MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

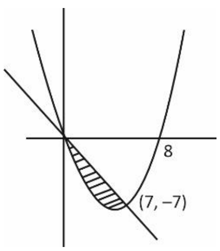
1. The area (in square units) of the region $\{(x, y) : x^2 - 8x \le y \le -x\}$ is :

(1) $\frac{343}{6}$ (2) $\frac{241}{6}$

(3) $\frac{221}{6}$ (4) $\frac{323}{6}$

Answer (1)

Sol.



Point of intersection of $y = x^2 - 8x$ and $y = -x$ is (0, 0) are (7, –7)

required are = $\int_0^7 (-x - x^2 + 8x)dx$

$= \int_0^7 (7x - x^2)dx$

$= \left[\frac{7x^2}{2} - \frac{x^3}{2}\right]_0^7$

$= \frac{343}{2} - \frac{343}{3}$

$= \frac{343}{6}$

2. Sum of $1 + \frac{1}{2}\left(1^2 + 2^2\right) + \frac{1}{3}\left(1^2 + 2^2 + 3^2\right) + \ldots$ upto 10 terms is

(1) $\frac{313}{2}$ (2) $\frac{315}{2}$

(3) 313 (4) 315

Answer (2)

Sol. $T_n = \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6}$

$= \frac{(n+1)(2n+1)}{6}$

$S_n = \sum_{n=1}^{10} T_n = \frac{1}{6}\sum_{n=1}^{10} 2n^2 + 3n + 1$

$= \frac{1}{6}\left[2 \times \frac{10 \times 11 \times 21}{6} + \frac{3 \times 10 \times 11}{2} + 10\right]$

$= \frac{315}{2}$

3. $\int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + |x| + 1} dx$ is equal to

(1) $\ln 3 - \frac{\pi}{3\sqrt{3}}$ (2) $\ln 3 + \frac{\pi}{3\sqrt{3}}$

(3) $\frac{\pi}{3\sqrt{3}}$ (4) $\frac{-\pi}{3\sqrt{3}}$

Answer (2)

Sol. $\int_0^1 \frac{2|x| + 2}{x^2 + |x| + 1} dx$

$2\int_0^1 \frac{x+1}{x^2 + x + 1} dx = \int_0^1 \frac{2x+1}{x^2 + x + 1} dx + \int_0^1 \frac{1}{x^2 + x + 1} dx$

$$= \ln\left|x^2+x+1\right|\Big]_0^1 + \int_0^1 \frac{1}{\left(x+\frac{1}{x}\right)^2+\frac{3}{4}}dx$$

$$= \ln 3 + \frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\Bigg]_0^1$$

$$= \ln 3 + \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= \ln 3 + \frac{\pi}{3\sqrt{3}}$$

4. If matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$ and matrix $[b_{ij}] = B = A^{99} - I_{3\times3}$, then the value of $\left(\frac{b_{31}+b_{32}}{b_{21}}\right)$ is equal to

(1) 147

(2) 149

(3) 160

(4) 159

Answer (2)

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 3 & 0 \end{bmatrix} = C \text{ (say)}$$

$A = (I + C)$

$A^{99} = (I + C)^{99}$

$$C^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}, C^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3\times3}$$

$A^{99} = I^{99} + {}^{99}C_{98} I^{98} C^1 + {}^{99}C_{97} I^{97} C^2 + 0 + 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 99\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 3 & 0 \end{bmatrix} + {}^{99}C_{27}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}$$

$$A^{99} - I \begin{bmatrix} 1 & 0 & 0 \\ 297 & 1 & 0 \\ 43956 & 297 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{99} - I = B = \begin{bmatrix} 1 & 0 & 0 \\ 297 & 1 & 0 \\ 43956 & 297 & 1 \end{bmatrix}$$

$\Rightarrow b_{31} = 43956$

$b_{32} = 299$, $b_{21} = 297$

$$\Rightarrow \frac{b_{31}+b_{32}}{b_{21}} = \frac{43956}{297} + 1$$

$= 149$

5. The value of x for which

$$\sin^{-1}\left(\frac{2}{3}\sqrt{1-x^2}\right) = \cot^{-1}\left(2\sqrt{x}\right) \text{ is}$$

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$

(3) $\frac{1}{8}$ (4) $\frac{1}{9}$

Answer (1)

Sol. $\sin^{-1}\left(\frac{2}{3}\sqrt{1-x^2}\right) = \cot^{-1}(2\sqrt{x})$

$$\sin^{-1}\left(\frac{2}{3}\sqrt{1-x^2}\right) = \sin^{-1}\left(\frac{1}{\sqrt{1+4x}}\right)$$

$$\frac{2}{3}\sqrt{1-x^2} = \frac{1}{\sqrt{1+4x}}$$

Squaring both sides

$$\frac{4}{9}(1-x^2) = \frac{1}{1+4x}$$

$4(1 - x^2)(1 + 4x) = 9$

$4(1 + 4x - x^2 - 4x^3) = 9$

$4 + 16x - 4x^2 - 16x^3 = 9$

$16x^3 + 4x^2 - 16x + 5 = 0$

$x = \frac{-5}{4}, \frac{1}{2}$

but x cannot be negative

$\Rightarrow x = \frac{1}{2}$

6. Consider two vectors

$\vec{a} = 2\hat{i} + 3\hat{j} + 3\hat{k}$,

$\vec{b} = 6\hat{i} + 3\hat{j} + 3\hat{k}$

If $2\vec{a} + 3\vec{b}$ and $\vec{a} - \vec{b}$ are two adjacent sides of a triangle, then the square of area of triangle is

(1) 1800 (2) 900

(3) 902 (4) 680

Answer (1)

Sol. $\vec{a} = 2\hat{i} + 3\hat{j} + 3\hat{k}$

$\vec{b} = 6\hat{i} + 3\hat{j} + 3\hat{k}$

Let $\vec{u} = 2\vec{a} + 3\vec{b} \;= 2(2\hat{i} + 3\hat{j} + 3\hat{k}) + 3(6\hat{i} + 3\hat{j} + 3\hat{k})$

$= 22\hat{i} + 15\hat{j} + 15\hat{k}$

$\vec{v} = \vec{a} - \vec{b} = -4\hat{i}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 22 & 19 & 15 \\ -4 & 0 & 0 \end{vmatrix}$$

$= -4(15\hat{j} - 15\hat{k}) = 60\hat{j} - 60\hat{k}$

$|\vec{u} \times \vec{v}| = \sqrt{(60)^2 + (-60)^2} = \sqrt{7200}$

Area = $\frac{1}{2}|\vec{u} \times \vec{v}|$

$= \frac{1}{2}\sqrt{7200}$

sq. of area = $\frac{1}{4} \times 7200 = 1800$

7. Consider the function $f: R \to R$ such that

$f(x) = \frac{2x^2 + x + 2}{3x^2 + 2x + 3}$, then $f(x)$ is

(1) one-one & onto (2) one-one & into

(3) many one & onto (4) many one & into

Answer (4)

Sol. $f(x) = \frac{2x^2 + x + 2}{3x^2 + 2x + 3}$

$\Rightarrow y = \frac{2x^2 + x + 2}{3x^2 + 2x + 3}$

$x^2(3y - 2) + x(2y - 1) + 3y - 2 = 0$

since $x \in R$

$\Rightarrow D \geq 0$

$(2y - 1)^2 - 4(3y - 2)^2 \geq 0$

$(2y - 1)^2 - (6y - 4)^2 \geq 0$

$(8y - 5)(-4y + 3) \geq 0$

$(8y - 5)(4y - 3) \leq 0$

$\Rightarrow y \in \left[\frac{5}{8}, \frac{3}{4}\right]$

$\Rightarrow$ range $\neq$ codomain $\Rightarrow$ into and also

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty^-} f(x) = \frac{2}{3}$ and $f(x)$ is continuous

$\Rightarrow$ function is many one

8. The shortest distance between skew lines

$L_1 : \frac{x+5}{2} = \frac{y-1}{12} = \frac{z-2}{-5}$ and

$L_2 : \frac{x-2}{3} = \frac{y-2}{4} = \frac{z-4}{-1}$ is

(1) $\frac{14}{\sqrt{1107}}$ (2) $\frac{13}{\sqrt{1017}}$

(3) $\frac{21}{\sqrt{1701}}$ (4) $\frac{19}{\sqrt{7101}}$

Answer (2)

Sol. $SD = \dfrac{(\vec{a}_2 - \vec{a}_1)(\vec{b}_1 - \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}$

$$= \frac{\begin{vmatrix} 2+5 & 2-1 & 4-2 \\ 2 & 12 & -5 \\ 3 & 4 & -1 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 12 & -5 \\ 3 & 4 & -1 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 7 & 1 & 2 \\ 2 & 12 & -5 \\ 3 & 4 & -1 \end{vmatrix}}{\sqrt{8^2 + 13^2 + 28^2}}$$

$$= \frac{13}{\sqrt{1017}}$$

9. A circle has centre in the I[st] Quadrant and touches the x-axis at a distance of 3 units from the origin and cutoff an intercept of $6\sqrt{3}$ on y-axis. Then, then length of chord having equation $x - y = 1$ intercepted by the circle is

(1) $2\sqrt{7}$ (2) $3\sqrt{7}$

(3) $\sqrt{7}$ (4) $4\sqrt{7}$

Answer (4)

Sol. Center of the circle = (3, *r*)

$(x-3)^2 + (y-r)^2 = r^2$

$x^2 + y^2 - 6x - 2ry + 9 = 0$

$2\sqrt{r^2 - 9} = 6\sqrt{3}$

$\sqrt{r^2 - 9} = 3\sqrt{3}$

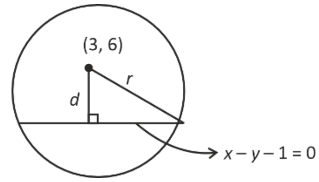
$r^2 - 9 = 27$

$r^2 = 36$

$r = 6$

$(x-3)^2 + (y-6)^2 = 36$

center (3, 6)



$d = \left|\dfrac{3-6-1}{\sqrt{2}}\right| = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$

Chord length $= 2\sqrt{r^2 - d^2}$

$= 2\sqrt{36 - 8}$

$= 2\sqrt{28} = 4\sqrt{7}$

10. An ellipse having eccentricity $\dfrac{1}{\sqrt{3}}$ and equation of its directrix is $x = 2\sqrt{2}$. A hyperbola whose eccentricity is equal to the length of semi-major axis of ellipse and its length of latus rectum is equal to length of minor axis of ellipse, then the distance between the foci of hyperbola is

(1) $\dfrac{16\sqrt{3}}{5}$ (2) $\dfrac{16\sqrt{6}}{15}$

(3) $\dfrac{17\sqrt{3}}{6}$ (4) $\dfrac{15\sqrt{3}}{6}$

Answer (2)

Sol. Equation = $\dfrac{1}{\sqrt{3}}$

Equation of directrix of ellipse is $x = \pm\dfrac{a}{e}$

$x = 2\sqrt{2}$

$\therefore \dfrac{a}{e} = 2\sqrt{2}$

$a = 2\sqrt{2} \times \dfrac{1}{\sqrt{3}} = \dfrac{2\sqrt{2}}{\sqrt{3}}$

$e_E^2 = 1 - \dfrac{b^2}{a^2}$

$$\frac{1}{3}=1-\frac{3b^2}{8} \Rightarrow \frac{3b^2}{8}=1-\frac{1}{3}=\frac{2}{3}$$

$$b^2=\frac{2}{3}\times\frac{8}{3}=\frac{16}{9}$$

$$b=\frac{4}{3}$$

$$e_H=2\sqrt{\frac{2}{3}}$$

$$LR_H=\frac{2b_H^2}{a_H}=2b=\frac{8}{3}$$

$$b_H^2=\frac{4}{3}a_H$$

$$e_H^2=1+\frac{b_H^2}{a_H^2}$$

$$\frac{8}{3}=1+\frac{4a_H}{3a_{H^2}}$$

$$\frac{5}{3}=\frac{4}{3a_H}$$

$$\Rightarrow a_H=\frac{4}{5}$$

Distance between foci of hyperbola is $2a_He_H$

$$=2\times\frac{4}{5}\times\frac{2\sqrt{2}}{\sqrt{3}}$$

$$=\frac{16\sqrt{6}}{15}$$

11. If the mean and variance of the observations 2, 4, α, 8, β, 12, 14 (where $\alpha < \beta$) are 8 and 16 respectively. Then, the equation whose roots are $3\alpha + 2$ and $4\beta + 1$ is

(1) $x^2 - 61x + 820 = 0$ (2) $x^2 - 60x + 340 = 0$

(3) $x^2 - 60x + 81 = 0$ (4) $x^2 - 61x + 810 = 0$

Answer (1)

Sol. Mean = $\frac{40+\alpha+\beta}{7}$

$$\frac{40+\alpha+\beta}{7}=8$$

$\alpha + \beta = 16$

Variance = 16

$$\frac{424+\alpha^2+\beta^2}{7}-64=16$$

$\alpha^2 + \beta^2 = 136$

$\alpha = 6$, $\beta = 10$

Equation whose roots are $3\alpha + 2$ and $4\beta + 1$.

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$x^2 - 61x + 820 = 0$

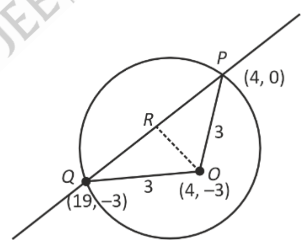
12. The line $x - y = 4$ intercept the circle $(x - 4)^2 + (y + 3)^2 = 9$ at point P and Q. There is a point $M(\alpha, \beta)$ on circle such that $MP = MQ$. Then the value of $|6\alpha + 8\beta|$ is equal to

(1) $3\sqrt{2}$ (2) $4\sqrt{3}$

(3) $3\sqrt{3}$ (4) $2\sqrt{3}$

Answer (1)

Sol.



Slope of given line, $m_1 = 1$

slope of 1st line $m_2 = -1$

Equation of Ist line (OR) :

$y + 3 = -1(x - 4)$

$x + y = 1$

Pont of intersection of $x + y = 1$ and $(x-4)^2 + (y+3)^2 = 9$

$$(x-4)^2 + (1+x)+3)^2 = 9$$

$$(x-4)^2 + (4+x)^2 = 9$$

$$2(x-4)^2 = 9 \Rightarrow x - 4\frac{\pm 3}{\sqrt{2}}$$

$$x = 4 \pm \frac{3}{\sqrt{2}}$$

$$\therefore y = -3 \pm \frac{3}{\sqrt{2}}$$

$\therefore$ Possible co-ordinates of M are

$$\left(4+\frac{3}{\sqrt{2}}, -3-\frac{3}{\sqrt{2}}\right) \text{ and } \left(4-\frac{3}{\sqrt{2}}, -3+\frac{3}{\sqrt{2}}\right)$$

$$\therefore |6\alpha + 8\beta| = 3\sqrt{2}$$

13. If $\cos 3\theta + 2\cos 2\theta = -2$, then sum of all possible solutions in $[0, 2\pi]$ is

(1) π (2) 2π

(3) 3π (4) 4π

Answer (4)

Sol. $\cos 3\theta + 2\cos 2\theta + 2 = 0$

$4\cos^3\theta - 3\cos\theta + 2(2\cos^2\theta - 1) + 2 = 0$

$\cos\theta(4\cos^2\theta - 3 + 4\cos\theta) = 0$

$\cos\theta(4\cos^2\theta + 4\cos\theta - 3) = 0$

$\cos\theta(4\cos^2\theta + 6\cos\theta - 2\cos\theta - 3) = 0$

$\cos\theta(2\cos\theta - 1)(2\cos\theta + 3) = 0$

So, $\cos\theta = 0$ or $\cos\theta = \frac{1}{2}$

$\Rightarrow \theta \in \frac{\pi}{2}, \frac{3\pi}{2}$ $\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$$\sum\theta = \frac{\pi}{2} + \frac{3\pi}{2} + \frac{\pi}{3} + \frac{5\pi}{3}$$

$= 4\pi$

14. Let $\lim_{x\to 2} \frac{\tan(x-2)\left(rx^2 + (p-2)x - 2p\right)}{(x-2)^2} = 5$. If both roots of the equation $rx^2 - px + q = 0$ lie in interval (0, 2) then set of values of q is $(\alpha, \beta]$ then $16(\beta - \alpha)^2$ is equal to

(1) 2 (2) 1

(3) 4 (4) 8

Answer (2)

Sol. $\lim_{x\to 2}\left(\frac{\tan(x-2)}{x-2}\right)\left(\frac{rx^2 - (p-2)x + 2p}{x-2}\right) = 5$

$= \lim_{x\to 2}\left(\frac{rx^2 - px + 2x + 2p}{x-2}\right) = 5$

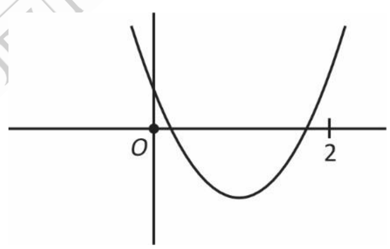
$4r - 4 = 0$ $\Rightarrow$ $r = 1$

$\lim_{x\to 2}\frac{2rx + (p-2)}{1} = 5$

$4 + p - 2 = 5$ $\Rightarrow$ $p = 3$

$\Rightarrow$ $x^2 - 3x + q = 0$ have both roots in (0, 2).

$\Rightarrow$ (i) $D \ge 0$ $\Rightarrow 9 - 4q \ge 0$ $\Rightarrow q \le \frac{9}{4}$



$f(0) > 0, f(2) > 0 \Rightarrow q > 0, q > 2$

$\Rightarrow$ $q \in \left(2, \frac{9}{4}\right]$

$$(\beta - \alpha)^2 = \left(\frac{9}{4} - 2\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21.

22.

23.

24.

25.

❑ ❑ ❑