Chapter 5

Rotational Motion

CENTRE OF MASS OF A RIGID BODY (CONTINUOUS MASS DISTRIBUTION)

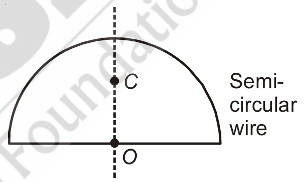
Mathematically position coordinates of the centre of mass of rigid body are given by

$$x_{cm} = \frac{\int x\,dm}{\int dm};\quad y_{cm} = \frac{\int y\,dm}{\int dm};\quad z_{cm} = \frac{\int z\,dm}{\int dm}$$

Centre of mass of some commonly used objects.

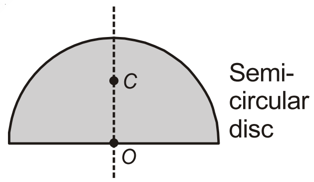
1. Semicircular wire of radius R.

$$OC = \frac{2R}{\pi}\text{, where } C \text{ is centre of mass}$$



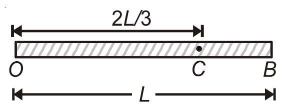
2. Semicircular disc of radius R

$$OC = \frac{4R}{3\pi}$$



3. Non-uniform rod of length L. The linear mass density varies linearly from zero at O to maximum at B.

$$OC = \frac{2L}{3}$$



VELOCITY AND ACCELERATION OF CENTRE OF MASS

Velocity of Centre of Mass

The instantaneous velocity of centre of mass is given by

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots . m_n\vec{v}_n}{\sum_{i=1}^{n} m_i};\qquad \text{or}\qquad \vec{v}_{cm} = \frac{\vec{P}_{system}}{M_{system}}$$

Where $\vec{P}_{system}$ is the total linear momentum of the system of particles.

Acceleration of Centre of Mass

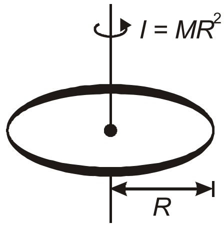
Differentiating $\vec{v}_{cm}$ w.r.t. time we get $\vec{a}_{cm}$ as

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \dots m_n\vec{a}_n}{\sum_{i=1}^{n} m_i}; \quad \text{or} \quad \vec{a}_{cm} = \frac{\Sigma\vec{F}_{ext}}{M_{system}}$$

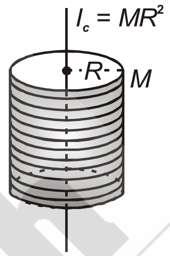
Where $\Sigma\vec{F}_{ext}$ is the vector sum of forces acting on the particles of system.

MOMENT OF INERTIA OF DIFFERENT OBJECTS

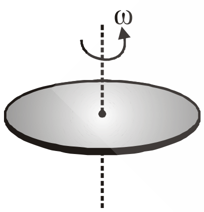
For an axis perpendicular to the plane of the ring



A hollow cylinder

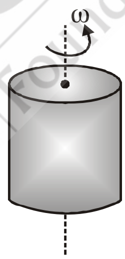


The axis perpendicular to the plane of the disc.



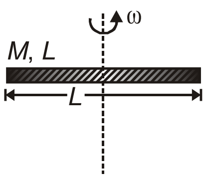
$$I_{cm} = \frac{MR^2}{2}$$

A solid cylinder



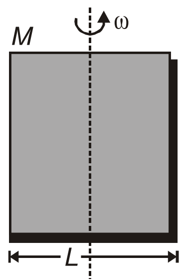
$$I_{cm} = \frac{MR^2}{2}$$

A thin rod



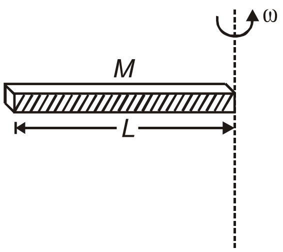
$$I_c = \frac{ML^2}{12}$$

A plate



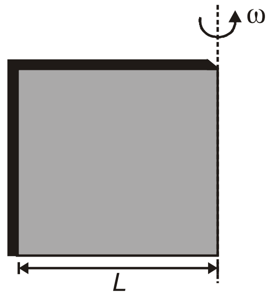
$$I_c = \frac{ML^2}{12}$$

A thin rod about a perpendicular axis through its end



$$I = \frac{ML^2}{3}$$

A plate about one edge



$$I = \frac{ML^2}{3}$$

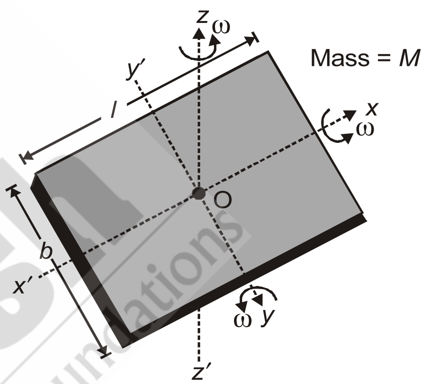
A Rectangular Plate

(a) $I_{xx'} = \frac{Mb^2}{12}$

(b) $I_{yy'} = \frac{Ml^2}{12}$

(c) $I_{zz'} = I_{xx'} + I_{yy'}$

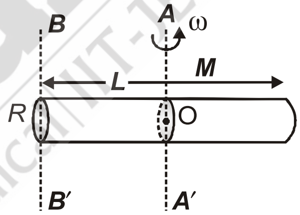
(d) $I_{zz'} = \frac{M(l^2 + b^2)}{12}$



A Thick Rod (Solid cylinder)

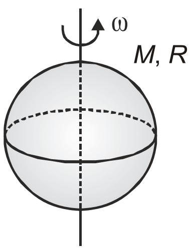
The axis is perpendicular to the rod and passing through the centre of mass

$$I_{AA'} = \frac{ML^2}{12} + \frac{MR^2}{4};\ I_{BB'} = \frac{ML^2}{3} + \frac{MR^2}{4}$$



A Solid Sphere

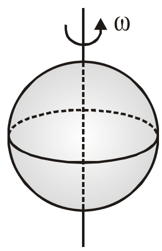
About its diameter



$$I_{cm} = \frac{2}{5}MR^2$$

A Hollow Sphere

About its diameter



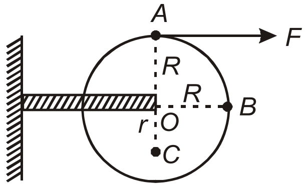
$$I_{cm} = \frac{2}{3}MR^2$$

RIGID BODY ROTATION

In this section, rotation of a body about a stationary fixed axis has been discussed

1. Rotating Disc

A tangential force F is applied at the periphery, as a result disc is rotating above an axis passing through its CM, normal to plane of disc



$\tau = F \times R$ [about O]

$$I = \frac{1}{2}mR^2$$

$$\alpha = \frac{\tau}{I} = \frac{2F}{MR}$$

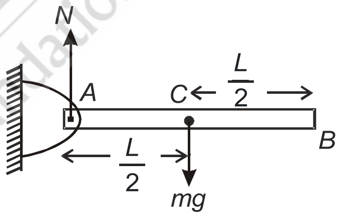
(1) Tangential acceleration of A is $a_A = R\alpha = \frac{2F}{M}$ (along horizontal)

(2) Tangential acceleration of B is $a_B = R\alpha = \frac{2F}{M}$ (vertically downwards)

(3) Tangential acceleration of C is $a_C = r\alpha = \frac{2Fr}{MR}$ (along horizontal, opposite to the direction of tangential acceleration of A)

2. Hinged Rod

The rod is released from rest from horizontal position



$\tau = mg \times \frac{L}{2}$ (about A)

$$I = \frac{ML^2}{3}$$

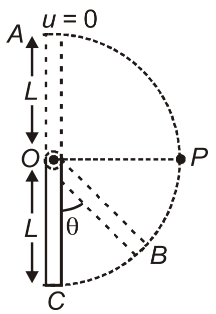
$$\alpha = \frac{\tau}{I} = \frac{3g}{2L}$$

(1) Linear acceleration of COM C is $a_{cm} = \frac{L}{2}\alpha = \frac{3g}{4}$. Also, $N = mg - ma_{cm} = \frac{mg}{4}$

(2) Linear acceleration of point B is $a_B = L\alpha = \frac{3g}{2}$

(3) The rod is released from unstable equilibrium position {from position A}

(a) When at B, $Mg\frac{L}{2}(1+\cos\theta) = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$



$$\omega = \sqrt{\frac{6g}{L}}\cos\frac{\theta}{2}$$

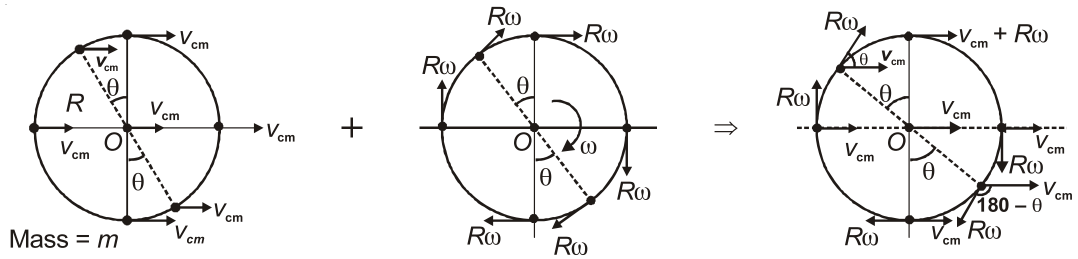
(b) at C, $\theta = 0°$

$$\omega = \sqrt{\frac{6g}{l}}$$

(c) at P, $\theta = 90°$, $\omega = \sqrt{\frac{3g}{l}}$

ROLLING OF A BODY

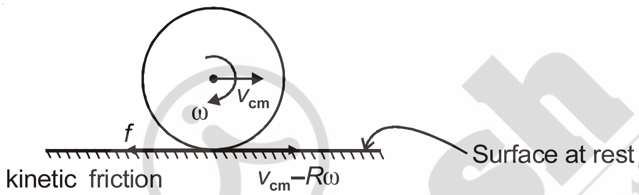
Rolling is combination of Pure Translation and Pure Rotation Motion about Centre of Mass.



Pure translation **Pure rotation (about cm)** **Rolling**

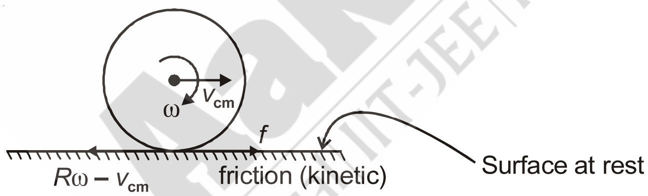
Case - I : Forward slipping

$v_{cm} > R\omega$



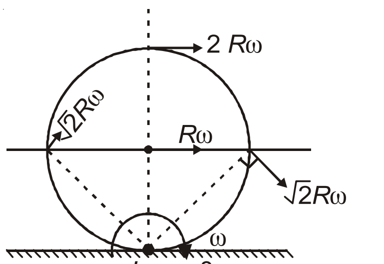
Case - II : Backward slipping

$v_{cm} < R\omega$

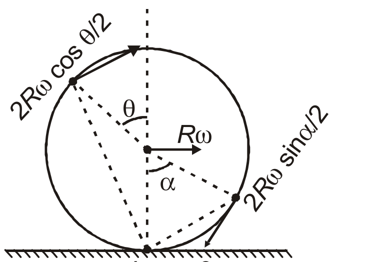


***Case - III :* Pure Rolling**

$v_{cm} = R\omega$



I $v = 0$
(instantaneous centre of rotation)



I $v = 0$
(instantaneous centre of rotation)

Note : *In this case, friction is static. It may or may not be zero.*

Kinetic energy of the body during pure rolling (E)

E = Translational KE + Rotational KE

$= E_T + E_R$

$= \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$

$= \frac{1}{2}\left(m + \frac{I_{cm}}{R^2}\right)v_{cm}^2 = \frac{1}{2}mv_{cm}^2\left(1 + \frac{K^2}{R^2}\right)$ (where k is radius of gyration)

$$E = E_T\left(1 + \frac{K^2}{R^2}\right)$$

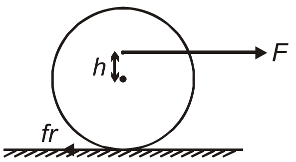
Similarly, $E = E_R\left(1 + \frac{R^2}{K^2}\right)$

Type of body	K	Fraction of total energy translational $X = \left(1+\frac{K^2}{R^2}\right)^{-1}$	Fraction of total energy rotational $Y = \left(1+\frac{R^2}{K^2}\right)^{-1}$	$\frac{Y}{X}$
1. Ring or hollow cylinder	R	$\frac{1}{2} = 0.5 = 50\%$	$\frac{1}{2} = 0.5 = 50\%$	1 : 1
2. Spherical Shell	$\sqrt{\frac{2}{3}}R$	$\frac{3}{5} = 0.6 = 60\%$	$\frac{2}{5} = 0.4 = 40\%$	2 : 3
3. Disc or solid cylinder	$\frac{R}{\sqrt{2}}$	$\frac{2}{3} = 0.666 = 66.67\%$	$\frac{1}{3} = 0.333 = 33.33\%$	1 : 2
4. Solid sphere	$\sqrt{\frac{2}{5}}R$	$\frac{5}{7} = 0.714 = 71.4\%$	$\frac{2}{7} = 0.286 = 28.6\%$	2 : 5

Note : *Above values X and Y are independent of mass and radius of the body. They only depend on the type of body.*

Applications

1. A force is applied at the distance h from centre of mass as shown in figure



$$a_{C.M} = \frac{F\left(1 + \frac{h}{R}\right)}{M\left(1 + \frac{K^2}{R^2}\right)}$$

$f_r = \frac{F(K^2 - hR)}{K^2 + R^2} \le \mu N$ (must be less than μmg for Rolling)

If $h < \frac{K^2}{R}$ friction is backward

$h > \frac{K^2}{R}$ friction become forward

2. If force is applied at centre of mass then (h = 0)

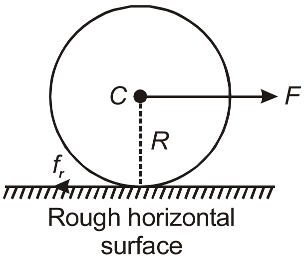
So, $a = \frac{F}{M\left(1 + \frac{K^2}{R^2}\right)}$ and $f_r = \frac{FK^2}{K^2 + R^2} = \frac{F}{\left(1 + \frac{R^2}{K^2}\right)}$

3. If force is applied at highest point ($h = R$)

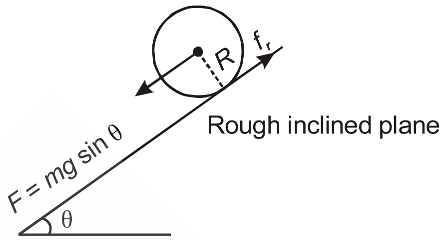
$$a_{\text{C.M.}} = \frac{2F}{M\left(1+\frac{K^2}{R^2}\right)}$$

$$f_r = \left(\frac{R^2 - K^2}{K^2 + R^2}\right)F \text{, forward direction}$$

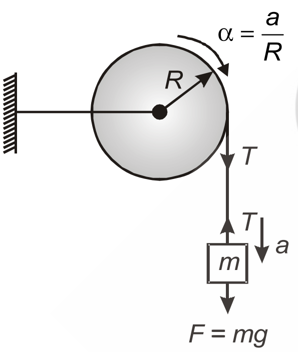
4. (i)



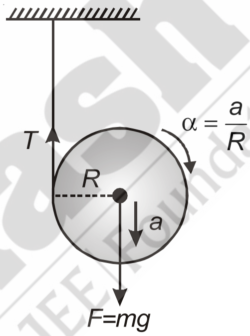
(ii)



(iii)



(iv)



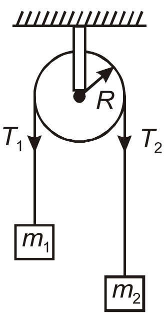
For all the four situations shown above,

$$a = \frac{F}{m + \frac{I_{\text{c.m.}}}{R^2}}$$

$$f_r \text{ or } T = \frac{I_{\text{c.m.}}a}{R^2}$$

In the situations described above, the linear acceleration of the moving object can be calculated by same formula, the value of F, and moment of inertia will depend on the kind of problem.

Also consider the following situation.



$$a = \frac{F}{(m_1 + m_2) + \frac{I}{R^2}}$$

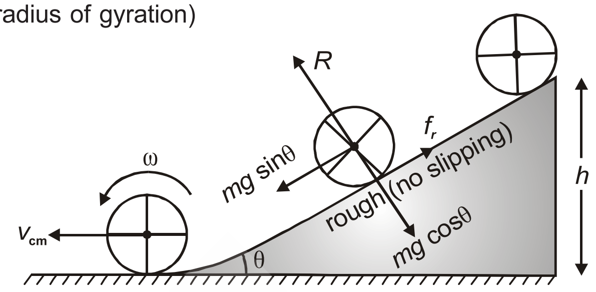
$$T_2 - T_1 = \frac{Ia}{R^2}$$

Here, $F = (m_2 - m_1)\,g$ = Net pulling force

Rolling of a Body on an Inclined Plane

By conservation of energy, $\underset{\text{(Total energy)}}{mgh} = \underset{\text{(Rotatory)}}{\frac{1}{2}I\omega^2} + \underset{\text{(Translatory)}}{\frac{1}{2}mv_{cm}^2}$

Let $\beta = 1 + \frac{I_{cm}}{MR^2} = 1 + \frac{K^2}{R^2}$ (where k is radius of gyration)



1. $a_{cm} = \dfrac{mg\sin\theta}{m + \dfrac{I_{cm}}{R^2}} = \dfrac{g\sin\theta}{1 + \dfrac{K^2}{R^2}} = \dfrac{g\sin\theta}{\beta}$

2. $v_{cm} = \sqrt{\dfrac{2gh}{\beta}} = \sqrt{2gh \Big/ 1 + \dfrac{K^2}{R^2}}$

3. $\text{Time} = \dfrac{1}{\sin\theta}.\sqrt{\beta\dfrac{2h}{g}}$ $\quad\because\quad$ Force of friction $f_r = \dfrac{mg\sin\theta}{1 + \dfrac{R^2}{K^2}}$

   i.e., $t \propto \sqrt{\beta}$

4. Force of friction $f_r = \dfrac{mg\sin\theta}{1 + \dfrac{R^2}{K^2}}$

5. Instantaneous power $P = (mg \sin\theta)v$

6. Maximum angle of inclination for pure rolling, $\theta_{max} = \tan^{-1}\left(\mu \times \left(1 + \dfrac{R^2}{K^2}\right)\right)$

   Ring : $\theta_{max} = \tan^{-1}(2\mu)$,

   Spherical Shell : $\theta_{max} = \tan^{-1}(2.5\ \mu)$

   Disc : $\theta_{max} = \tan^{-1}(3\mu)$

   Solid sphere : $\theta_{max} = \tan^{-1}(3.5\ \mu)$.

ANGULAR MOMENTUM

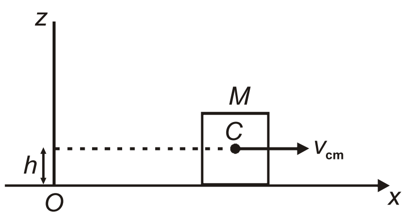
The general formula for angular momentum about any point is

$$\vec{L} = M\vec{v}_{cm}R + I\vec{\omega}$$

Case - I : Pure translation

$|\vec{L}_O| = Mv_{cm}h$

$L_C = 0$



Case - II : Rolling body

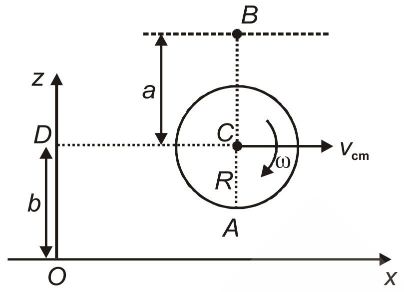
$L_c = I_c\omega$

$L_A = I_C\omega + Mv_{cm}R$

$L_O = I_c\omega + Mv_{cm}b$

$L_B = I_c\omega - Mv_{cm}a$

$L_D = I_c\omega$



Case - III : Centre of mass is fixed

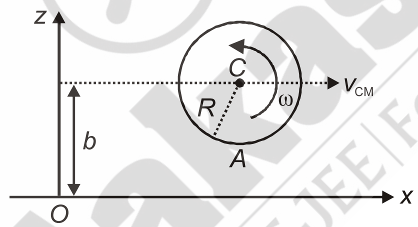
Put $v_{cm} = 0$ in the above results so $L_0 = L_A = L_C = I_C\omega$

Case - IV

$L_c = - I_c\omega$

$L_A = - I_C\omega + Mv_{cm}R$

$L_O = - I_c\omega + Mv_{cm}b$

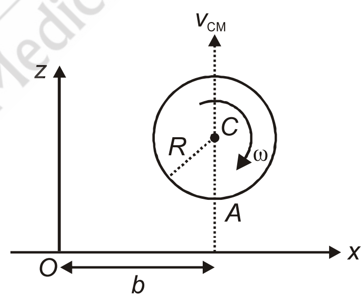


Case - V

$L_c = I_c\omega$

$L_A = I_C\omega$

$L_O = I_c\omega - Mv_{cm}b$



Note : *In all above situations, anticlockwise sense has been assigned a negative sign.*

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