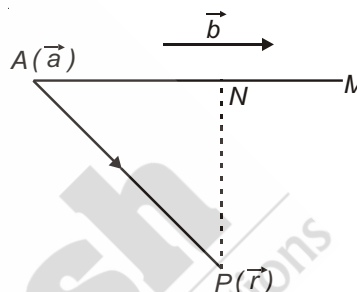
$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

3. Perpendicular distance of a point from a line

Let straight line passes through $A(\vec{a})$ and is $\parallel$ to $\vec{b}$. Then perpendicular distance of point $P(\vec{r})$ from the

$$\text{line} = \frac{|(\vec{r} - \vec{a}) \times \vec{b}|}{|\vec{b}|}$$

$$\text{Another form : } \left[|\vec{r} - \vec{a}|^2 - \left\{ \frac{(\vec{r} - \vec{a}) \cdot \vec{b}}{|\vec{b}|} \right\}^2 \right]^{1/2}$$



4. Perpendicular distance of a point from a plane

Case I : When plane passes through a point $\vec{a}$ and is $\parallel$ to $\vec{b}$ and $\vec{c}$. Distance of point $P(\vec{r})$ from the

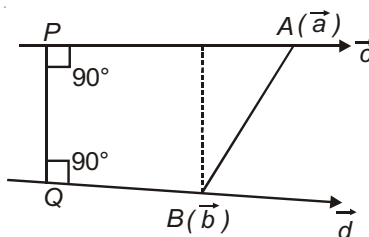
$$\text{plane} = \frac{(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|}$$

Case II : When plane passes through the points $\vec{a}, \vec{b}$ and $\vec{c}$.

$$\frac{(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b})}{|\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}|}$$

5. Shortest distance between two non-intersecting lines :

Let there be two non-intersecting lines passing through $\vec{a}$ and $\vec{b}$ and are $\parallel$ to $\vec{c}$ and $\vec{d}$. If PQ be the shortest distance between them, then



$PQ = \text{projection of } \vec{AB} \text{ on the vector } \vec{n} \text{ (where } \vec{n} = \vec{c} \times \vec{d} \text{)}$

$$\text{Then } PQ = \frac{(\vec{b} - \vec{a}) \cdot (\vec{c} \times \vec{d})}{|\vec{c} \times \vec{d}|} = \frac{[\vec{b} - \vec{a} \ \vec{c} \ \vec{d}]}{|\vec{c} \times \vec{d}|}$$

6. Vector equation of the bisector of the angle between two straight lines :

Let the two lines AB and AC meet at A whose p.v. is $\vec{a}$. Let $\vec{b}$ and $\vec{c}$ be vectors $\parallel$ to AB and AC respectively. Also let P be a point on internal bisector of $\angle BAC$ where $\vec{OP} = \vec{r}$.

From P draw line $\parallel AB$ which meets AC at M .

Now $\angle PAM = \angle PAB = \angle APM$

$\Rightarrow AM = PM = t$ (say)

$\vec{AM}$ is collinear with $\vec{c}$ and $\vec{MP}$ is collinear with $\vec{b}$.

Then $\vec{AM} = t \frac{\vec{c}}{|\vec{c}|}, \vec{MP} = t \frac{\vec{b}}{|\vec{b}|}$

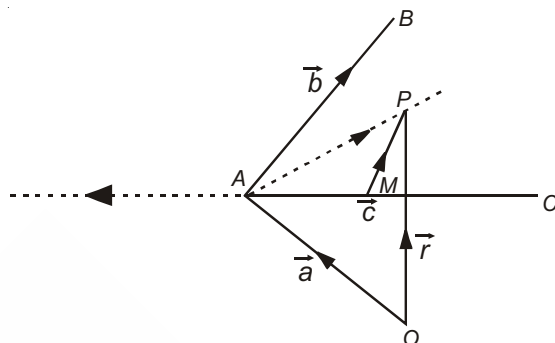
Now in $\triangle APM$, $\vec{AP} = \vec{AM} + \vec{MP} = t \left(\frac{\vec{c}}{|\vec{c}|} + \frac{\vec{b}}{|\vec{b}|} \right)$

In $\triangle OAP$, $\vec{OP} = \vec{OA} + \vec{AP}$

$\Rightarrow \vec{r} = \vec{a} + t \left(\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|} \right)$, required equation of

internal bisector

Equation of external bisector is given by $\vec{r} = \vec{a} + t \left(\frac{\vec{b}}{|\vec{b}|} - \frac{\vec{c}}{|\vec{c}|} \right)$



□ □ □