Chapter 9

Oscillations and Waves

Periodic Function

If $f(t + T) = f(t)$ then function 'f' is periodic with period T.

Harmonic Motion

When oscillatory motion of a particle can be expressed in terms of sine or cosine functions, it is said to be a harmonic motion.

SIMPLE HARMONIC MOTION

When a motion can be expressed in terms of a single sine or cosine (sinusoidal) function, the motion is said to be Simple Harmonic Motion (SHM). For SHM, force $\propto$-(displacement)

$\Rightarrow \quad F \propto -x$

$\Rightarrow \quad F = -kx$ [Restoring Force]

$\Rightarrow \quad a = -\dfrac{k}{m}x$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad \text{or} \quad \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$ (This equation represents the differential equation of S.H.M.)

Velocity and acceleration of a particle executing S.H.M.

If $\quad x = A\sin\omega t$, A is amplitude (maximum displacement from mean position)

$\Rightarrow \quad v = \dfrac{dx}{dt} = A\omega\cos\omega t$

or, $\quad v = A\omega\sin\left(\omega t + \dfrac{\pi}{2}\right)$, maximum speed = $A\omega$

i.e., **velocity leads displacement by** $\dfrac{\pi}{2}$. (This is always true in SHM)

(a) Dependence of velocity v with displacement from mean position (x)

$$v = \omega\sqrt{A^2 - x^2}$$

(b) Acceleration

$$a = \frac{dv}{dt} = -A\omega^2 \sin\omega t \text{ , maximum acceleration } = A\omega^2$$

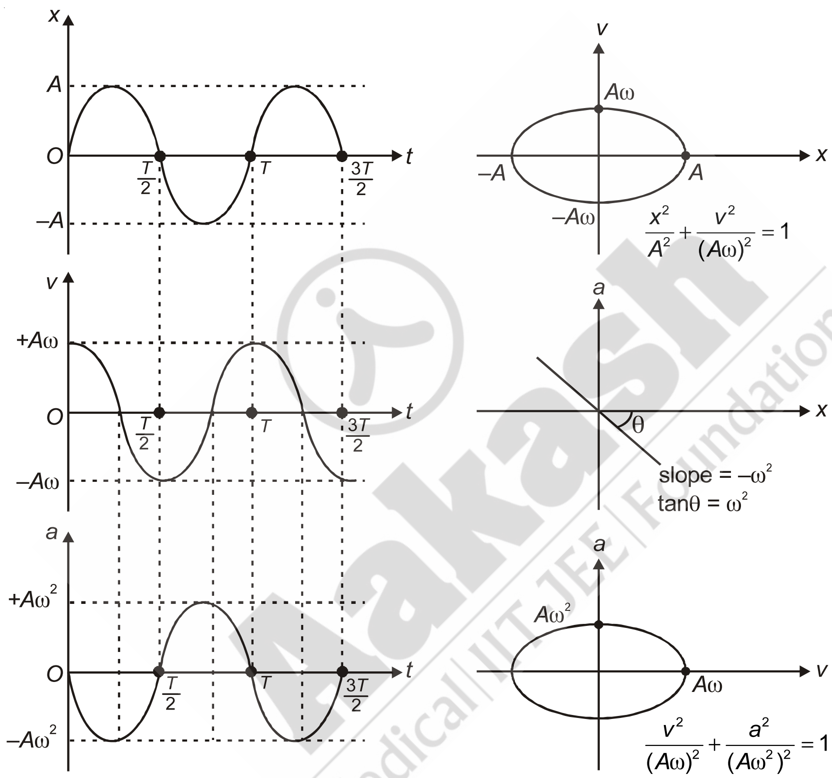
$\Rightarrow$ $a = A\omega^2 \sin(\omega t + \pi)$

i.e., **acceleration leads velocity by $\frac{\pi}{2}$. Acceleration and displacement are in opposite phase**.

(c) Dependence of acceleration with position, is $a = -\omega^2 x$

Graphical representation of variation of position, velocity and acceleration

(For $x = A \sin \omega t$)



Energy in SHM

Salient points regarding energy in SHM :

Oscillating quantity	Time period	Frequency
Displacement	T	f
KE	$T/2$	$2f$
PE	$T/2$	$2f$
|KE ~ PE|	$T/4$	$4f$
Total Energy	∞	0

1. $KE_{avg} = \frac{1}{4}m\omega^2A^2$.

2. $KE_{max} = \frac{1}{2}m\omega^2A^2$ at mean position.

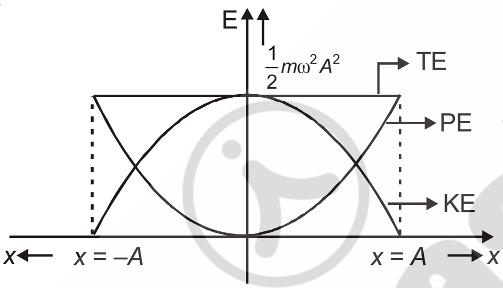
3. KE_{min} = zero at extreme position.

4. $PE_{avg} = \frac{1}{4}m\omega^2A^2$.

5. $PE_{max} = \frac{1}{2}m\omega^2A^2$ at extreme position.

6. Total energy, $E = \frac{1}{2}m\omega^2A^2$ which is constant *i.e.* doesn't depend on x

7. Both kinetic and potential energy vary parabolically with x.



8. PE = KE at $x = \pm\frac{A}{\sqrt{2}}$ and $t = \frac{T}{8}$. (Starting from mean position towards $+x$).

SIMPLE PENDULUM

Time period of oscillation of simple pendulum of length l for small angular amplitude is given by $T = 2\pi\sqrt{\frac{l}{g}}$

where g is the effective acceleration due to gravity, directed along the length of pendulum when it is at mean position.

SOME IMPORTANT POINTS :

On changing various factors, T changes as :

1. If length 'l' is changed, $\frac{\Delta T}{T} = \frac{1}{2}.\frac{\Delta l}{l}$

2. If gravity 'g' is changed, $\frac{\Delta T}{T} = -\frac{1}{2}.\frac{\Delta g}{g}$

Simple Pendulum in Lift

Effective $g' = |\vec{g} + \vec{a}|$, where $\vec{a}$ is pseudo acceleration. $T = 2\pi\sqrt{\frac{l}{g'}}$

Simple Pendulum of Length Comparable to the Radius of Earth

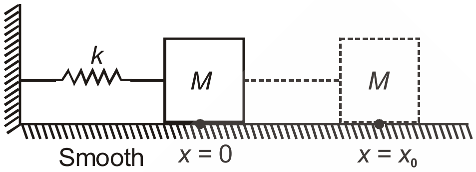
Time period of such a pendulum is given by,

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{l}+\frac{1}{R_e}\right)}}$$

OSCILLATION OF SPRING

Horizontal Oscillations

The spring is pulled/pushed from $x = 0$ to $x = x_0$ and released.



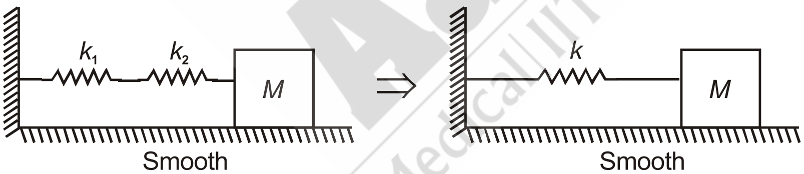
The block executes SHM

(1) Amplitude of oscillation = x_0

(2) Time period $T = 2\pi\sqrt{\frac{M}{k}}$

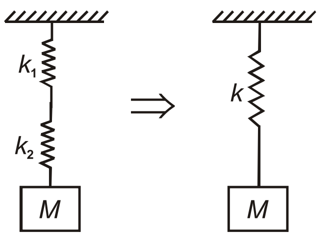
COMBINATIONS OF SPRINGS

Series Combination :

(1) 

$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$ or Effective spring constant, $k = \frac{k_1 k_2}{k_1 + k_2}$, $T = 2\pi\sqrt{\frac{M}{k}}$

(2) $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

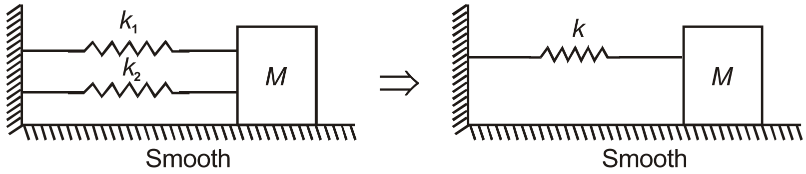


Effective spring constant, $k = \frac{k_1 k_2}{k_1 + k_2}$

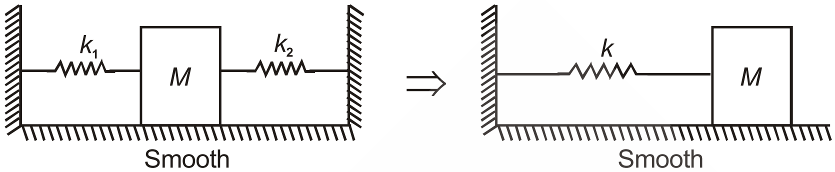
$T = 2\pi\sqrt{\frac{M}{k}}$

Parallel Combination

(1) Effective spring constant, $k = k_1 + k_2$, $T = 2\pi\sqrt{\frac{M}{k}}$

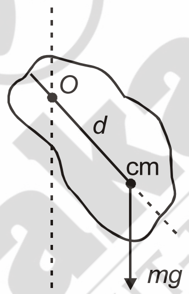


(2) Effective spring constant, $k = k_1 + k_2$ $T = 2\pi\sqrt{\frac{M}{k}}$



Physical Pendulum

Figure shows an extended body (called physical pendulum) pivoted about point O, which is at a distance d from its centre of mass.



Time period of oscillation, $T = 2\pi\sqrt{\frac{I}{mgd}}$ I = moment of inertia of the body about pivoted point.

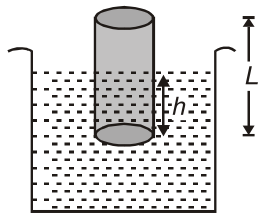
d is the distance of centre of mass from suspension point

Oscillation of a Floating Cylinder

If σ = density of cylinder material

ρ = density of fluid ($\sigma < \rho$)

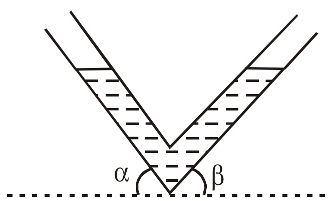
then, $T = 2\pi\sqrt{\frac{\sigma L}{\rho g}} = 2\pi\sqrt{\frac{h}{g}}$



Oscillations of a Liquid in a Tube

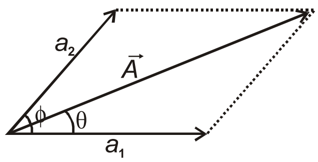
$$T = 2\pi\sqrt{\frac{l}{g(\sin\alpha + \sin\beta)}}$$

l = total length of liquid column



Superposition of SHMs

Consider two SHMs along the same line



If $\quad y_1 = a_1 \sin \omega t$

$\quad y_2 = a_2 \sin (\omega t + \phi)$

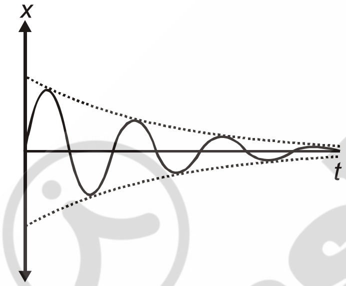
then, equation of resultant SHM is given by,

$$y = y_1 + y_2 = A \sin (\omega t + \theta)$$

where, $A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos\phi}$ & $\theta = \tan^{-1}\left(\dfrac{a_2 \sin\phi}{a_1 + a_2 \cos\phi}\right)$

Damped Oscillations

If there is any dissipative force like viscous force in SHM, then the amplitude of the particle decreases with time such type of oscillations are known as damped oscillations.



(i) Differential equation for damped oscillation $m\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = 0$,

where b = Coefficient of damping.

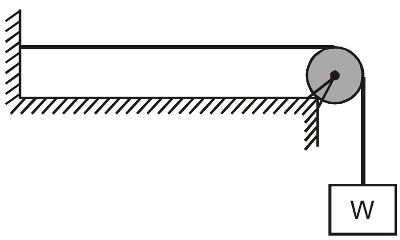
(ii) Displacement-time equation, $x = A(t) \sin(\omega' t + \phi)$

(iii) Amplitude of damped oscillation, $A(t) = A_0 e^{-\frac{b}{2m}t}$, where A_0 = Initial amplitude.

(iv) Angular frequency of damped oscillation, $\omega' = \sqrt{\omega_0^2 - \left(\dfrac{b}{2m}\right)^2}$, where $\omega_0 = \sqrt{\dfrac{k}{m}}$ = natural frequency.

Speed of mechanical waves

(a) Transverse wave in a stretched string



T = tension in the string

μ = mass per unit length

D = diameter of string

ρ = density

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{\text{stress}}{\text{density}}} = \frac{2}{D}\sqrt{\frac{T}{\pi\rho}}$$

$$\boxed{\mu = A\rho = \frac{\pi D^2}{4}\rho}$$

$$v = \sqrt{\frac{T}{A\rho}}$$

(b) Transverse wave in a long bar is given by

$v = \sqrt{\frac{Y}{\rho}}$ where Y = Young's modulus, ρ = Density of material

(c) Longitudinal Waves

(i) In liquid $v = \sqrt{\frac{\beta}{\rho}}$ β = bulk modulus of elasticity

ρ = density

(ii) In gases $v = \sqrt{\frac{\beta}{\rho}}$. β is bulk modulus of the gas. This value is not a fixed value for gases

Case - I : Suggested by Newton

Taking isothermal process $\beta = P \Rightarrow v = \sqrt{\frac{P}{\rho}}$

Put P = 1 atm, ρ = 1.23 kg/m^3 $\Rightarrow$ v = 280 m/s (more than 15% error)

Case - II : Corrected by Laplace

For Adiabatic $\beta = \gamma P$

$\Rightarrow$ $v = \sqrt{\frac{\gamma P}{\rho}}$. Taking γ = 1.4, we get, $\boxed{\text{For air } v = 20\sqrt{T}}$

v = 330 m/s

Note : *Propagation of sound in air is adiabatic.*

Factors affecting speed of sound

$$v = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{\gamma P}{\rho}}$$

(1) v is independent of pressure (If temperature is kept constant)

(2) $v \propto \frac{1}{\sqrt{\rho}}$ or $v \propto \frac{1}{\sqrt{M}}$ (If temperature is kept constant)

(3) Velocity of a wave depends on medium, not on the frequency of source

(4) $v \propto \sqrt{T}$

(5) Velocity of sound in humid air is more because its density is less than that of dry air.

(6) Velocity of sound in humid hydrogen is less than in dry hydrogen due to similar reason.

SOUND WAVES

These are mechanical and longitudinal waves. They propagate in form of compressions and rarefactions. Particle displacements can be represented by wave function

$S = A \sin(\omega t - kx)$

As particles oscillate, pressure variation takes place according to the wave function.

$\Delta P = \Delta P_0 \cos(\omega t - kx)$, ΔP_0 = maximum pressure variation

Characteristic of Sound

Loudness : Sensation of sound produced in human ear is due to amplitude. It depends upon intensity, density of medium, presence of surrounding bodies,

(a) Intensity of Wave

$$I = 2\pi^2 f^2 A^2 \rho v$$

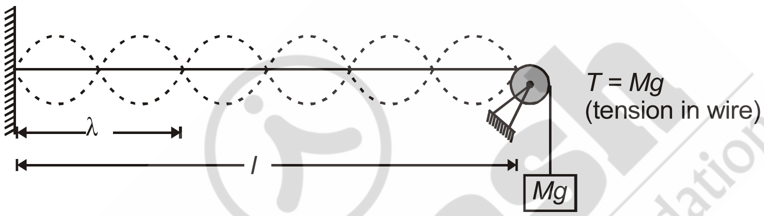
$$= \frac{1}{2}\omega^2 A^2 \rho v$$

$I \propto f^2$ and $I \propto A^2$

(b) Intensity Level or (Sound Level) (β)

$$\beta = 10\log_{10}\left(\frac{I}{I_0}\right) \qquad \begin{bmatrix} I_0 = \text{minimum intensity of audible sound} = 10^{-12}\ \text{W/m}^2 \\ I = \text{measured intensity} \end{bmatrix}$$

Sonometer : In this case, transverse stationary waves are formed.



The wire vibrates in n loops, then

$$l = \frac{n\lambda}{2} \text{ or } \lambda = \frac{2l}{n}$$

Velocity $v = \sqrt{\frac{T}{\mu}}$ where 'μ' is mass per unit length of wire.

$$\Rightarrow \quad \nu_n = \frac{v}{\lambda} = \frac{nv}{2l} = \frac{n}{2l}\sqrt{\frac{T}{\mu}}$$

Pipe length l	Fundamental Mode	Ist Overtone	$(n-1)^{th}$ overtone	Ratio of Successive frequency
Open	$\nu = \frac{V}{2l}$ I^{st} Harmonic	$\nu = \frac{V}{l}$ 2^{nd} Harmonic	$\nu = n\frac{V}{2l}$ n^{th} Harmonic	1 : 2 : 3 : 4
Closed	$\nu = \frac{V}{4l}$ I^{st} Harmonic	$\nu = \frac{3V}{4l}$ 3^{rd} Harmonic	$\nu = (2n-1)\frac{V}{4l}$ $(2n-1)^{th}$ Harmonic	1 : 3 : 5 : 7

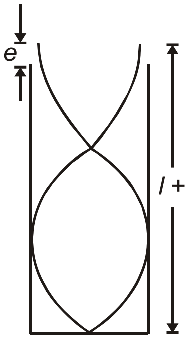
Note : *Even numbered (i.e., 2^{nd}, 4^{th} .....) harmonics do not exist in closed organ pipe.*

End correction (e) :

The antinodes are formed slightly outside the open end. The distance of antinode from open end of the pipe is called end correction. It depends on radius of pipe. ($e = 0.6\ r$)

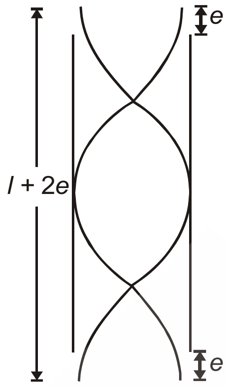
Thus, we have,

For closed organ pipe



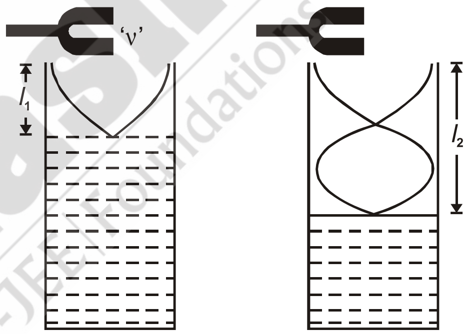
$$\nu = \frac{(2n-1)V}{4(l+e)}$$

For open organ pipe



$$\nu = \frac{nV}{2(l+2e)}$$

Resonance Tube:



If resonance is obtained first at length l_1.

then at length l_2, then

$\boldsymbol{\lambda = 2(l_2 - l_1)}$

$\Rightarrow$ distance between two successive lengths is $\frac{\lambda}{2}$

Interference

Consider two waves of same frequency and wavelength,

$y_1 = a_1 \sin(\omega t - kx),\ I_1 = Ca_1^2$

$y_2 = a_2 \sin(\omega t - kx + \phi),\ I_2 = Ca_2^2$

Equation of resultant wave is,

$$y = y_1 + y_2 = A \sin(\omega t - kx + \theta), \text{ where } A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\phi} \text{ and } \theta = \tan^{-1}\left(\frac{a_2\sin\phi}{a_1 + a_2\cos\phi}\right)$$

Resultant Intensity is given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$$

DOPPLER'S EFFECT

If a wave source and a observer are moving relative to each other, the frequency observed by the receiver (f) is different from the actual source frequency (f_0) given by,

$$f = f_0\left(\frac{v \pm v_0}{v \mp v_s}\right)$$

where v = speed of sound, v_0 = speed of observer, v_s = speed of source

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